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On the adjustment problem for linear programs

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Abstract

We propose a generalization of the inverse problem which we will call the *adjustment problem.* For an optimization problem with linear objective function and its restriction defined by a given subset of feasible solutions, the adjustment problem consists in finding the least costly perturbations of the original objective function coefficients, which guarantee that an optimal solution of the perturbed problem is also feasible for the considered restriction. We describe a method of solving the adjustment problem for continuous linear programming problems when variables in the restriction are required to be binary.

Keywords: combinatorial optimization, linear programming, inverse optimization, adjustment problem.

1 Introduction

Consider an optimization problem (P) with linear objective function and let F be a given subset of the set of feasible solutions of (P). Assume that one wants to adjust the objective function coefficients in (P), such that an optimal solution of the perturbed problem belongs to F. The *adjustment problem* introduced in (Libura, 2001) seeks among admissible perturbations of the coefficients, one that is least costly according to some given norm.

If for example (P) is the minimum spanning tree problem in a given graph, then we may look for such perturbations of lengths of edges, that there exists a minimum spanning tree for the perturbed graph that is a Hamiltonian path. Similarly, given a (continuous) linear programming problem, we may be interested in such perturbations of the objective function coefficients, which would guarantee that there is an optimal solution of the perturbed problem, satisfying additional restrictions, e.g. integrality restrictions.

When the restricted solution set F contains only a single element x^o, then the adjustment problem becomes the so-called *inverse problem* with respect to x^o. The inverse problem and some of its variants have attracted recently significant attention (see e.g. Ahuja and Orlin, 2001; Heuberger, 2004; Zhang and Liu, 2002).

The adjustment problem appears to be more difficult than the standard inverse problem. The latter problem is in fact equivalent to the problem of finding a projection of the original objective vector on so-called stability region of the solution x^o (see e.g. Greenberg, 1998; Libura, 1996). If (P) is a continuous or integer linear program, then the stability region of x^o is a polyhedral convex cone. Also the adjustment problem for $|F| > 1$ is equivalent to the problem of finding a projection of the objective vector; in this case onto a union of convex cones, which is not necessarily a convex region.

In this paper we formulate and apply a mixed-integer linear programming model to solve the adjustment problem for a continuous linear programming problem (P), when F is given by linear and integrality constraints, such that all the variables, for which we are allowed to adjust the objective function coefficients, are binary.

The paper is organized as follows: Section 2 contains the formal definition of the adjustment problem. In Section 3 we state the adjustment problem for continuous linear programming problems. In Section 4 we describe a reformulation of the adjustment problem in the case of zero-one variables. Section 5 contains examples illustrating the approach.

2 Notation and preliminaries

Let $c \in \mathbb{R}^n$ and $X \subseteq \mathbb{R}^n$. We will consider an optimization problem with linear objective function:

$$v(c, X) = \max\{c^{\mathsf{T}}x : x \in X\} \tag{1}$$

and its restriction for a given subset $F \subseteq X$ of feasible solutions:

$$\max\{c^{\mathsf{T}}x : x \in F\}. \tag{2}$$

The *adjustment problem* related to F and a given set $\Delta \subseteq \mathbb{R}^n$ is stated as follows:

$$a(F, \Delta) = \min\{\|\delta\| : \; v(c+\delta, X) = v(c+\delta, F), \; \delta \in \Delta\}, \tag{3}$$

where $\|\delta\|$ denotes a norm of δ. In this paper we will consider mainly l_1 or l_∞ norms, i.e., $\|\delta\| = \|\delta\|_1 = \sum_{i=1}^n |\delta_i|$ or $\|\delta\| = \|\delta\|_\infty = \max_{i=1,\dots,n} |\delta_i|$. Nevertheless, the described approach may be used also when $\|\delta\|$ is a so-called weighted l_1 or l_∞ norm for some given vector $w = (w_1, \dots, w_n)^\mathrm{T} \in \mathbb{R}^n$, $w > 0$. In this case we have $\|\delta\| = \|\delta\|_{1,w} = \sum_{i=1}^n |\delta_i|/w_i$ or $\|\delta\| = \|\delta\|_{\infty,w} = \max_{i=1,\dots,n} |\delta_i|/w_i$. The set $\Delta \subseteq \mathbb{R}^n$ describes all admissible perturbations of the original vector of weights c. In the following we will assume that $\Delta = \mathbb{R}^n$ or Δ is a bounded subset of $\mathbb{R}^n$.

Given sets $\Delta \subseteq \mathbb{R}^n$ and $F \subseteq X$ we will call the optimal value $a(F, \Delta)$ of the problem (3), the *adjustment cost* with respect to F and Δ.

When $F = \{x^o\}$, $x^o \in \mathbb{R}^n$, then the adjustment problem becomes the standard *inverse problem* with respect to x^o:

$$i(x^o, \Delta) = a(\{x^o\}, \Delta) = \min\{\|\delta\| : \; v(c+\delta, X) = (c+\delta)^\mathrm{T} x^o, \; \delta \in \Delta\}. \tag{4}$$

If an optimal solution of the adjustment problem exists, then directly from the definitions of the adjustment problem and the inverse problem it follows, that such a solution provides also an optimal solution for the inverse problem with respect to some element of F. Moreover, $a(F, \Delta) \leq i(x, \Delta)$ for any $x \in F$. Thus we have the following fact:

Proposition 1 *For $F \subseteq X$ and $\Delta \subseteq \mathbb{R}^n$,*

$$a(F, \Delta) = \min\{i(x, \Delta) : \; x \in F\}. \tag{5}$$

This proposition might suggest, that the adjustment problem can be solved in two phases: by finding an optimal solution x^* of the restriction (2) and then by solving the inverse problem with respect to x^*. But even if a solution of that inverse problem exists, such approach may fail as the following simple example shows.

Consider an optimization problem: $\max\{c^\mathrm{T} x : \; x \in X\}$, where $c = (4, 5)^\mathrm{T}$, $X = \{x = (x_1, x_2)^\mathrm{T} \in \mathbb{R}^2 : 2x_1 + x_2 \leq 2, \; 0 \leq x_1 \leq 1, \; 0 \leq x_2 \leq 1\}$, and its restriction for $F = X \cap \mathbb{Z}^2$, where $\mathbb{Z}$ denotes the set of integers. Thus the original problem is a continuous linear programming problem and the restriction consists in imposing integrality conditions on variables x_1, x_2. It is easy to see, that the vector $x^* = (0, 1)^\mathrm{T}$ is a unique optimal solution of the restricted problem

$$\max\{4x_1 + 5x_2 : \; 2x_1 + x_2 \leq 2, \; 0 \leq x_1 \leq 1, \; 0 \leq x_2 \leq 1, \; x_1, x_2 \in \mathbb{Z}\}.$$

Solving the inverse problem with respect to x^* for the l_1 norm and $\Delta = \mathbb{R}^2$ we obtain $i(x^*, \mathbb{R}^2) = a(\{x^*\}, \mathbb{R}^2) = 4$ and $\delta^* = (-4, 0)^{\mathrm{T}}$. But this is not a solution of the adjustment problem with respect to F, because in this case we have $a(F, \mathbb{R}^2) = 3$ with $\delta^a = (0, -3)^{\mathrm{T}}$ as an optimal vector of perturbations.

The optimal solutions of both these problems are shown in Figure 1, where $S(0,0)$, $S(0,1)$ and $S(1,0)$ denote, respectively, the stability regions of solutions belonging to $F = \{(0,0)^{\mathrm{T}}, (0,1)^{\mathrm{T}}, (1,0)^{\mathrm{T}}\}$. Solving the inverse problem with respect to $x^* = (0,1)^{\mathrm{T}}$ we obtain $c^* = c + \delta^*$ as a projection of the vector $c = (4,5)^{\mathrm{T}}$ on a convex cone $S(0,1)$. But an optimal solution of the adjustment problem with respect to F corresponds to the vector $c^a = c + \delta^a$, which is a projection of c onto a non-convex set $S(0,0) \cup S(0,1) \cup S(1,0)$.

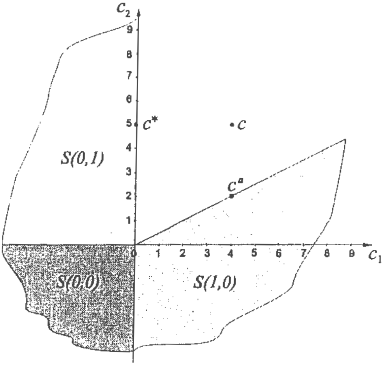

Figure 1: Optimal solutions of the inverse problem and the adjustment problem.

3 The adjustment problem for linear programming problems

Let

$$X = \{x \in \mathbb{R}^n : Ax \le b,\ x \ge 0\},$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. In this case the initial problem (1) is a linear programming problem

$$\max\{c^{\mathrm{T}}x : Ax \le b,\ x \ge 0\}. \tag{6}$$

The following lemma (see e.g. Padberg, 1995) states well known optimality conditions for problem (6):

Lemma 1 *A feasible solution $x^o \in X$ is an optimal solution of problem* (6) *if and only if there exists $y \in \mathbb{R}^m$ such that $y \geq 0$ and*

(i) $A^{\mathsf{T}}y \geq c$,

(ii) $b^{\mathsf{T}}y = c^{\mathsf{T}}x^o$.

Given $\Delta \subseteq \mathbb{R}^n$ and a feasible solution x^o for (6), it follows from Lemma 1 that the inverse problem with respect to x^o can be stated as the following mathematical programming problem:

$$\begin{aligned} i(x^o, \Delta) \quad = \quad & \min \|\delta\| \\ & A^{\mathsf{T}}y - \delta \geq c \\ & b^{\mathsf{T}}y - \delta^{\mathsf{T}}x^o = c^{\mathsf{T}}x^o \\ & y \geq 0 \\ & \delta \in \Delta. \end{aligned} \tag{7}$$

Observe that if $\Delta = \mathbb{R}^n$ or Δ is a polyhedron in $\mathbb{R}^n$, then for the l_1 and l_∞ norm problem (7) can be easily stated as a linear programming problem.

Let $F = \{x \in X : Cx \leq d\} \cap S$, where $C \in \mathbb{R}^{p \times n}$, $d \in \mathbb{R}^p$ and S is some specified subset of $\mathbb{R}^n$. Thus the restriction of the original problem (1) is defined by adding new linear constraints

$$Cx \leq d$$

and requiring that feasible solutions belong to the set S. In the following we will usually assume that $S = \mathbb{Z}^n$ or we will simply take $S = \mathbb{R}^n$.

The adjustment problem with respect to F and Δ can be formulated now as the following mathematical programming problem:

$$\begin{aligned} a(F, \Delta) \quad = \quad & \min \|\delta\| \\ & A^{\mathsf{T}}y - \delta \geq c \\ & b^{\mathsf{T}}y - c^{\mathsf{T}}x - \delta^{\mathsf{T}}x = 0 \\ & Ax \leq b \\ & Cx \leq d \\ & x, y \geq 0, \; x \in S \\ & \delta \in \Delta. \end{aligned} \tag{8}$$

Observe that (8) is no longer easily stated as a continuous linear program, even when $\Delta = S = \mathbb{R}^n$, due to the nonlinear term $\delta^{\mathsf{T}}x$.

4 Adjustment problem with binary variables

Consider now a special case of the adjustment problem. Namely, assume that $F \subseteq \{0,1\}^n$, i.e., all the variables are binary in F. We are faced with such a situation if $S = \{0,1\}^n$ or when $S = \mathbb{Z}^n$ and – moreover – constraints $Ax \le b$, $Cx \le d$, $x \ge 0$, contain or imply bounds $x \le \mathbf{1}$, where $\mathbf{1} \in \mathbb{R}^n$ denotes a vector of ones.

If the set of feasible solutions of the restricted problem fulfills the requirement $F \subseteq \{0,1\}^n$, then the nonlinear term $\delta^{\mathsf{T}}x$ in (8) may be formally linearized in a standard way using additional variables and constraints. In the following we will describe this reformulation.

It will be convenient to express the vector $\delta = (\delta_1, \ldots, \delta_n)^{\mathsf{T}} \in \mathbb{R}^n$ as a difference of two nonnegative vectors δ^+, $\delta^- \in \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \ge 0\}$. Let

$$\delta = \delta^+ - \delta^-,$$

where

$$\delta^+ = (\delta_1^+, \ldots, \delta_n^+)^{\mathsf{T}}, \ \ \delta_i^+ = \max\{0, \delta_i\}, \ i = 1, \ldots, n,$$
$$\delta^- = (\delta_1^-, \ldots, \delta_n^-)^{\mathsf{T}}, \ \ \delta_i^- = \max\{0, -\delta_i\}, \ i = 1, \ldots, n.$$

Thus we have

$$\delta^{\mathsf{T}}x = \sum_{i=1}^{n} (\delta_i^+ x_i - \delta_i^- x_i).$$

We will introduce new nonnegative variables z_i^+, z_i^-, $i = 1, \ldots, n$, satisfying the following conditions:

$$z_i^+ = \delta_i^+ x_i, \quad i = 1, \ldots, n, \tag{9}$$

$$z_i^- = \delta_i^- x_i, \quad i = 1, \ldots, n. \tag{10}$$

Now we can replace the constraint

$$b^{\mathsf{T}}y - c^{\mathsf{T}}x - \delta^{\mathsf{T}}x = 0$$

in problem (8) with the following linear constraint:

$$b^{\mathsf{T}}y - c^{\mathsf{T}}x - \mathbf{1}^{\mathsf{T}}z^+ + \mathbf{1}^{\mathsf{T}}z^- = 0,$$

where $z^+ = (z_1^+, \ldots, z_n^+)^{\mathsf{T}} \in \mathbb{R}^n_+$ and $z^- = (z_1^-, \ldots, z_n^-)^{\mathsf{T}} \in \mathbb{R}^n_+$.

For any new variable z_i^+, z_i^-, $i = 1, \ldots, n$, we have to add also constraints which would guarantee that equations (9) and (10) hold. Let us take for example the equation $z_i^+ = \delta_i^+ x_i$ for some index $i \in \{1, \ldots, n\}$. This equation is equivalent to two implications:

$$\begin{aligned} x_i = 0 \quad &\Longrightarrow \quad z_i^+ = 0, \\ x_i = 1 \quad &\Longrightarrow \quad z_i^+ = \delta_i^+, \end{aligned}$$

which can be modeled in a standard way (see e.g. Williams, 1993) by adding the following new constraints:

$$\begin{aligned} z_i^+ - Mx_i &\leq 0, \\ -\delta_i^+ + z_i^+ &\leq 0, \\ \delta_i^+ - z_i^+ + Mx_i &\leq M, \end{aligned}$$

where M a is sufficiently large constant satisfying the inequality $\delta_i^+ \leq M$ for any $i = 1, \ldots, n$. If the set of admissible perturbations Δ is bounded, then a suitable value of M can be calculated directly from the description of Δ. If $\Delta = \mathbb{R}^n$, then we can simply take $M = \|c\|_1$. Indeed, in this case $y = 0$ and $\delta = -c$ provides a feasible solution of (8) for any $x \in F$ and thus there exists an optimal solution of (8), in which $|\delta_i| \leq \|c\|_1$.

Finally, the adjustment problem may be stated in the following form:

$$\begin{aligned} a(F, \Delta) \quad = \quad & \min \; \|\delta^+\| + \|\delta^-\| \\ & A^{\mathsf{T}}y - \delta^+ + \delta^- \geq c \\ & b^{\mathsf{T}}y - c^{\mathsf{T}}x - \mathbf{1}^{\mathsf{T}}z^+ + \mathbf{1}^{\mathsf{T}}z^- = 0 \\ & z^+ - Mx \leq 0 \\ & -\delta^+ + z^+ \leq 0 \\ & \delta^+ - z^+ + Mx \leq M \\ & z^- - Mx \leq 0 \\ & -\delta^- + z^- \leq 0 \\ & \delta^- - z^- + Mx \leq M \\ & \delta^+ - \delta^- \in \Delta \\ & x \in F \subseteq \{0, 1\}^n \\ & y,\, z^+, z^-, \delta^+, \delta^- \geq 0. \end{aligned} \tag{11}$$

Thus for a linear programming problem (1) and $\Delta = \mathbb{R}^n$ or Δ given as a polyhedron in $\mathbb{R}^n$, the adjustment problem with $F \subseteq \{0, 1\}^n$ and l_1 or l_∞ norm in $\mathbb{R}^n$ can be stated as a mixed-integer linear programming problem.

We will finish this section with some comments concerning the described approach.

A similar approach can be used for initial linear programs with constraints of the equality type. Also, only slight modifications are needed when the problem (1) is stated as a minimization problem.

Observe that in some cases we may relax the requirement, that all of the variables in the considered restriction of the initial problem must be binary. In fact, we want to linearize the terms $\delta_i^+ x_i$ and $\delta_i^- x_i$ only for such variables for which we are allowed to change the coefficient in the objective function. Formally, the set of these variables is determined by the set Δ. Frequently, the considered formulation of the optimization problem contains some auxiliary variables, which do not appear in the objective function. Thus, it is quite natural that these variables do not appear in the statement of the adjustment problem either, and may be omitted in the linearization step (see Example 2).

The choice of an appropriate norm in the adjustment problem (11) depends on that, what we actually want to minimize: the sum of perturbations, the maximum value of the perturbation or the maximum percentage of necessary changes in the objective function coefficients. For example, in the latter case the weighted $l_{\infty,w}$ norm with $w = c$ is appropriate.

In the following section we will give two comprehensive examples, which illustrate the described approach. An 1.133 GHz Pentium III computer and CPLEX 6.5 package were used to solve the MIP models arising for these examples. In all cases computing times for the adjustment problems were below 0.2 sec.

5 Examples

Example 1

Consider a weighted digraph G shown in Figure 2. The following path of length 17 (given as a subset of arcs) is the shortest path from vertex s to vertex t in G:

$$p = \{ (s, 2), (2, 4), (4, 5), (5, 3), (3, 6), (6, t) \}.$$

This path is indicated with bold lines in Figure 2.

Assume that we are interested in paths from s to t which pass through vertex 1 and we want to find the smallest possible modification of arc lengths which would guarantee that there is such a path among the shortest paths in the modified network. Therefore we have to solve the adjustment problem

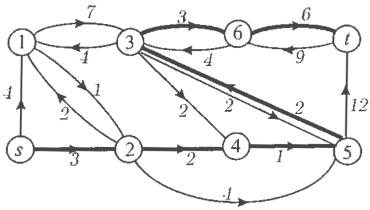

Figure 2: Digraph G from Example 1 with indicated lengths of arcs.

related to the original problem and its appropriate restriction requiring that the path from s to t contains the vertex 1.

It is well known that the shortest path problem can be stated as a linear programming problem (see e.g. Nemhauser and Wolsey, 1988).

Let $V = \{v_1, \ldots, v_8\} = \{s, 1, 2, 3, 4, 5, 6, t\}$, $E = \{a_1, \ldots, a_{17}\} = \{$(s,1), (s,2), (1,2), ..., (6,t), (t,6) $\}$, $c = (4, 3, 1, 7, 2, 2, 4, 4, 2, 2, 3, 1, 2, 12, 4, 6, 9)^\mathsf{T}$, and let A denote the incidence matrix of digraph $G = (V, E, c)$. Then for $b = (1, 0, 0, 0, 0, 0, 0, -1)^\mathsf{T}$ the set of vertices of the polyhedron X, where

$$X = \{x \in \mathbb{R}^{17}_{+} : Ax = b\},$$

forms a set of characteristic vectors of paths from s to t in digraph G. Any vertex of X is a binary vector, because the matrix A is totally unimodular. An optimal solution of the original problem, which corresponds to the path $p = \{(s, 2), (2, 4), (4, 5), (5, 3), (3, 6), (6, t)\}$, is given by the following vector:

$$x^o = (0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0)^\mathsf{T}.$$

If we are interested in paths from s to t passing through the vertex 1 (observe that the path p does not fulfill this condition), then we are faced with a restriction of the shortest path problem and the new set of feasible solutions is formed by additional constraints. We can require for example that at least one arc leaving the vertex 1 belongs to the feasible path. This leads to the following feasible set in a restriction of the original problem:

$$F = \{x \in X \cap \{0, 1\}^{17} : x_3 + x_4 \geq 1\}.$$

Assume that $\Delta = \mathbb{R}^{17}$ and let $\delta^+(i, j)$ and $\delta^-(i, j)$ denote, respectively, an increase and a decrease of the length of arc $(i, j) \in E$. Solving the adjustment problem (11) with l_1 norm we obtain the following optimal solution:

$\delta^+(s, 2) = 2$ and $\delta^+(i, j) = 0$ for $(i, j) \in E \setminus \{(s, 2)\}$;
$\delta^-(i, j) = 0$ for $(i, j) \in E$.

Observe that the shortest path in the modified network given by vector

$$x = (1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0)^{\mathrm{T}},$$

now passes through vertex 1. This optimal path in digraph G with modified lengths of arcs is shown in Figure 3.

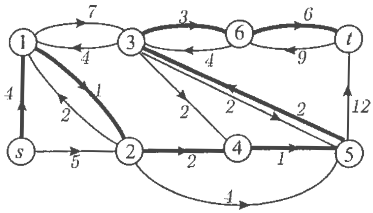

Figure 3: Digraph G from Example 1 with modified lengths of arcs and an optimal path from s to t indicated with bold lines.

The optimal solution of the adjustment problem can be interpreted as follows: To guarantee that some shortest path from s to t in the modified digraph G passes through the vertex 1, it is enough to increase the weight of arc $a_2 = (s, 2)$ by $\delta^+(s, 2) = 2$ while the lengths of all the other arcs remain unchanged. Moreover, this is the smallest possible (in the sense of l_1 norm) perturbation of lengths of arcs to achieve this goal.

Solving the adjustment problem with l_∞ and $l_{\infty,c}$ norm we obtain a minimum adjustment cost equal to 0.5 and 3/19, respectively, but more arc lengths have to be adjusted.

□

Example 2

Consider a weighted graph $G = (V, E, c)$, where $V = \{1, 2, 3, 4, 5\}$, $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$ and $c = (4, 1, 4, 5, 3, 7, 8)^{\mathrm{T}}$. A subset of edges $T = \{\{1, 2\}, \{1, 3\}, \{3, 4\}, \{3, 5\}\}$ forms the minimum spanning tree in this graph; its weight is equal to 15. The graph G and the minimum spanning tree T are shown in Figure 4.

Our goal will be to find such perturbations $\delta \in \mathbb{R}^7$ of the weight vector c, that some minimum spanning tree in the perturbed graph is a Hamiltonian path in this graph. We will solve an appropriate adjustment problem.

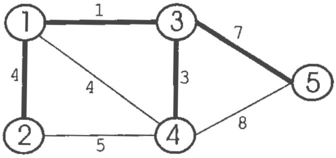

Figure 4: Graph G and its minimum spanning tree indicated with bold lines.

It is well known, that the minimum spanning tree problem in graph G can be formulated in various ways as a continuous linear programming problem. In the following we will use the formulation based on the minimum cost multicommodity flow problem (see e.g. Magnanti and Wolsey, 1995).

Let $G = (V, E, c)$ be a connected weighted graph, where $V = \{1, \dots, n\}$, $E \subseteq \{\{i, j\} : i, j \in V\}$, $c \in \mathbb{R}^{|E|}$. We use the following notation:

$K = V \setminus \{1\}$,

$D = \{(i, j) \in V \times V : \{i, j\} \in E\}$.

Define for $(i, j) \in D$ nonnegative variables $x_{ij} \in \mathbb{R}$ and introduce for $k \in K$ and $(i, j) \in D$ auxiliary nonnegative variables $f_{ij}^k \in \mathbb{R}$. The following continuous linear programming problem (12) may be used to calculate the minimum weight spanning tree T^* in graph $G = (V, E, c)$. Namely, given an optimal solution x^*, f^* of problem (12), the optimal tree T^* is composed of the edges $\{i, j\} \in E$ for which $x_{ij}^* + x_{ji}^* > 0$.

$$
\min \sum_{\{i,j\} \in E} c_{ij} \cdot (x_{ij} + x_{ji})
$$

$$
\begin{aligned}
\sum_{(i,j) \in D} x_{ij} &= n - 1 & & \\
\sum_{(j,1) \in D} f_{j1}^k - \sum_{(1,j) \in D} f_{1j}^k &= -1, & & k \in K \\
\sum_{(j,k) \in D} f_{jk}^k - \sum_{(k,j) \in D} f_{kj}^k &= 1, & & k \in K \\
\sum_{(j,i) \in D} f_{ji}^k - \sum_{(i,j) \in D} f_{ij}^k &= 0, & & i, k \in K,\ i \neq k \\
x_{ij} - f_{ij}^k &\geq 0, & & k \in K,\ (i, j) \in D \\
x_{ij} &\geq 0, & & (i, j) \in D \\
f_{ij}^k &\geq 0, & & k \in K,\ (i, j) \in D.
\end{aligned}
\tag{12}
$$

Let $l \in \mathbb{R}$, $v_1^k \in \mathbb{R}$ for $k \in K$, $v_k^k \in \mathbb{R}$ for $k \in K$, $v_i^k \in \mathbb{R}$ for $i, k \in K$, $i \neq k$, and $w_{ij}^k \geq 0$ for $k \in K$, $(i,j) \in D$, denote dual variables for consecutive groups of constraints in problem (12). Then the dual problem of linear program (12) can be stated as follows:

$$\max \sum_{k \in K} (v_k^k - v_1^k) + (n-1) \cdot l$$

$$\begin{array}{rcll} v_j^k - v_i^k - w_{ij}^k & \leq & 0, & k \in K,\ (i,j) \in D \\ \sum_{k \in K} w_{ij}^k + l & \leq & c_{ij}, & \{i,j\} \in E \\ \sum_{k \in K} w_{ji}^k + l & \leq & c_{ij}, & \{i,j\} \in E \\ w_{ij}^k & \geq & 0, & k \in K,\ (i,j) \in D \\ v_i^k & \in \mathbb{R}, & & i \in V,\ k \in K \\ l & \in \mathbb{R}. & & \end{array} \tag{13}$$

Consider now this restriction of the problem (12): the obtained spanning tree should also be a Hamiltonian path. This may be achieved by requiring that $x_{ij} \in \{0,1\}$ for $(i,j) \in D$, and by adding the following set of constraints:

$$\sum_{j \in V} (x_{ij} + x_{ji}) \leq 2, \qquad i \in V. \tag{14}$$

Now we may state the adjustment problem (11). Observe, that we will need to linearize only nonlinear terms for variables x_{ij}, $(i,j) \in D$.

Let $\delta^+(i,j)$ and $\delta^-(i,j)$ denote, respectively, an increase and a decrease of the weight c_{ij} of edge $\{i,j\} \in E$. Solving the adjustment problem with l_1 norm and $\Delta = \mathbb{R}^7$ we obtain the following optimal perturbations of weights of edges in graph G:

$\delta^+(3,5) = 1$ and $\delta^+(i,j) = 0$ for $\{i,j\} \in E \setminus \{3,5\}$,

$\delta^-(i,j) = 0$ for $\{i,j\} \in E$.

Thus, the adjustment cost is equal to 1 and only a weight of the single edge has to be perturbed. Figure 5 shows the graph G with modified weights of edges and the minimum spanning tree in this graph, which is now a Hamiltonian path.

If we solve the adjustment problem with $l_{\infty,c}$ norm, then we obtain a minimum adjustment cost equal to 1/15. Thus, modifying in an appropriate way weights of edges in the original graph G by no more than approximately

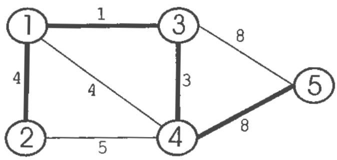

Figure 5: Graph G with modified weights of edges and its minimum spanning tree indicated with bold lines.

6.7% we can guarantee that the perturbed graph has the minimum spanning tree which is also a Hamiltonian path.

□

6 Conclusions

The adjustment problem is a straightforward generalization of the inverse problem. Nevertheless it leads to rather different optimization problems. For example in case of a linear optimization problem, the inverse problem (for a given feasible solution x) can be stated as a convex optimization problem due to the convexity of the stability region of any feasible solution. In contrast to this, the adjustment problem (for a given subset of feasible solutions F, such that $|F| > 1$) confronts us with a nonconvex optimization problem, which may lead to substantial difficulties.

The paper has presented a solution method for the adjustment problem when applied to a continuous linear program and its restriction given by additional linear and integrality constraints, such that all the adjustable variables are binary. In this case the adjustment problem can be formulated as a mixed-integer linear programming problem. Such a solution method for the adjustment problem has appeared quite satisfactory for small examples, but the computational efficiency of this MIP reformulation for larger instances requires further investigations.

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References

[1] Ahuja RK, Orlin JB. Inverse optimization. Operations Research 2001; 49; 771-783.

[2] Heuberger C. Inverse combinatorial optimization: A survey on problems, methods, and results. Journal of Combinatorial Optimization 2004;8; 329-361.

[3] Greenberg H. An annotated bibliography for post-solution analysis in mixed integer and combinatorial optimization. In: Woodruff D (Ed.), Advances in computational and stochastic optimization, logic programming and heuristic search. Kluwer Academic Publishers: Dordrecht; 1998. p. 97-148.

[4] Libura M. Optimality conditions and sensitivity analysis for combinatorial optimization problems. Control and Cybernetics 1996;25; 1165-1180.

[5] Libura M. On the inverse problem and some related problems in combinatorial optimization. Research Report RB/52/2001, Systems Research Institute, Polish Academy of Sciences, Warszawa; 2001.

[6] Magnanti TL, Wolsey LA. Optimal trees. In: Ball MO et al. (Eds), Network models, Handbook in operations research and management science, vol.7. North-Holland: Amsterdam; 1995. p. 503-615.

[7] Nemhauser GL, Wolsey LA. Integer and combinatorial optimization. John Wiley & Sons: New York; 1988.

[8] Padberg M. Integer programming and extensions. Springer-Verlag: Berlin, Heidelberg, New York; 1995.

[9] Williams HP. Model building in mathematical programming. John Wiley & Sons: Chichester; 1993.

[10] Zhang J, Liu Z. A general model of some inverse combinatorial optimization problems and its solutions method under l_∞ norm. Journal of Combinatorial Optimization 2002;6; 207-227.