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On the accuracy function and robust solutions for combinatorial optimization problems

M. Libura

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 8373578

fax: (+48) (22) 8372772

Kierownik Pracowni zgłaszający pracę:
prof. dr hab. inż. Krzysztof C. Kiwiel

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Marek Libura

Systems Research Institute
Polish Academy of Sciences
Newelska 6, 01-447 Warszawa, Poland
E-mail: Marek.Libura@ibspan.waw.pl

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Abstract

We consider so-called generic combinatorial optimization problem defined for a finite ground set with specified positive weights of its elements. The set of the feasible solutions of the problem is given as a family of subsets of the ground set. We want to find a subset for which the sum of its elements is minimum among all feasible subsets.

It is assumed that the set of the feasible solutions is fixed, but the weights of all elements may be perturbed simultaneously and independently up to a given percentage of their nominal values. For such a model of perturbations and for an arbitrary feasible solution of the problem we consider so-called *accuracy function*. Its value for a given percentage level of weights perturbations is equal to the maximum relative error of this solution.

A feasible solution is called *robust* for a specified maximum level of perturbations if the value of its accuracy function is minimum among all feasible solutions. The maximum percentage level of perturbations for which an initially optimal solution remains robust is called the *robustness radius* of this solution.

In this paper we obtain lower bounds for the robustness radius in case of single as well as multiple optimal solutions.

Keywords: combinatorial optimization, inexact data, accuracy function, robust solutions.

1 Introduction

Let $E = \{e_1, \ldots, e_n\}$ be a finite *ground set* and let for $e \in E$, $c(e) > 0$ denotes the *weight* of element e. Consider a family $\mathcal{F} \subseteq 2^E \setminus \{\emptyset\}$ of nonempty subsets of E, called *feasible solutions*, and let for $X \in \mathcal{F}$ and $c = (c(e_1), \ldots, c(e_n))^{\mathrm{T}}$,

$$w(c, X) = \sum_{e \in X} c(e)$$

denotes the weight of solution X. The *generic combinatorial optimization problem*

$$v(c) = \min\{w(c, X) : \ X \in \mathcal{F}\} \tag{1}$$

seeks for a feasible solution of minimum weight. Various discrete optimization problems, like the traveling salesman problem, the minimum spanning tree problem, the shortest spanning tree problem, the linear 0-1 programming problem, can be stated in this general form. In the following we assume that the set of the feasible solutions $\mathcal{F}$ is fixed, but the vector of weights c may be perturbed or is given with errors. Namely, we assume that $c \in C(c^o, \delta)$, where for $c^o \in \mathbb{R}$, $c^o > 0$, and $\delta \in [0, 1)$,

$$C(c^o, \delta) = \{d \in \mathbb{R}^n : \ |c^o - d| \leq c^o \cdot \delta\}.$$

Thus, there is some initial vector of weights $c^o > 0$ and for a given parameter $\delta \in [0, 1)$ the maximum perturbation of any weight does not exceed $\delta \cdot 100\%$ of its initial value.

For a given feasible solution $X \in \mathcal{F}$ and $c \in C(c^o, \delta)$, the quality of this solution can be measured by its *relative error* $\varepsilon(c, X)$, where

$$\varepsilon(c, X) = \frac{w(c, X) - v(c)}{v(c)}. \tag{2}$$

Observe that $\varepsilon(c, X) \geq 0$ for arbitrary $X \in \mathcal{F}$, $c \in C(c^o, \delta)$, and $\varepsilon(c, X) = 0$ if and only if X is an optimal solution in (1).

For a given feasible solution $X \in \mathcal{F}$ and $\delta \in [0, 1)$ the *accuracy function* $a(X, \delta)$ introduced in [3] gives the maximum value of the relative error $\varepsilon(c, X)$ for $c \in C(c^o, \delta)$:

$$a(X, \delta) = \max\{\varepsilon(c, X) : \ c \in C(c^o, \delta)\}. \tag{3}$$

It is shown in [3] that for an arbitrary feasible solution X, $a(X, \delta)$ is a nondecreasing and convex function of δ. Also general formulae, which allow to compute the value of $a(X, \delta)$, are given in [3, 4].

The accuracy function has a finite number of breakpoints in the interval $[0, 1)$. If X is an optimal solution in (1), then $a(X, 0) = 0$, but when δ grows, then $a(X, \delta)$ may become positive, which means that X is not longer an optimal solution in (1) for some $c \in C(c^o, \delta)$. From the practical point of view it is of special interest to know the first breakpoint of the accuracy function, corresponding to the least value of δ for which $a(X, \delta)$ becomes positive. This value is called the *accuracy radius* of the solution X and formally is defined as follows:

$$r^a(X) = \sup\{\delta \in [0, 1) : \ a(X, \delta) = 0\}. \tag{4}$$

In [4] a general formula, which allows to find the exact value of the accuracy radius is given, and an approach to calculate a lower bound for this value is described.

Example

Consider an undirected graph $G = (V, E)$, where $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$.

Let $\mathcal{F}$ be a family of subsets of E corresponding to all spanning trees in G, and let $c^o = (14, 11, 14, 15, 13, 18, 17)^T$ be a vector of the initial weights of edges in G. Then the combinatorial optimization problem (1) for $c = c^o$ is just the minimum spanning tree problem in G. A subset of edges $X = \{\{1, 2\}, \{1, 3\}, \{3, 4\}, \{4, 5\}\}$ is an optimal solution for this problem. The graph G and the minimum spanning tree X are shown in Figure 1.

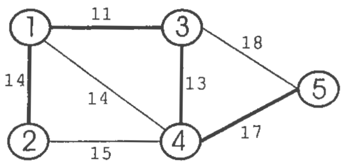

Figure 1: Graph G and its minimum spanning tree indicated with bold lines.

In Figure 2 the accuracy function of the solution X is shown for $\delta \in [0, 0.5]$. From this picture one can read that the solution X remains optimal if the maximum percentage perturbation of any weight do not exceed approximately 2.8%; this value corresponds to the accuracy radius, which in this case is equal to $1/35$. For larger values of perturbations the solution X may become suboptimal and, for example, for $\delta = 0.3$, i.e., when the maximum perturbations of weights are equal 30% of their initial values, the maximum relative error of X reaches 60%.

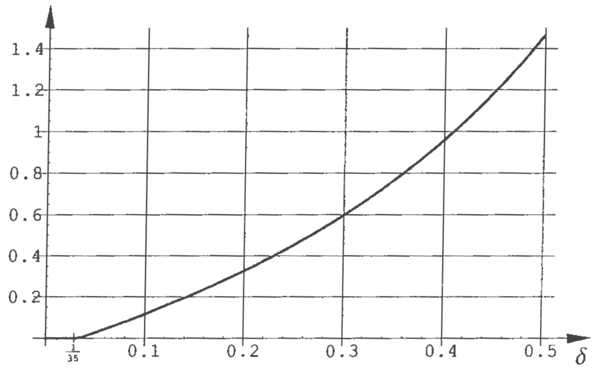

Figure 2: The accuracy function of the optimal spanning tree X.

□

In the framework of so-called *robust optimization* (see e.g. [2]) the set $C(c^o, \delta)$ for a given fixed value of δ is interpreted as a set of possible scenarios. Then the value of $a(X, \delta)$ provides a so-called *worst-case relative regret* of the solution X over the set of possible scenarios. In the robust optimization one wants to find such a feasible solution, that its worst-case relative regret is the minimum among the feasible solutions of problem (1). Therefore we will consider the following function of $\delta \in [0, 1)$:

$$z(\delta) = \min_{X \in \mathcal{F}} a(X, \delta). \tag{5}$$

We will call this function the minimum relative regret function or – for short – the *regret function.* A feasible solution X will be called *robust* for a given $\delta \in [0, 1)$ if and only if $a(X, \delta) = z(\delta)$.

It is obvious that the solution $X \in \mathcal{F}$ is robust for a given δ if $a(X, \delta) = 0$. Thus, if X is an optimal solution for $\delta = 0$, then it remains robust in the interval $[0, r^a(X)]$. But it may be robust also for larger values of δ (see example below). On the other hand, a solution which is non-optimal for $\delta = 0$ may became a robust solution for larger values of perturbations.

If X is an optimal solution in (1) for $\delta = 0$, then the maximum value of δ for which X remains robust is called the *robustness radius* of X and denoted by $r^r(X)$. Formally:

$$r^r(X) = \sup\{\delta \in [0, 1) : \ a(X, \delta) = z(\delta)\}. \tag{6}$$

The exact value of $r^r(X)$ (or a nontrivial lower bound on it) appears interesting from the practical point of view. If X is a single optimal solution in (1), then $r^r(X)$ gives the maximum percentage perturbations of the weight coefficients, that the solution X guarantees the minimum relative regret among all feasible solutions. If there are multiple feasible solutions in (1), then an optimal solution with the largest robustness radius may be regarded as preferable to be implemented. In this case the value of the robustness radius provides the decision maker with an additional information, which allows to distinguish multiple optimal solutions from the robustness point of view.

Example (continued)

In Figure 4 the regret function for the minimum spanning tree problem in graph G from Figure 1 is shown. According to (5) this function is a pointwise minimum of the accuracy functions for all spanning trees in graph G. In this case the regret function is determined by the following three spanning trees: $X = \{\{1,2\},\{1,3\},\{3,4\},\{4,5\}\}$, $X' = \{\{1,2\},\{1,3\},\{2,4\},\{4,5\}\}$ and $X'' = \{\{1,2\},\{2,4\},\{3,5\},\{4,5\}\}$; all other feasible solutions may be neglected in (5). Corresponding accuracy functions for solutions X, X' and X'' are shown in Figure 3.

From Figure 4 one can see that the solution X remains robust behind its accuracy radius. The robustness radius of this solution is determined by the value of $\delta = \delta'$, for which the accuracy functions of X and X' are equal. In our example δ' is equal approximately 0.23. This means that the solution X remains robust if the maximum percentage perturbation of any weight do not exceed approximately 23% of its nominal value.

For larger values of δ the solution X' becomes a robust solution and it remains robust till approximately $\delta'' = 0.43$. For larger level of perturbations, again, we have a new robust solution, this time it is X''.

□

Calculating the exact value of the robustness radius is a difficult task. Therefore in the next section we give some simple bounds for the accuracy functions and derive corresponding bounds for the robustness radius and the regret function.

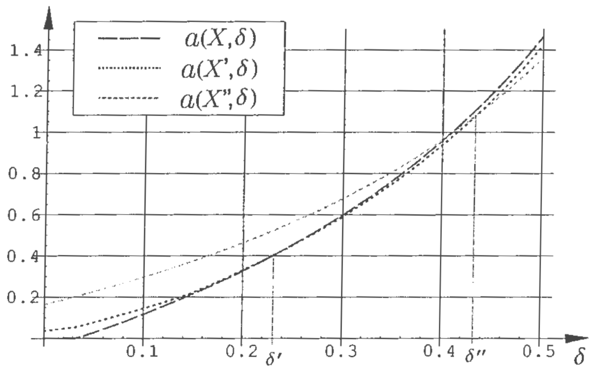

Figure 3: Accuracy functions for spanning trees X, X' and X''.

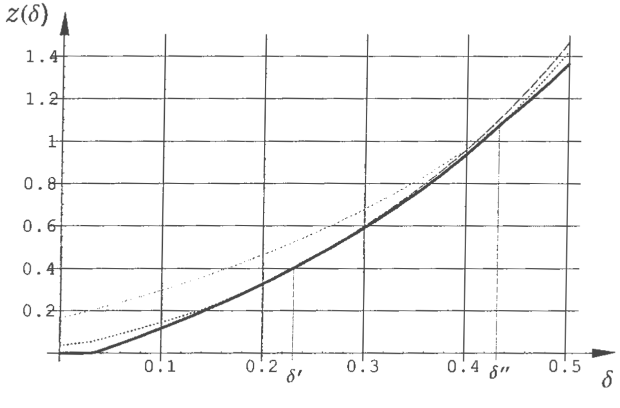

Figure 4: The regret function for the minimum spanning tree problem.

2 Bounds for the regret function and the robustness radius

In [3] it is shown, that for $X \in \mathcal{F}$ and $\delta \in [0,1)$ the accuracy function of X is expressed by the following formula:

$$a(X,\delta) = \max_{Y \in \mathcal{F}} \frac{w(c^o, X) - w(c^o, Y) + \delta\, w(c^o, X \otimes Y)}{(1-\delta)\, w(c^o, Y)}, \tag{7}$$

where $X \otimes Y = (X \cup Y) \setminus (X \cap Y)$. It will be convenient to rewrite (7) in the following equivalent form:

$$a(X,\delta) = \max_{Y \in \mathcal{F}} \frac{(1+\delta) w(c^o, X) - (1-\delta) w(c^o, Y) - 2\,\delta\, w(c^o, X \cap Y)}{(1-\delta)\, w(c^o, Y)}. \tag{8}$$

Lemma 1 gives an upper bound for the accuracy function of an arbitrary feasible solution in problem (1).

Lemma 1 *For $X \in \mathcal{F}$ and $\delta \in [0,1)$,*

$$a(X,\delta) \le \frac{2\delta}{1-\delta} + \frac{1+\delta}{1-\delta} \cdot a(X,0). \tag{9}$$

Proof For arbitrary $X, Y \in \mathcal{F}$ we have $w(c^o, X \cap Y) \ge 0$ and

$$w(c^o, X) \le w(c^o, Y) + w(c^o, X) - v(c^o).$$

Thus, after replacing in (8) $w(c^o, X)$ with $w(c^o, Y) + w(c^o, X) - v(c^o)$ and removing $2\delta\, w(c^o, X \cap Y)$, we obtain:

$$\begin{aligned} a(X,\delta) &\le \max_{Y \in \mathcal{F}} \left\{ \frac{(1+\delta) w(c^o, Y) - (1-\delta) w(c^o, Y)}{(1-\delta) w(c^o, Y)} \right. \\ &\qquad \left. + \frac{1+\delta}{1-\delta} \cdot \frac{w(c^o, X) - v(c^o)}{w(c^o, Y)} \right\} \\ &= \frac{2\delta}{1-\delta} + \frac{1+\delta}{1-\delta} \cdot \max_{Y \in \mathcal{F}} \frac{w(c^o, X) - v(c^o)}{w(c^o, Y)} \\ &= \frac{2\delta}{1-\delta} + \frac{1+\delta}{1-\delta} \cdot a(X,0). \end{aligned}$$

■

Corollary 1 *For $\delta \in [0, 1)$,*

$$z(\delta) \leq \frac{2\delta}{1-\delta}. \tag{10}$$

Proof For arbitrary $X \in \mathcal{F}$, $a(X, 0) \geq 0$. Moreover, if X^o is an optimal solution in (1) for $c = c^o$, then $a(X^o, 0) = 0$. So, from the definition of the regret function and from (9) we have immediately

$$z(\delta) \leq \min_{X \in \mathcal{F}} \left\{ \frac{2\delta}{1-\delta} + \frac{1+\delta}{1-\delta} \cdot a(X, 0) \right\} = \frac{2\delta}{1-\delta} + \frac{1+\delta}{1-\delta} \cdot a(X^o, 0) = \frac{2\delta}{1-\delta}.$$

■

The following lemma provides a simple lower bound for the accuracy function of any feasible solution X.

Lemma 2 *For $X \in \mathcal{F}$ and $\delta \in [0, 1)$,*

$$a(X, \delta) \geq \frac{1+\delta}{1-\delta} \cdot a(X, 0). \tag{11}$$

Proof For a given solution X and arbitrary $Y \in \mathcal{F}$ we have the following inequality:

$$a(X, \delta) \geq \frac{(1+\delta)w(c^o, X) - (1-\delta)w(c^o, Y) - 2\delta w(c^o, X \cap Y)}{(1-\delta)w(c^o, Y)}.$$

Taking $Y = X^o$, where X^o is an optimal solution for $c = c^o$, we have:

$$a(X, \delta) \geq \frac{(1+\delta)w(c^o, X) - (1-\delta)v(c^o) - 2\delta w(c^o, X \cap X^o)}{(1-\delta)v(c^o)}.$$

Replacing $w(c^o, X \cap X^o)$ with $v(c^o) = w(c^o, X^o) \geq w(c^o, X \cap X^o)$ we obtain:

$$\begin{aligned} a(X, \delta) &\geq \frac{(1+\delta)w(c^o, X) - (1-\delta)v(c^o) - 2\delta v(c^o)}{(1-\delta)v(c^o)} \\ &= \frac{1+\delta}{1-\delta} \cdot \frac{w(c^o, X) - v(c^o)}{v(c^o)} \\ &= \frac{1+\delta}{1-\delta} \cdot a(X, 0). \end{aligned}$$

■

Assume now that X^o is a single optimal solution in problem (1) for $c = c^o$, and let $a_1 > 0$ denotes the relative error of the second-best solution in this problem. If we know the exact value of a_1 or some positive lower bound for a_1, then the bounds for the accuracy function provided by Lemma 1 and Lemma 2 allow to calculate a lower bound for the robustness radius of X^o. The following fact holds:

Theorem 1 *If X^o is a single optimal solution in problem* (1) *for $c = c^o$, and $a_1 = a(X^1, 0)$, where X^1 is a second-best solution in* (1), *then*

$$r^\tau(X^o) \geq \begin{cases} \frac{a_1}{2-a_1} & if \quad a_1 < 1, \\ 1 & otherwise. \end{cases} \tag{12}$$

Proof Consider the following two convex functions of δ on the interval $[0, 1)$: $f'(\delta) = \frac{2\delta}{1-\delta}$, which – according to Lemma 1 – is an upper bound on $a(X^o, \delta)$ and $f''(\delta) = \frac{1+\delta}{1-\delta} \cdot a_1$, which – according to Lemma 2 – is a lower bound on $a(X^1, \delta)$. From Lemma 2 it follows, that f'' provides also a lower bound for the accuracy function of any feasible solution $Y \in \mathcal{F} \setminus \{X^o\}$. Thus, the solution X^o remain robust for all such $\delta \in [0, 1)$ that $f'(\delta) \leq f''(\delta)$. If $a_1 \geq 1$, then this inequality holds for any $\delta \in [0, 1)$ which means that $r^\tau(X^o) = 1$. For $a_1 < 1$ the inequality $f'(\delta) \leq f''(\delta)$ is valid for $\delta \leq \frac{a_1}{2-a_1}$ and this value provides a lower bound on the robustness radius of X^o. ■

Consider now the case when there are multiple optimal solutions in problem (1). Let Ω, where $|\Omega| = p > 1$, denote the set of all optimal solutions in problem (1) for $c = c^o$ and let

$$b = \min\{w(c^o, X) : \ X \in \mathcal{F} \setminus \Omega\} - v(c^o).$$

All of the solutions belonging to Ω give the same optimal objective value for $\delta = 0$, but they may differ from the robustness point of view. It is obvious that any solution in Ω is robust for $\delta = 0$, but an interesting question arises, how to select an optimal solution which remains robust in some neighborhood of $\delta = 0$.

From the formula (7) on the accuracy function it follows directly that for any $X \in \Omega$ we have $a(X, \delta) = 0$ for $\delta = 0$, and $a(X, \delta) > 0$ for $\delta > 0$. Moreover, the following lemma states that for some neighborhood of $\delta = 0$ the accuracy function of any solution belonging to Ω depends only on other solutions from this set, and does not depend on any feasible solution belonging to the set $\mathcal{F} \setminus \Omega$.

Lemma 3 *For $X \in \Omega$ and $\delta \le \delta' = b/d$, where $d = \min\{w(c^o, E),\ v(c^o) + \max_{Y \in \mathcal{F} \setminus \Omega} w(c^o, Y)\}$,*

$$a(X, \delta) = \frac{2\delta}{(1-\delta)} - \frac{2\delta}{(1-\delta)v(c^o)} \cdot \min_{Y \in \Omega} w(c^o, X \cap Y). \tag{13}$$

Proof For arbitrary $X \in \Omega$ and $Y \in \mathcal{F} \setminus \Omega$ we have $w(c^o, Y) - w(c^o, X) \ge b$ and $w(c^o, X \otimes Y) \le d$, which implies that for $\delta \le \delta'$ the following inequality holds:

$$w(c^o, X) - w(c^o, Y) + \delta w(c^o, X \otimes Y) \le 0.$$

But this means that then the maximum over the set $\mathcal{F}$ in (7) can be replaced with the maximum over the set Ω. Thus, for arbitrary $X \in \Omega$ and $\delta \le \delta'$,

$$\begin{aligned}
a(X, \delta) &= \max_{Y \in \mathcal{F}} \frac{w(c^o, X) - w(c^o, Y) + \delta w(c^o, X \otimes Y)}{(1-\delta) w(c^o, Y)} \\
&= \max_{Y \in \Omega} \frac{w(c^o, X) - w(c^o, Y) + \delta w(c^o, X \otimes Y)}{(1-\delta) w(c^o, Y)} \\
&= \max_{Y \in \Omega} \frac{\delta w(c^o, X \otimes Y)}{(1-\delta) v(c^o)} \\
&= \frac{\delta}{(1-\delta) v(c^o)} \cdot \max_{Y \in \Omega} w(c^o, X \otimes Y) \\
&= \frac{\delta}{(1-\delta) v(c^o)} \cdot \max_{Y \in \Omega} \left(w(c^o, X) + w(c^o, Y) - 2w(c^o, X \cap Y)\right) \\
&= \frac{2\delta}{(1-\delta) v(c^o)} \cdot \left(v(c^o) - \min_{Y \in \Omega} w(c^o, X \cap Y)\right) \\
&= \frac{2\delta}{(1-\delta)} - \frac{2\delta}{(1-\delta) v(c^o)} \cdot \min_{Y \in \Omega} w(c^o, X \cap Y).
\end{aligned}$$

■

Lemma 3 allows to formulate a *necessary* condition for a solution from the set Ω to be robust in the neighborhood of $\delta = 0$. Directly from the definition of the regret function and from (13) we have the following corollary:

Corollary 2 *If an optimal solution $X^o \in \Omega$ remains robust in some neighborhood of $\delta = 0$, then the following condition must hold:*

$$\min_{Y \in \Omega} w(c^o, X^o \cap Y) = \max_{X \in \Omega} \min_{Y \in \Omega} w(c^o, X \cap Y). \tag{14}$$

Proof A solution $X^o \in \mathcal{F}$ is robust for a given $\delta \in [0,1)$ if and only if $a(X^o,\delta) = z(\delta) = \min_{X\in\mathcal{F}} a(X,\delta)$. When for $X^o \in \Omega$ the condition (14) does not hold, i.e., $\min_{Y\in\Omega} w(c^o, X^o \cap Y) < \max_{X\in\Omega} \min_{Y\in\Omega} w(c^o, X\cap Y)$, then for $\delta > 0$ it follows from (13) that

$$a(X^o,\delta) > \min_{X\in\Omega} a(X,\delta) \geq \min_{X\in\mathcal{F}} a(X,\delta)$$

and therefore X^o is not a robust solution.

■

Next we will show that the condition (14) is also a *sufficient* condition for the robustness of X^o in some neighborhood of $\delta = 0$ and we will give a corresponding lower bound for the robustness radius of X^o. Observe that if we know all of the p elements of the set Ω, then such a robust solution can be selected in $O(p^2)$ calculations of weights $w(c^o, X\cap Y)$ for $X, Y \in \Omega$, followed by $O(p^2)$ comparisons.

The following theorem is an analogue of Theorem 1 for the case of multiple optimal solutions.

Theorem 2 *If $X^o \in \Omega$ and X^o satisfies the condition* (14), *then*

$$r^\tau(X^o) \geq \min\{\delta', \delta''\}, \tag{15}$$

where

$$\delta' = \begin{cases} \frac{b}{2v(c^o)-b} & if \quad b < v(c^o), \\ 1 & otherwise, \end{cases}$$

and

$$\delta'' = \frac{b}{\min\{w(c^o,E),\ v(c^o) + \max_{X\in\mathcal{F}\setminus\Omega} w(c^o,X)\}}.$$

Proof From Lemma 1 we have that for $\delta \in [0,1)$ the function $f'(\delta) = \frac{2\delta}{1-\delta}$ provides an upper bound for $a(X^o,\delta)$, and from Lemma 2 it follows that $f''(\delta) = \frac{1+\delta}{1-\delta} \cdot \frac{b}{v(c^o)}$ is a lower bound on $a(Y,\delta)$ for any $Y \in \mathcal{F}\setminus\Omega$. But for $\delta < \delta'$, $f'(\delta) \leq f''(\delta)$, which implies that then $a(X^o,\delta) \leq \min_{Y\in\mathcal{F}\setminus\Omega} a(Y,\delta)$.

If, moreover, X^o satisfies (14), then from Lemma 3 it follows that for $\delta < \delta''$, $a(X^o,\delta) \leq \min_{Y\in\Omega} a(Y,\delta)$. Thus, for $\delta < \min\{\delta',\delta''\}$ the inequality $a(X^o,\delta) \leq \min_{Y\in\mathcal{F}} a(Y,\delta)$ is valid, which means that X^o is a robust solution on the interval $[0, \min\{\delta',\ \delta''\}\ )$ and (15) holds.

■

3 Conclusions

In this paper we consider the generic combinatorial optimization problem with inexact data. It is assumed that any coefficient in the objective function may differ from their nominal value by at most a given percentage $\delta \cdot 100\%$. Thus, in the framework of so-called robust optimization with interval data, the parameter $\delta \in [0, 1)$ determines a particular set of scenarios.

We exploit previous results concerning the accuracy function to derive lower bounds for perturbations, for which a given optimal solution, obtained for nominal parameters, remains robust.

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