

7/2005

**Raport Badawczy**

**RB/63/2005**

**Research Report**

**New existence result  
for 3-D shape memory model**

**I. Pawłow**

**Instytut Badań Systemowych  
Polska Akademia Nauk**

**Systems Research Institute  
Polish Academy of Sciences**



# **POLSKA AKADEMIA NAUK**

## **Instytut Badań Systemowych**

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 8373578

fax: (+48) (22) 8372772

Kierownik Pracowni zgłaszający pracę:  
Prof. dr hab. inż. Kazimierz Małanowski

Warszawa 2005

## CHAPTER 1

NEW EXISTENCE RESULT FOR 3-D SHAPE MEMORY  
MODEL

Irena Pawłow\* and Wojciech M. Zajączkowski\*\*

\* *Systems Research Institute, Polish Academy of Sciences,  
Newelska 6, 01-447 Warsaw, Poland  
E-mail: pawlow@ibspan.waw.pl  
and Institute of Mathematics and Cryptology,  
Cybernetics Faculty, Military University of Technology,  
S. Kaliskiego 2, 00-908 Warsaw, Poland*

\*\* *Institute of Mathematics, Polish Academy of Sciences,  
Śniadeckich 8, 00-956 Warsaw, Poland  
E-mail: wz@impan.gov.pl  
and Institute of Mathematics and Cryptology,  
Cybernetics Faculty, Military University of Technology,  
S. Kaliskiego 2, 00-908 Warsaw, Poland*

The paper presents a new existence result for three-dimensional (3-D) shape memory model which has the form of a nonlinear thermoelasticity system with viscosity  $\nu > 0$  and capillarity  $\kappa > 0$ . In contrast to the previous authors results, proved under assumption  $0 < 2\sqrt{\kappa} < \nu$ , here we admit  $\kappa > 0$  and  $\nu > 0$  possibly arbitrarily small. With such assumption the obtained existence result becomes more adequate for shape memory problems where viscosity effects are negligible small. Moreover, we consider a broader class of boundary conditions.

The main new part of the present paper constitutes solvability analysis of the initial-boundary-value problems for viscoelasticity system with capillarity.

Keywords: nonlinear thermoelasticity, viscosity, capillarity, shape memory, global existence

AMS Subject Classification: 35K50, 35K60, 35Q72, 74B20

## 1. Introduction

The goal of this paper is to present a new existence result for three-dimensional (3-D) shape memory model which has been previously studied

by the authors under more restrictive assumptions in [10], [11]. The model, firstly introduced and studied in [13], [14], [9], has the form of the following nonlinear thermoelasticity system with viscosity  $\nu > 0$  and strain-gradient coefficient (called capilarity)  $\varkappa > 0$ :

$$\begin{aligned} \mathbf{u}_{tt} - \nu Q \mathbf{u}_t + \varkappa Q^2 \mathbf{u} &= \nabla \cdot \mathbf{F}_\varepsilon(\varepsilon, \theta) + \mathbf{b} && \text{in } \Omega^T = \Omega \times (0, T), \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \mathbf{u}_t|_{t=0} = \mathbf{u}_1 &&& \text{in } \Omega, \\ \mathbf{B}(\partial_x) \mathbf{u} = \mathbf{0} &&& \text{on } S^T = S \times (0, T), \end{aligned} \tag{1.1}$$

$$\begin{aligned} c_0(\varepsilon, \theta) \theta_t - k_0 \Delta \theta &= \theta F_{,\theta\varepsilon}(\varepsilon, \theta) + \nu(\mathbf{A}\varepsilon_t) \cdot \varepsilon_t + g && \text{in } \Omega^T, \\ \theta|_{t=0} = \theta_0 &&& \text{in } \Omega, \\ \mathbf{n} \cdot \nabla \theta = 0 &&& \text{on } S^T, \end{aligned} \tag{1.2}$$

where

$$c_0(\varepsilon, \theta) = c_\nu - \theta F_{,\theta\theta}(\varepsilon, \theta), \tag{1.3}$$

and  $\mathbf{B}(\partial_x) \mathbf{u}$  stands for one of the following two types of boundary conditions

$$\mathbf{u} = \mathbf{0}, \quad \mathbf{Q}\mathbf{u} = \mathbf{0} \quad \text{on } S^T, \tag{1.4}$$

or

$$\mathbf{u} = \mathbf{0}, \quad (\mathbf{A}\varepsilon(\mathbf{u}))\mathbf{n} = \mathbf{0} \quad \text{on } S^T.$$

Here  $\Omega \subset \mathbb{R}^3$  is a bounded domain with a smooth boundary  $S$ , occupied by a solid body in a reference configuration with constant mass density ( $\rho = 1$ );  $\mathbf{n}$  is the unit outward normal vector to  $S$ ;  $T > 0$  is an arbitrary fixed time;  $\mathbf{u} : \Omega^T \rightarrow \mathbb{R}^3$  is the displacement and  $\theta : \Omega^T \rightarrow \mathbb{R}_+$  is the absolute temperature. The second order tensors

$$\varepsilon = \varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \quad \text{and} \quad \varepsilon_t = \varepsilon(\mathbf{u}_t) = \frac{1}{2}(\nabla \mathbf{u}_t + (\nabla \mathbf{u}_t)^T)$$

denote respectively the linearized strain and the strain rate. The operator  $Q$  stands for the linearized elasticity operator defined by

$$\mathbf{Q}\mathbf{u} = \nabla \cdot (\mathbf{A}\varepsilon(\mathbf{u})) = \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla(\nabla \cdot \mathbf{u}), \tag{1.5}$$

where  $\mathbf{A} = (A_{ijkl})$  is the fourth order elasticity tensor representing linear isotropic Hooke's law

$$\mathbf{A}\varepsilon(\mathbf{u}) = \lambda \text{tr} \varepsilon(\mathbf{u}) \mathbf{I} + 2\mu \varepsilon(\mathbf{u}), \tag{1.6}$$

$\mathbf{I}$  is the identity tensor, and  $\lambda, \mu$  are the Lamé constants such that  $\mu > 0$  and  $3\lambda + 2\mu > 0$ .

Correspondingly, the fourth order operator  $Q^2 = QQ$  is given by

$$Q^2 \mathbf{u} = \nabla \cdot (\mathbf{A}\boldsymbol{\varepsilon}(\mathbf{Q}\mathbf{u})) = \mu^2 \Delta^2 \mathbf{u} + (\lambda + \mu)(\lambda + 3\mu) \nabla \nabla \cdot (\Delta \mathbf{u}). \quad (1.7)$$

Moreover,  $F(\boldsymbol{\varepsilon}, \theta)$  denotes the elastic energy which is a nonconvex (multi-well) function of  $\boldsymbol{\varepsilon}$  with the shape strongly depending on  $\theta$ . The remaining quantities in (1.1), (1.2) have the following meaning:  $c_0(\boldsymbol{\varepsilon}, \theta)$  is the specific heat coefficient,  $c_v, k_0, \nu$  and  $\kappa$  are positive numbers denoting respectively thermal specific heat, heat conductivity, viscosity and capillarity.

System (1.1), (1.2) describes balance laws for the linear momentum and the internal energy. The underlying free energy density has the Landau-Ginzburg form

$$f(\boldsymbol{\varepsilon}(\mathbf{u}), \nabla \boldsymbol{\varepsilon}(\mathbf{u}), \theta) = -c_v \theta \log \theta + F(\boldsymbol{\varepsilon}(\mathbf{u}), \theta) + \frac{\kappa}{2} |\mathbf{Q}\mathbf{u}|^2 \quad (1.8)$$

with the three terms representing respectively thermal, elastic and strain-gradient (capillarity) energy. The corresponding stress tensor is given by

$$\mathbf{S} = \frac{\delta f}{\delta \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}), \nabla \boldsymbol{\varepsilon}(\mathbf{u}), \theta) + \mathbf{S}^v = F_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}), \theta) - \kappa \mathbf{A}\boldsymbol{\varepsilon}(\mathbf{Q}\mathbf{u}) + \nu \mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}_t), \quad (1.9)$$

where  $\delta f / \delta \boldsymbol{\varepsilon} = f_{,\boldsymbol{\varepsilon}} - \nabla \cdot f_{,\nabla \boldsymbol{\varepsilon}}$  denotes the first variation with respect to  $\boldsymbol{\varepsilon}$ , and  $\mathbf{S}^v = \nu \mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}_t)$  is the viscous stress according to Hooke's-like law. For thermodynamical background of the model we refer to [9], [14].

We add few remarks on model (1.1), (1.2) and its solvability. Firstly, we point out that dynamics (1.1)<sub>1</sub> is in accordance with the so-called viscosity-capillarity criterion justified by several authors, among them Slemrod [15], Abeyaratne-Knowles [1] as a proper model for dynamics of phase transitions in van der Waals fluids and for propagating phase boundaries in solids. By this criterion, originally formulated in case of one space dimension, a proper constitutive relation for the stress has the form (see e.g. [1], eq. (2.8))

$$s = F_{,u_x}(u_x) - \kappa u_{xxx} + \nu u_{xt} \quad (1.10)$$

where  $u_x$  is the strain,  $F(u_x)$  is a nonconvex double-well elastic energy, and  $\nu \geq 0$  and  $\kappa \geq 0$  are the viscosity and the strain-gradient coefficient, respectively. We can see that equation (1.9) generalizes stress-strain relation (1.10) to the case of three space dimensions.

Secondly, we remark that in case of vanishing viscosity  $\nu = 0$ , problem (1.1), (1.2) represents a 3-D analog of the well-known Falk model for one-dimensional martensitic phase transitions of the shear type (see [6], [4]). Unfortunately, either our previous theory [10], [11] or the present one do not cover the case  $\nu = 0$ . The existence proofs in [10], [11] as well as

the earlier one in [14] were based on the following condition between the viscosity and capillarity coefficients

$$0 < 2\sqrt{\kappa} < \nu. \quad (1.11)$$

Such condition allows for the decomposition of elasticity system (1.1)<sub>1</sub> into two second order parabolic problems

$$\begin{aligned} w_t - \beta Qw &= \nabla \cdot F_{,\epsilon}(\epsilon, \theta) + b && \text{in } \Omega^T, \\ w|_{t=0} &= u_1 - \alpha Qu_0 && \text{in } \Omega, \\ w &= 0 && \text{on } S^T, \end{aligned} \quad (1.12)$$

$$\begin{aligned} u_t - \alpha Qu &= w && \text{in } \Omega^T, \\ u|_{t=0} &= u_0 && \text{in } \Omega, \\ u &= 0 && \text{on } S^T, \end{aligned} \quad (1.13)$$

where  $\alpha, \beta$  are numbers satisfying

$$\alpha + \beta = \nu, \quad \alpha\beta = \kappa.$$

Due to condition (1.11) these numbers are real and positive,  $\alpha, \beta \in \mathbb{R}_+$ . The decomposition (1.12), (1.13) was the main idea underlying the existence proofs in the above mentioned papers. It is known, however, that in structural phase transitions in shape memory alloys strain-gradient effect is observable but not the viscous one (see e.g. [4]). For that reason condition (1.11) is not appropriate for shape memory models.

In view of that it is of importance to construct an existence theory with relaxed condition (1.11). In the present paper we replace (1.11) by

$$\kappa > 0 \quad \text{and} \quad \nu > 0, \quad (1.14)$$

allowing the viscosity to be arbitrarily small but positive. In such a case system (1.1) is parabolic and the theory of parabolic equations can be applied (see [16], [5]).

We mention a similar study due to Yoshikawa [20] which is also concerned with the existence of solutions to problem (1.1), (1.2) under assumption (1.14). On the contrary to the present paper, however, the result in [20] concerns model (1.1), (1.2) with simplified energy equation (1.2)<sub>1</sub>.

This simplification consists in neglecting the nonlinear term  $-\theta F_{,\theta\theta}(\epsilon, \theta)$  in the specific heat coefficient  $c_0(\epsilon, \theta)$  by assuming that  $c_0(\epsilon, \theta) = c_v = \text{const} > 0$ . Obviously, such simplification destroys the thermodynamic structure of the model but makes the mathematical analysis much simpler.

We add that the same simplification was used in the first result on the global in time unique solvability of system (1.1), (1.2) in 2-D and 3-D cases obtained in [14].

The existence result due to Yoshikawa [20] generalizes that in [14] by admitting weaker assumptions on the data, in particular (1.14) instead of (1.11), and a more general solutions class. The technique used in [20] is different from the classical methods for parabolic systems applied in [14], [10], [11]. It is based on the so-called maximal regularity theory for abstract parabolic equations.

The authors papers [10] and [11] generalize the result of [14] respectively in 2-D and 3-D case by removing the above mentioned simplification of the energy equation. We stress that the presence of a non-linearity in the leading coefficient of the heat conduction equation introduces essential difficulties in the existence proof.

As it has been already mentioned in case of one-space dimension problem (1.1), (1.2) with  $\kappa > 0$  and  $\nu = 0$  is identical with the Falk model. In such a case, in contrast to the three-dimensional one, there are several results on the existence and uniqueness of solutions, in particular due to Sprekels and Zheng [17], Aiki [2] and Yoshikawa [19]. The latter paper includes up-to-date list of references related to 1-D Falk's model. For a survey of diffused-interface models of shape alloys and the related mathematical results we refer to [12]. We mention also that problem (1.1), (1.2) without capillarity but with the viscosity, i.e.  $\kappa = 0$ ,  $\nu > 0$ , with simplified energy equation discussed above, has been studied by Zimmer [18].

Finally, we add a remark concerning boundary conditions in (1.1)<sub>2</sub>. In [14] and later in [10], [11] the no-displacement boundary condition  $\mathbf{u} = 0$  on  $S^T$  was chosen in order to apply the result due to Nečas [7] on the ellipticity property of the operator  $Q$  whereas the condition  $Q\mathbf{u} = 0$  on  $S^T$  resulted in a compatibility with parabolic decomposition (1.12), (1.13).

In the present paper, apart from  $\mathbf{u} = 0$ ,  $Q\mathbf{u} = 0$  on  $S^T$ , we admit the other type of boundary conditions (1.4)<sub>2</sub>.

The plan of the paper is as follows.

In Section 2 we formulate the assumptions and state the existence and uniqueness theorems. These theorems generalize the results of [11], Theorems 2.1, 2.2, by admitting assumption (1.14) and a broader class of boundary conditions (1.4).

In Section 3 we examine the solvability of the initial-boundary-value problem defined by the differential operator on the left-hand side of (1.1)<sub>1</sub> with initial conditions (1.1)<sub>2</sub> and boundary conditions (1.4). We show that the

differential operator is parabolic in the sense of Solonnikov and that the initial and boundary conditions satisfy the Shapiro-Lopatinskij conditions (complementarity condition).

In Section 4 we present auxiliary results on the solvability of parabolic problems of fourth and second order. These results play a key role in the new existence proof.

Section 5 presents the outline of the existence proof.

We use following notations:

$$f_{,i} = \frac{\partial f}{\partial x_i}, \quad i = 1, 2, 3, \quad f_i = \frac{df}{dt}, \quad \varepsilon = (\varepsilon_{ij})_{i,j=1,2,3},$$

$$F_{,e}(\varepsilon, \theta) = \left( \frac{\partial F(\varepsilon, \theta)}{\partial \varepsilon_{ij}} \right)_{i,j=1,2,3}, \quad F_{,\theta}(\varepsilon, \theta) = \frac{\partial F(\varepsilon, \theta)}{\partial \theta},$$

where space and time derivatives are material.

Vectors (tensors of the first order), tensors of the second order (referred simply to as tensors) and tensors of higher order are denoted by bold letters. Tensors of the second order represent linear transformations of vectors into vectors;  $S^T$ ,  $\text{tr}S$ ,  $S^{-1}$  and  $\det S$ , respectively, denote the transpose, trace, inverse, and determinant of a tensor  $S$ .

A dot designates the inner product, irrespective of the space in question:  $\mathbf{u} \cdot \mathbf{v}$  is the inner product of vectors  $\mathbf{u} = (u_i)$  and  $\mathbf{v} = (v_i)$ ,  $S \cdot \mathbf{R} = \text{tr}(S^T \mathbf{R})$  is the inner product of tensors  $S = (S_{ij})$  and  $\mathbf{R} = (R_{ij})$ ,  $\mathbf{A}^m \cdot \mathbf{B}^m$  is the inner product of the  $m$ -th order tensors  $\mathbf{A}^m = (A_{i_1 \dots i_m}^m)$  and  $\mathbf{B}^m = (B_{i_1 \dots i_m}^m)$ . In Cartesian components,

$$\begin{aligned} (S\mathbf{u})_i &= S_{ij}u_j, & (S^T)_{ij} &= S_{ji}, & \text{tr}S &= S_{ii}, \\ \mathbf{u} \cdot \mathbf{v} &= u_i v_i, & S \cdot \mathbf{R} &= S_{ij}R_{ij}, \\ \mathbf{A}^m \cdot \mathbf{B}^m &= A_{i_1 \dots i_m}^m B_{i_1 \dots i_m}^m. \end{aligned}$$

Here and throughout the summation convention over repeated indices is used. By  $\mathbf{A} = (A_{ijkl})$  we denote the fourth order elasticity tensor which represents a symmetric linear transformation of symmetric tensors into symmetric tensors. We write  $(\mathbf{A}\boldsymbol{\varepsilon})_{ij} = A_{ijkl}\varepsilon_{kl}$ .

The symbols  $\nabla$  and  $\nabla \cdot$  denote the material gradient and the divergence. For the divergence we use the convention of the contraction over the last index, e.g.  $(\nabla \cdot S)_i = \partial S_{ij} / \partial x_j$ .

We use the Sobolev spaces notation of [8]. Throughout the paper  $c$  and  $c(T)$  denote generic constants, different in various instances, depending on the data of the problem and domain  $\Omega$ . The argument  $T$  indicates time horizon dependence which is always of the form  $T^a$ ,  $a \in \mathbb{R}_+$ .



## 2. Assumptions and main results

Problem (1.1), (1.2) is studied under the following assumptions (A1)–(A5) (the same as in [11]):

(A1) Domain  $\Omega \subset \mathbb{R}^3$  with the boundary of class  $C^4$ . The  $C^4$ -regularity is needed to apply the classical regularity result for parabolic systems.

(A2) The coefficients of the operator  $Q$  satisfy

$$\mu > 0, \quad 3\lambda + 2\mu > 0.$$

These conditions assure the following properties:

(i) Coercivity and boundedness of the operator  $A$

$$\underline{c}|\varepsilon|^2 \leq (A\varepsilon) \cdot \varepsilon \leq \bar{c}|\varepsilon|^2, \quad (2.1)$$

where  $\underline{c} = \min\{3\lambda + 2\mu, 2\mu\}$ ,  $\bar{c} = \max\{3\lambda + 2\mu, 2\mu\}$ ;

(ii) Strong ellipticity of the operator  $Q$  (see [14], Sec.7). Thanks to this property the following estimate due to Nečas [7] holds true

$$c\|u\|_{W_2^2(\Omega)} \leq \|Qu\|_{L_2(\Omega)} \quad \text{for } \{u \in W_2^2(\Omega) \mid u|_{\partial\Omega} = 0\}; \quad (2.2)$$

(iii) Parabolicity in general (Solonnikov) sense of system defined by the differential operator on the left-hand side of (1.1)<sub>1</sub> (see Lemma 3.1).

The next assumption concerns the structure of the elastic energy.

(A3) Function  $F(\varepsilon, \theta) : S^2 \times [0, \infty) \rightarrow \mathbb{R}$  is of class  $C^3$ , where  $S^2$  denotes the set of symmetric second order tensors in  $\mathbb{R}^3$ . We assume the splitting

$$F(\varepsilon, \theta) = F_1(\varepsilon, \theta) + F_2(\varepsilon),$$

where  $F_1$  and  $F_2$  are subject to the following conditions:

(A3-1) Conditions on  $F_1(\varepsilon, \theta)$

(i) concavity with respect to  $\theta$

$$-F_{1,\theta\theta}(\varepsilon, \theta) \geq 0 \quad \text{for } (\varepsilon, \theta) \in S^2 \times [0, \infty). \quad (2.3)$$

(ii) Nonnegativity

$$F_1(\varepsilon, \theta) \geq 0 \quad \text{for } (\varepsilon, \theta) \in S^2 \times [0, \infty).$$

(iii) Boundedness of the norm

$$\|F_1\|_{C^3(S^2 \times [0, \infty))} < \infty.$$

- (iv) Growth conditions. There exist a positive constant  $c$  and numbers  $s, K_1 \in (0, \infty)$  such that

$$|\partial_\varepsilon^i \partial_\theta^j F_1| \leq c(1 + \theta^{s-i} |\varepsilon|^{K_1-j}), \quad 0 \leq i+j \leq 2, \quad i, j \in \mathbb{N},$$

and  $i = 2, j = 1$ ,

for large values of  $\theta$  and  $\varepsilon_{ij}$ , where admissible ranges of  $s$  and  $K_1$  are given by

$$0 < s < \frac{2}{3}, \quad 0 < K_1 < \frac{15}{4}.$$

Moreover, in case  $K_1 > 1$  the numbers  $s$  and  $K_1$  are linked by the equality

$$15s + 4K_1 = 15.$$

(A3-2) Conditions on  $F_2(\varepsilon)$

- (i) Nonnegativity

$$F_2(\varepsilon) \geq 0 \quad \text{for } \varepsilon \in S^2.$$

- (ii) Boundedness of the norm

$$\|F_2\|_{C^2(S^2)} < \infty.$$

- (iii) Growth conditions

$$|\partial_\varepsilon^i F_2| \leq c(1 + |\varepsilon|^{K_2-i}), \quad 0 \leq i \leq 2, \quad i \in \mathbb{N},$$

for large values of  $\varepsilon_{ij}$ , where

$$0 < K_2 \leq \frac{9}{2}.$$

Before formulating the assumptions on the data we note some consequences of assumption (A3-1) which are of importance for the existence proof. In view of (A3-1) (i), by definition of  $c_0(\varepsilon, \theta)$ ,

$$0 < c_v \leq c_0(\varepsilon, \theta) \quad \text{for } (\varepsilon, \theta) \in S^2 \times [0, \infty). \quad (2.4)$$

Moreover, (A3-1) (iii) and (iv) imply the bounds

$$\begin{aligned} |c_0(\varepsilon, \theta)|, \quad |c_{0,\theta}(\varepsilon, \theta)| &\leq c(1 + |\varepsilon|^{K_1}), \\ |c_{0,\varepsilon}(\varepsilon, \theta)| &\leq c(1 + |\varepsilon|^{\max\{0, K_1-1\}}) \quad \text{for } (\varepsilon, \theta) \in S^2 \times [0, \infty). \end{aligned} \quad (2.5)$$

From (A3-1)(i) and (ii) it follows that

$$F_1(\varepsilon, \theta) - \theta F_{1,\theta}(\varepsilon, \theta) \geq 0 \quad \text{for } (\varepsilon, \theta) \in S^2 \times [0, \infty), \quad (2.6)$$

and owing to (A3-2) (i),

$$(F_1(\epsilon, \theta) - \theta F_{1,\theta}(\epsilon, \theta)) + F_2(\epsilon) \geq 0 \quad \text{for } (\epsilon, \theta) \in S^2 \times [0, \infty), \quad (2.7)$$

what means that the elastic part of the internal energy is nonnegative.

The later bound is used in derivation of energy estimate.

(A4) The data satisfy

$$\begin{aligned} b &\in L_p(\Omega^T), \quad 5 < p < \infty, \\ g &\in L_q(\Omega^T), \quad 5 < q < \infty, \quad \text{and } g \geq 0 \text{ a.e. in } \Omega^T, \\ u_0 &\in W_p^{4-2/p}(\Omega), \quad u_1 \in W_p^{2-2/p}(\Omega), \quad 5 < p < \infty, \\ \theta_0 &\in W_q^{2-2/q}(\Omega), \quad 5 < q < \infty, \quad \text{and } \theta_* = \min_{\Omega} \theta_0 > 0. \end{aligned}$$

Moreover the initial data are supposed to satisfy the compatibility conditions for the classical solvability of parabolic problems.

We note that by Sobolev's imbeddings,

$$\theta_0 \in C^{1,\alpha_0}(\Omega), \quad \epsilon_0 \in C^{2,\alpha'_0} \quad \text{with } 0 < \alpha_0, \alpha'_0 < 1. \quad (2.8)$$

Similarly as in [11] we introduce an additional technical assumption which requires a special separable form of  $F_1(\epsilon, \theta)$ :

(A5) Function  $F_1(\epsilon, \theta)$  has the form

$$F_1(\epsilon, \theta) = \sum_{i=1}^N \tilde{F}_{1i}(\theta) \tilde{F}_{2i}(\epsilon),$$

where  $N \in \mathbb{N}$  is a finite number, and in accordance with (A3-1) (i)–(iii), (A5-1)

$$\begin{aligned} \tilde{F}_{1i} &\in C^2([0, \infty)), & \tilde{F}_{2i} &\in C^2(S^2), & i &= 1, \dots, N, \\ \tilde{F}_{1i}(\theta) &\geq 0, & \tilde{F}_{2i}(\epsilon) &\geq 0, & i &= 1, \dots, N, \\ -\tilde{F}_{1i,\theta\theta}(\theta) &\geq 0, & & & i &= 1, \dots, N. \end{aligned}$$

Moreover, functions  $\tilde{F}_i(\theta)$ ,  $i = 1, \dots, N$ , are given by

(A5-2)

$$\tilde{F}_{1i}(\theta) = \begin{cases} 0 & \text{for } 0 \leq \theta \leq \theta_1, \\ \varphi_i(\theta) & \text{for } \theta_1 \leq \theta \leq \theta_2, \\ \theta^{s_i} & \text{for } \theta \geq \theta_2, \end{cases}$$

where numbers  $s_i$ ,  $i = 1, \dots, N$ , and  $\theta_1, \theta_2$  satisfy the following conditions

$$0 < s_i \leq s < 1, \quad 1 < \theta_1 < \theta_2 \leq \theta_1^{1/s_i} \quad \text{for } i = 1, \dots, N.$$

The requirements in (A5-1) imply that

$$\begin{aligned} \varphi_i(\theta_1) &= \theta_1, & \varphi'_i(\theta_1) &= 1, & \varphi''_i(\theta_1) &= 0, \\ \varphi_i(\theta_2) &= \theta_2^{s_i}, & \varphi'_i(\theta_2) &= s_i \theta_2^{s_i-1}, & \varphi''_i(\theta_2) &= s_i(s_i-1)\theta_2^{s_i-2}, \\ \theta^{s_i} &\leq \varphi_i(\theta) \leq \theta, & -\varphi'_i(\theta) &\geq 0, & & \text{for } \theta \in (\theta_1, \theta_2), \end{aligned}$$

where  $i = 1, \dots, N$ . We note that functions  $\tilde{F}_{1i}(\theta)$  in (A5-2) satisfy the growth conditions (A3-1) (iv) which now read as follows

$$\begin{aligned} |\partial_\theta^j \tilde{F}_{1i}| &\leq c(1 + \theta^{s-j}), \\ |\partial_\varepsilon^j \tilde{F}_{2i}| &\leq c(1 + |\varepsilon|^{K_1-j}), \end{aligned}$$

where  $i = 1, \dots, N$  and  $j = 0, 1, 2$ .

We point out that in [11] the above separable form of  $F_1(\varepsilon, \theta)$  has been used to prove the key  $L_\infty$ -norm estimate for  $\theta$ . In our present argumentation this part of the proof will remain unchanged. We add also that such separable form of  $F_1(\varepsilon, \theta)$  is conformable with the known Falk-Konopka elastic energy model (for more detailed account see [11]).

The main result of the present paper is the following existence theorem.

**Theorem 2.1:** Let assumptions (A1)–(A5) be satisfied and the coefficients  $\varkappa, \nu$  fulfil condition (1.14). Then for any  $T > 0$  there exists a solution  $(u, \theta)$  to problem (1.1), (1.2) with boundary conditions (1.4) in the space

$$V(p, q) \equiv \{(u, \theta) | u \in W_p^{4,2}(\Omega^T), \theta \in W_q^{2,1}(\Omega^T), 5 < p \leq q < \infty\}, \quad (2.9)$$

such that

$$\|u\|_{W_2^{4,2}(\Omega^T)} \leq c(T), \quad \|\theta\|_{W_2^{2,1}(\Omega^T)} \leq c(T), \quad (2.10)$$

with a positive constant  $C(T)$  depending on the data of the problem and  $T^a, a \in \mathbb{R}_+$ . Moreover, there exists a positive finite number  $\omega$  satisfying

$$[g + \nu(A\varepsilon_i \cdot \varepsilon_i) \exp(\omega t) + [\omega c_0(\varepsilon, \theta) + F_{,\varepsilon\varepsilon}(\varepsilon, \theta) \cdot \varepsilon_i] \theta_* \geq 0 \quad \text{in } \Omega^T,$$

such that

$$\theta \geq \theta_* \exp(-\omega t) \quad \text{in } \Omega^T. \quad (2.11)$$

We point out that this theorem generalizes the result in [11], Theorem 2.1, by admitting a weaker assumption on the coefficients  $\varkappa, \nu$  and a broader class of boundary conditions.

We remark also that solutions specified in Theorem 2.1 enjoy, by virtue of Sobolev's imbeddings, the following properties:

$$u, u_i, \varepsilon, \nabla \varepsilon, \nabla^2 \varepsilon, \varepsilon_i, \theta, \nabla \theta \quad (2.12)$$

are Hölder continuous in  $\Omega^T$  and satisfy the corresponding a priori bounds with constant  $c(T)$ .

For completeness we recall also the uniqueness result which follows by repeating the arguments used in [10] in the study of problem (1.1), (1.2), (1.4)<sub>1</sub> in 2-D case. The proof is based on a direct comparison of two solutions, the use of energy estimates together with the regularity properties (2.12). The parabolic decomposition of elasticity system (1.1) is not applied in the uniqueness proof.

**Theorem 2.2:** Let the assumptions of Theorem 2.1 be satisfied and in addition suppose that

$$(A6) \quad F(\epsilon, \theta) : S^2 \times [0, \infty) \rightarrow \mathbb{R} \text{ is of class } C^4, \text{ and } g \in L_\infty(\Omega^T).$$

Then the solution  $(u, \theta) \in V(p, q)$  to problem (1.1), (1.2) is unique.

### 3. Parabolicity of the elasticity system with viscosity and capillarity

We consider the following problem

$$\begin{aligned} u_{tt} - \nu Q u_t + \kappa Q^2 u &= f && \text{in } \Omega^T, \\ u|_{t=0} &= u_0, \quad u_t|_{t=0} = u_1, && \text{in } \Omega, \\ B(\partial_x)u &= 0 && \text{on } S^T, \end{aligned} \tag{3.1}$$

where  $Q$  is the linear elasticity operator defined by (1.5) and  $B(\partial_x)u$  stands for one of the following two types of boundary conditions

$$u = 0, \quad Qu = 0 \quad \text{on } S^T, \tag{3.2}$$

or

$$u = 0, \quad (A\epsilon(u))n = 0 \quad \text{on } S^T.$$

In view of (1.5), (1.7) system (3.1)<sub>1</sub> can be expressed in the explicit form

$$\begin{aligned} u_{tt} + \Delta(-\nu\mu u_t + \kappa\mu^2\Delta u) \\ + \nabla\nabla \cdot [-\nu(\lambda + \mu)u_t + \kappa(\lambda + \mu)(\lambda + 3\mu)\Delta u] &= f \end{aligned} \tag{3.3}$$

or, equivalently, in the matrix form

$$\sum_{j=1}^3 l_{kj}(\partial_t, \partial_x)u_j = f_k, \quad k = 1, 2, 3, \tag{3.4}$$

where

$$\begin{aligned} l_{kj}(\partial_t, \partial_x) &= \delta_{kj}[\partial_t^2 + \Delta(-\nu\mu\partial_t + \kappa\mu^2\Delta)] \\ &+ \partial_{x_k}\partial_{x_j}[-\nu(\lambda + \mu)\partial_t + \kappa(\lambda + \mu)(\lambda + 3\mu)\Delta]. \end{aligned} \tag{3.5}$$

By assumption (A2),  $\mu > 0$ ,  $\lambda + \mu > 0$ .

We write (3.4) in the short form

$$\mathcal{L}(\partial_t, \partial_x)u = f, \quad (3.6)$$

where  $\mathcal{L}(\partial_t, \partial_x)$  is the matrix operator with elements  $\{l_{kj}(\partial_t, \partial_x)\}_{k,j=1,2,3}$ . Moreover, let  $L = \det \mathcal{L}$ .

**Lemma 3.1:** *System (3.1)<sub>1</sub> is parabolic.*

**Proof:** By the Fourier-Laplace transform

$$\hat{u}(\xi, p) = \int_0^{\infty} e^{pt} dt \int_{\mathbb{R}^3} e^{i\xi \cdot x} u(x, t) dx,$$

system (3.6) takes the form

$$\sum_{j=1}^3 l_{kj}(p, i\xi) \hat{u}_j = \hat{f}_k, \quad k = 1, 2, 3.$$

Then

$$L = d^2[d + (\lambda + \mu)b|\xi|^2]$$

where

$$d = p^2 + \mu a|\xi|^2, \quad a = \nu p + \kappa \mu |\xi|^2, \quad b = \nu p + \kappa(\lambda + 3\mu)|\xi|^2.$$

The roots of equation  $L = 0$  are

- (i) double-root  $d = 0$ ;
- (ii)  $d + (\lambda + \mu)b|\xi|^2 = 0$ .

Solving (i) we get

$$p^2 + \nu \mu |\xi|^2 p + \kappa \mu^2 |\xi|^4 = 0,$$

so

$$p = \frac{-\nu \pm \sqrt{\nu^2 - 4\kappa}}{2} \mu |\xi|^2.$$

In case of (ii) we have

$$p^2 + (\lambda + 2\mu)\nu p |\xi|^2 + \kappa(\lambda + 2\mu)^2 |\xi|^4 = 0.$$

Hence,

$$p = \frac{-\nu \pm \sqrt{\nu^2 - 4\kappa}}{2} (\lambda + 2\mu) |\xi|^2.$$

In both cases there exists a real positive number  $\delta$  such that

$$\operatorname{Re} p \leq -\delta |\xi|^2.$$

This ends the proof.  $\square$

Now we examine the boundary conditions.

**Lemma 3.2:** *Boundary conditions (3.2)<sub>1</sub> and (3.2)<sub>2</sub> satisfy the Shapiro-Lopatinskij conditions for system (3.1) (complementarity condition).*

**Proof:** We examine system (3.1)<sub>1</sub> with the vanishing right-hand side and initial conditions,  $f = u_0 = u_1 = 0$ . Let us denote this problem by (P). First we examine problem (P) locally in the half-space  $x_3 > 0$ . Looking for solutions vanishing at  $x_3 \rightarrow \infty$  the Shapiro-Lopatinskij condition means that we have only a solution identically equal to zero (see [16], Chap. 2, §8). Hence, we can replace this condition by the coercivity argument. For this purpose we derive an estimate for weak solutions of problem (P). Multiplying (3.1)<sub>1</sub> by  $u_t$  and integrating over  $\Omega$  we get

$$\frac{1}{2} \frac{d}{dt} \int_{\bar{\Omega}} |u_t|^2 dx - \nu \int_{\bar{\Omega}} Q u_t \cdot u_t dx + \kappa \int_{\bar{\Omega}} Q^2 u \cdot u_t dx = 0.$$

Integrating by parts and then integrating with respect to time yields

$$\begin{aligned} & \frac{1}{2} \int_{\bar{\Omega}} |u_t(t)|^2 dx + \frac{\kappa}{2} \int_{\bar{\Omega}} |Qu(t)|^2 dx + \nu \int_{\bar{\Omega}'} (A\varepsilon(u_{t'})) \cdot \varepsilon(u_{t'}) dx dt' \\ & - \nu \int_{\bar{S}'} u_{t'} \cdot (A\varepsilon(u_{t'})) ndS dt' + \kappa \int_{\bar{S}'} u_{t'} \cdot (A\varepsilon(Qu)) ndS dt' \\ & - \kappa \int_{\bar{S}'} Qu \cdot (A\varepsilon(u_{t'})) ndS dt' = 0, \quad t \leq T. \end{aligned} \quad (3.7)$$

We see that any of the boundary conditions (3.2)<sub>1</sub> or (3.2)<sub>2</sub> imply that the boundary integrals in (3.7) vanish. Hence, in view of estimates (2.1), (2.2), it follows that

$$\|u_t\|_{L_\infty(0,T;L_2(\bar{\Omega}))} + \|u\|_{L_\infty(0,T;W_2^2(\bar{\Omega}))} + \|\varepsilon_t\|_{L_2(\bar{\Omega}')} \leq 0$$

what implies  $u = 0$  in  $\Omega^T$ . This concludes the proof.  $\square$

**4. Auxiliary existence results for parabolic problems of fourth and second order**

Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain with a sufficiently smooth boundary  $S$ . Let us consider the fourth order system

$$\begin{aligned} u_{tt} - \nu Q u_t + \kappa Q^2 u &= f && \text{in } \Omega^T, \\ u|_{t=0} &= u_0, \quad u_t|_{t=0} = u_1 && \text{in } \Omega, \\ B(\partial_x)u &= 0 && \text{on } S^T, \end{aligned} \tag{4.1}$$

where  $B(\partial_x)$  is given by (1.4)<sub>1</sub> or (1.4)<sub>2</sub>.

**Lemma 4.1:** *Let  $f \in L_p(\Omega^T)$ ,  $u_0 \in W_p^{4-2/p}(\Omega)$ ,  $u_1 \in W_p^{2-2/p}(\Omega)$ ,  $1 < p < \infty$ ,  $S \in C^4$ . Then there exists a unique solution  $u \in W_p^{4,2}(\Omega^T)$  of problem (4.1) such that*

$$\|u\|_{W_p^{4,2}(\Omega^T)} \leq c(\|f\|_{L_p(\Omega^T)} + \|u_0\|_{W_p^{4-2/p}(\Omega)} + \|u_1\|_{W_p^{2-2/p}(\Omega)}). \tag{4.2}$$

**Proof:** Since the complementarity condition is satisfied (see Lemma 3.2) we apply the results of [16]. This shows the assertion.  $\square$

Let us consider problem

$$\begin{aligned} u_{tt} - \nu Q u_t + \kappa Q^2 u &= \nabla \cdot \sigma + b && \text{in } \Omega^T, \\ u|_{t=0} &= u_0, \quad u_t|_{t=0} = u_1 && \text{in } \Omega, \\ B(\partial_x)u &= 0 && \text{on } S^T, \end{aligned} \tag{4.3}$$

where  $\sigma = (\sigma_{ij})_{i,j=1,2,3}$ ,  $b = (b_i)_{i=1,2,3}$ ,  $B(\partial_x)$  given by (1.4)<sub>1</sub> or (1.4)<sub>2</sub>.

**Lemma 4.2:** *Let  $\sigma \in L_p(\Omega^T)$ ,  $b \in L_p(\Omega^T)$ ,  $u_0 \in W_p^{3-2/p}(\Omega)$ ,  $u_1 \in W_p^{1-2/p}(\Omega)$ ,  $1 < p < \infty$ ,  $S \in C^3$ . Then solutions of problem (4.3) satisfy the inequality*

$$\begin{aligned} \|u\|_{W_p^{3,2}(\Omega^T)} &\leq c(\|\sigma\|_{L_p(\Omega^T)} \\ &+ \|b\|_{L_p(\Omega^T)} + \|u_0\|_{W_p^{3-2/p}(\Omega)} + \|u_1\|_{W_p^{1-2/p}(\Omega)}). \end{aligned} \tag{4.4}$$

**Proof:** Let  $G = (G_{ij})_{i,j=1,2,3}$  be the Green function of problem (4.3) with vanishing initial conditions  $u_0 = u_1 = 0$  on the half-space  $x_3 > 0$ . Then  $u$  admits the following representation

$$u(x, t) = \int_{\mathbb{R}_+^3 \times \mathbb{R}} G(x, t, x', t') (\nabla_{x'} \cdot \sigma + b) dx' dt'.$$



Since  $G|_{\sigma_3=0} = 0$  we obtain

$$u(x, t) = - \int_{\mathbb{R}_+^3 \times \mathbb{R}} \nabla_{\sigma'} G(x, t, x', t') \sigma dx' dt' + \int_{\mathbb{R}_+^3 \times \mathbb{R}} G b dx' dt', \quad (4.5)$$

where  $(\nabla_{\sigma'} G \sigma)_i = \sum_{j,k=1}^3 \partial_{x'_k} G_{ij} \sigma_{jk}$ . In view of (4.5), by [16] it follows that

$$\|u\|_{W_p^{3,2}(\mathbb{R}_+^3 \times \mathbb{R})} \leq c(\|\sigma\|_{L_p(\mathbb{R}_+^3 \times \mathbb{R})} + \|b\|_{L_p(\mathbb{R}_+^3 \times \mathbb{R})}).$$

Next, by using the regularizer technique and the extension of the initial data the assertion can be concluded.  $\square$

Now let us consider the parabolic problem

$$\begin{aligned} a(x, t)\theta_t - \Delta\theta &= f && \text{in } \Omega^T, \\ \theta|_{t=0} &= \theta_0 && \text{in } \Omega, \\ n \cdot \nabla\theta &= 0 && \text{on } S^T. \end{aligned} \quad (4.6)$$

**Lemma 4.3:** *Let  $f \in L_p(\Omega^T)$ ,  $\theta_0 \in W_p^{2-2/p}(\Omega)$ ,  $1 < p < \infty$ ,  $S \in C^2$  and the coefficient  $a \in C^{\alpha, \alpha/2}(\Omega^T)$ ,  $\alpha \in (0, 1)$  satisfies  $a^* \geq a \geq a_* > 0$ ,  $a_*$ ,  $a^*$  — constants,  $a_t \in L_2(0, T; L_2(\Omega))$ . Then there exists a unique solution  $\theta \in W_p^{2,1}(\Omega^T)$  of problem (4.6) such that*

$$\|\theta\|_{W_p^{2,1}(\Omega^T)} \leq \varphi\left(\frac{1}{\min a}, \max a, \|a\|_{C^{\alpha, \alpha/2}(\Omega^T)}, \|a_t\|_{L_2(\Omega^T)}, T\right) \cdot [\|f\|_{L_p(\Omega^T)} + \|\theta_0\|_{W_p^{2-2/p}(\Omega)} + \|\theta\|_{V_2^2(\Omega^T)}] \quad (4.7)$$

and

$$\|\theta\|_{V_2^2(\Omega^T)} := \text{ess sup}_{t \in (0, T)} \|\theta(t)\|_{L_2(\Omega)} + \|\nabla\theta\|_{L_2(\Omega^T)} \leq A(T), \quad (4.8)$$

where  $\varphi$  is an increasing positive function of its arguments, and  $A(T)$  is a positive function depending on the data  $f$ ,  $\theta_0$  and  $a$ ,  $\|a_t\|_{L_2(\Omega^T)}$ .

**Proof:** First we obtain energy inequality (4.8). Next by applying a partition of unity and using energy inequality (4.8) we obtain (4.7).  $\square$

### 5. Outline of the proof of Theorem 2.1

The idea of the proof is the same as in [11]. It is based on the Leray-Schauder fixed point theorem. In the present proof the essential role play the auxiliary results in Lemmas 4.1 – 4.3. Here we present the main steps, the details are given in [11].

**Step 1. The solution map.** We use the Leray-Schauder theorem in the following formulation:

**Theorem 5.1:** Let  $B$  be a Banach space. Assume that  $T : [0, 1] \times B \rightarrow B$  is a map with the following properties:

- (i) For any fixed  $\tau \in [0, 1]$  the map  $T(\tau, \cdot) : B \rightarrow B$  is completely continuous.
- (ii) For every bounded subset  $C$  of  $B$ , the family of maps  $T(\cdot, \chi) : [0, 1] \rightarrow B, \chi \in C$ , is uniformly equicontinuous.
- (iii) There is a bounded subset  $C$  of  $B$  such that any fixed point in  $B$  of  $T(\tau, \cdot), 0 \leq \tau \leq 1$ , is contained in  $C$ .
- (iv)  $T(0, \cdot)$  has precisely one fixed point in  $B$ .  
Then  $T(1, \cdot)$  has at least one fixed point in  $B$ .

In order to define the corresponding solution map we extend the definition of  $F_1(\varepsilon, \theta)$  to all  $\theta \in \mathbb{R}$  in such a way that it is of class  $C^3$ , and that

$$F_{1,\theta\theta}(\varepsilon, \theta) \geq 0 \quad \text{for all } (\varepsilon, \theta) \in S^2 \times (-\infty, 0).$$

With such extension the lower bound (2.4) on  $c_0(\varepsilon, \theta)$  remains valid for all  $(\varepsilon, \theta) \in S^2 \times \mathbb{R}$ .

The solution space is  $V(p, q)$  defined by (2.9). The solution map

$$T(\tau, \cdot) : (\bar{u}, \bar{\theta}) \in V(p, q) \rightarrow (u, \theta) \in V(p, q), \quad \tau \in [0, 1], \quad (5.1)$$

is defined by means of the following initial-boundary value problems:

$$\begin{aligned} u_{tt} - \nu Q u_t + \kappa Q^2 u &= \tau [\nabla \cdot F_{,\varepsilon}(\bar{\varepsilon}, \bar{\theta}) + b] && \text{in } \Omega^T, \\ u|_{t=0} &= \tau u_0, \quad u_t|_{t=0} = \tau u_1 && \text{in } \Omega, \\ B(\partial_\nu)u &= 0 && \text{on } S^T, \end{aligned} \quad (5.2)$$

$$\begin{aligned} c_0(\varepsilon, \bar{\theta}, \tau)\theta_t - k_0 \Delta \theta &= \tau [\bar{\theta} F_{,\theta\varepsilon}(\varepsilon, \bar{\theta}) \cdot \varepsilon_t + \nu(A\varepsilon_t) \cdot \varepsilon_t + g] && \text{in } \Omega^T, \\ \theta|_{t=0} &= \tau \theta_0 && \text{in } \Omega, \\ n \cdot \nabla \theta &= 0 && \text{on } S^T, \end{aligned} \quad (5.3)$$

where

$$c_0(\varepsilon, \bar{\theta}, \tau) = c_v - \tau \bar{\theta} F_{,\theta\theta}(\varepsilon, \bar{\theta}), \quad \bar{\varepsilon} = \varepsilon(\bar{u}).$$

Clearly, any fixed point of  $T(1, \cdot)$  in  $V(p, q)$  is equivalent to a solution  $(u, \theta)$  of problem (1.1), (1.2) in  $V(p, q)$ . Therefore, the proof resolves itself into checking properties (i)–(iv) of the solution map  $T(\tau, \cdot)$ .

**Step 2. Properties (i), (ii) and (iv) of the solution map.**

Property (i) follows by showing that for any fixed  $\tau \in [0, 1]$ ,  $T(\tau, \cdot)$  maps the bounded subsets into precompact subsets in  $V(p, q)$ . Let  $(\bar{u}^n, \bar{\theta}^n)$  be a bounded sequence in  $V(p, q)$  such that for  $n \rightarrow \infty$

$$\bar{u}^n \rightharpoonup \bar{u} \text{ weakly in } W_p^{4,2}(\Omega^T), \quad \bar{\theta}^n \rightharpoonup \bar{\theta} \text{ weakly in } W_q^{2,1}(\Omega^T), \quad 5 < p, q < \infty.$$

By repeating the arguments [11] and using Lemma 4.1 we conclude that for the values of  $T(\tau, \cdot)$  given by  $(u^n, \theta^n) = T(\tau, \bar{u}^n, \bar{\theta}^n)$ , the following convergences hold for  $n \rightarrow \infty$

$$\begin{aligned} u^n &\rightarrow u \text{ strongly in } W_p^{4,2}(\Omega^T), \quad 5 < p < \infty, \\ \theta^n &\rightarrow \theta \text{ strongly in } W_q^{2,1}(\Omega^T), \quad 5 < q < \infty, \end{aligned}$$

where  $(u, \theta) = T(\tau, \bar{u}, \bar{\theta})$ . This shows (i).

Property (ii) follows by direct comparison of two solutions  $(u, \theta)$  and  $(\bar{u}, \bar{\theta})$  corresponding to parameters  $\tau$  and  $\bar{\tau}$ , respectively. Applying Lemmas 4.1 and 4.3 we show that

$$\|u - \bar{u}\|_{W_p^{4,2}(\Omega^T)}, \quad \|\theta - \bar{\theta}\|_{W_q^{2,1}(\Omega^T)} \leq c|\tau - \bar{\tau}|, \quad 5 < p, q < \infty.$$

Property (iv) is obvious in view of Lemmas 4.1, 4.3 and the definition of  $T(\tau, \cdot)$ .

Further steps of the proof concern property (iii) of the solution map.

**Step 3. A priori bounds for a fixed point.** Without loss of generality we set  $\tau = 1$  and assume that  $(u, \theta) \in V(p, q)$  is a fixed point of  $T(1, \cdot)$ . We begin with proving that temperature is positive. Having this we establish energy estimates and then improve them recursively.

**Step 3.1. Positivity of temperature.**

**Lemma 5.1:** (see [10], Lemma 3.1) *Let*

$$\theta_* \equiv \min_{\Omega} \theta_0 > 0, \quad g \geq 0 \text{ in } \Omega^T,$$

and  $(u, \theta)$  be a solution to (1.1), (1.2) such that  $\varepsilon, \varepsilon_t \in L_\infty(\Omega^T)$ ,  $\theta \in L_\infty(\Omega^T)$ ,  $\theta_t \in L_1(0, T; L_q(\Omega))$ ,  $1 < q < \infty$ .

Then there exists a positive finite number  $\omega$  satisfying

$$[g + \nu(A\varepsilon_t) \cdot \varepsilon_t] \exp(\omega t) + [\omega c_0(\varepsilon, \theta) + F_{,\theta\varepsilon}(\varepsilon, \theta) \cdot \varepsilon_t] \theta_* \geq 0 \text{ in } \Omega^T,$$

such that

$$\theta \geq \theta_* \exp(-\omega t) \text{ in } \Omega^T. \tag{5.4}$$

We point out that the regularity assumptions in Lemma 5.1 are satisfied for solutions in the space  $V(p, q)$ .

### Step 3.2. Energy estimates.

**Lemma 5.2:** (see [10], Lemma 3.2). Let

$$\begin{aligned} u_0 &\in W_2^2(\Omega), \quad u_1 \in L_2(\Omega), \quad \theta_0 \in L_1(\Omega), \\ (F_1(\varepsilon_0, \theta_0) - \theta_0 F_{1,\theta}(\varepsilon_0, \theta_0)) + F_2(\varepsilon_0) &\in L_1(\Omega), \\ b &\in L_1(0, T; L_2(\Omega)), \quad g \in L_1(\Omega^T). \end{aligned}$$

Assume that  $\theta \geq 0$  in  $\Omega^T$  and the bound (2.7) holds. Then a solution  $(u, \theta)$  to (1.1), (1.2) satisfies estimate

$$\begin{aligned} \|\theta\|_{L_\infty(0, T; L_1(\Omega))} + \|u_1\|_{L_\infty(0, T; L_2(\Omega))}^2 + \|Qu\|_{L_\infty(0, T; L_2(\Omega))}^2 \\ + \|(F_1(\varepsilon, \theta) - \theta F_{1,\theta}(\varepsilon, \theta)) + F_2(\varepsilon)\|_{L_\infty(0, T; L_1(\Omega))} \leq c \end{aligned} \quad (5.5)$$

with a constant  $c$  depending only on the data.

We indicate the implications of estimate (5.5) which are of importance in the next step. Firstly, by property (2.2) of the operator  $Q$ ,

$$\|u\|_{L_\infty(0, T; W_2^1(\Omega))} \leq c,$$

consequently

$$\|\varepsilon\|_{L_\infty(0, T; W_2^1(\Omega)) \cap L_\infty(0, T; L_6(\Omega))} \leq c. \quad (5.6)$$

Secondly, (5.5) implies the bound

$$\|u\|_{W_{2,\infty}^{2,1}(\Omega^T)} \leq c,$$

so, in view of Sobolev's imbeddings,

$$\|\varepsilon\|_{W_{2,\infty}^{1,1/2}(\Omega^T) \cap L_{10}(\Omega^T)} \leq c. \quad (5.7)$$

Our aim is to prove estimates (2.10). This will be accomplished with the help of Lemmas 4.1–4.3. We point out that in view of the nonlinearity of the coefficient  $c_0(\varepsilon, \theta)$  to apply Lemma 4.3 we have to prove first Hölder-norm bounds for  $\varepsilon$  and  $\theta$ . To this end we proceed in a number of steps which provide the recursive improvement of estimates for  $\theta$  and  $\varepsilon$ .

**Step 3.3. The first temperature estimate.** According to Lemma 4.2 we have the following estimate for problem (1.1):

$$\begin{aligned} \|\varepsilon\|_{W_p^{2,1}(\Omega^T)} &\leq c \|u\|_{W_p^{2,3/2}(\Omega^T)} \\ &\leq c (\|F_{1,\varepsilon}(\varepsilon, \theta)\|_{L_r(\Omega^T)} + \|b\|_{L_r(\Omega^T)}) \\ &\quad + \|u_0\|_{W_p^{2-\alpha/r}(\Omega)} + \|u_1\|_{W_p^{1-\alpha/r}(\Omega)}, \quad 1 < p < \infty. \end{aligned} \quad (5.8)$$

In view of this estimate, repeating the arguments of [11], Lemma 4.3, which rely on multiplying equation (1.2)<sub>1</sub> by  $\theta$ , integrating over  $\Omega$ , and applying appropriate imbeddings and interpolation inequalities, we get

**Lemma 5.3:** (see [11], Lemma 4.9) *Suppose that assumption (A3-1) (iv) is satisfied and  $\|\varepsilon\|_{L_{10}(\Omega)} \leq c(T)$ . Then there exists a constant  $c(T)$  depending only on the data and  $T^a$ ,  $a \in \mathbb{R}_+$ , such that*

$$\|\theta\|_{L_\infty(0,T;L_2(\Omega))} + \|\nabla\theta\|_{L_2(\Omega^T)} \leq c(T). \tag{5.9}$$

We indicate the implications of (5.9). By virtue of the imbedding,

$$\|\theta\|_{L_{10/3}(\Omega^T)} \leq c(T). \tag{5.10}$$

Moreover, using the following estimate derived in the proof of Lemma 5.3 (see [11], eq. (52))

$$\|\varepsilon\|_{W_{20/7}^{2,1}(\Omega^T)} \leq c(T)(\|\nabla\theta\|_{L_2(\Omega^T)}^{3s/4} + 1), \tag{5.11}$$

we have

$$\|\varepsilon\|_{W_{20/7}^{2,1}(\Omega^T)} \leq c(T).$$

Hence, by Sobolev's imbedding,  $\varepsilon$  is Hölder continuous in  $\Omega^T$  and

$$\|\varepsilon\|_{C^{\alpha_1, \alpha_1/2}(\Omega^T)} \leq c(T) \quad \text{with } 0 < \alpha_1 < 1/4. \tag{5.12}$$

Due to (5.12), using the growth condition on  $F_{1,\varepsilon}$  in (A3-1) (iv) and estimate (5.8), we conclude that

$$\|\varepsilon\|_{W_p^{2,1}(\Omega^T)} \leq c(T)(\|\theta\|_{L_{p^*}^2(\Omega^T)}^2 + 1) \leq c(T) \quad \text{for } p = 10/(3s) > 5. \tag{5.13}$$

Further, thanks to (5.12), bounds (2.5) imply

$$|c_0(\varepsilon, \theta)| + |c_{0,\varepsilon}(\varepsilon, \theta)| + |c_{0,\theta}(\varepsilon, \theta)| \leq c(T) \quad \text{in } \Omega^T. \tag{5.14}$$

**Step 3.4. The second temperature estimate.** Multiplying equation (1.2)<sub>1</sub> by  $\theta_t$  and integrating over  $\Omega^t$  we obtain

**Lemma 5.4:** (see [11], Lemma 4.4)

*Suppose that*

$$\begin{aligned} 0 < s < 2/3 \quad g \in L_2(\Omega^T), \quad \nabla\theta_0 \in L_2(\Omega), \quad \text{and} \\ \|\theta\|_{L_{10/3}(\Omega^T)} \leq c(T), \quad \|\varepsilon\|_{C^{\alpha_1, \alpha_1/2}(\Omega^T)} \leq c(T), \\ \|\varepsilon_t\|_{L_p(\Omega)} \leq c(T) \quad \text{for } p = 10/(3s). \end{aligned}$$

*Then there exists a constant  $c(T) > 0$  such that*

$$\|\theta_t\|_{L_2(\Omega^T)} + \|\theta\|_{L_\infty(0,T;W_2^2(\Omega))} \leq c(T). \tag{5.15}$$

**Step 3.5. The third temperature estimate.** Writing equation (1, 2)<sub>1</sub> in the form

$$-k_0 \Delta \theta = -c_0(\varepsilon, \theta) \theta_t + \theta F_{,\theta \varepsilon}(\varepsilon, \theta) \cdot \varepsilon_t + \nu(A \varepsilon_t) \cdot \varepsilon_t + g \quad (5.16)$$

and using (5.14), (5.15), the right-hand side of (5.16) can be estimated in  $L_2(\Omega^T)$ -norm. Consequently, by virtue of the classical elliptic theory,

$$\|\theta\|_{L_2(0,T;W_2^2(\Omega))} \leq c(T), \quad \text{so} \quad \|\theta\|_{L_2(0,T;L_\infty(\Omega))} \leq c(T). \quad (5.17)$$

Furthermore, (5.17) and (5.15) imply that

$$\|\theta\|_{W_2^{2,1}(\Omega^T)} \leq c(T), \quad \|\nabla \theta\|_{W_2^{1,1/2}(\Omega^T)} \leq c(T),$$

so, by Sobolev's imbeddings,

$$\|\theta\|_{L_{10}(\Omega^T)} \leq c(T), \quad \|\nabla \theta\|_{L_{10/3}(\Omega^T)} \leq c(T). \quad (5.18)$$

**Step 3.6. The improvement of strain estimate.** Using (5.18)<sub>1</sub> we repeat estimate (5.13) to conclude that

$$\begin{aligned} \|\varepsilon\|_{W_2^{2,1}(\Omega^T)} &\leq c(T)(\|\theta\|_{L_{10}(\Omega^T)} + 1) \leq c(T), \\ \|\nabla \varepsilon\|_{W_2^{1,1/2}(\Omega^T)} &\leq c(T) \end{aligned}$$

for  $p = 10/s > 15$ . Consequently, by the imbedding,

$$\|\nabla \varepsilon\|_{C^{\alpha_2, \alpha_2/2}(\Omega^T)} \leq c(T) \quad \text{with} \quad 0 < \alpha_2 < 1 - s/2. \quad (5.19)$$

Now, recalling assumptions (A3-1) (iii), (iv) and (A3-2) (ii), (iii), and using (5.12), (5.18), (5.19), we get

$$\|\nabla \cdot F_{,\varepsilon}(\varepsilon, \theta)\|_{L_{10/3}(\Omega^T)} \leq c(T).$$

Hence, by virtue of Lemma 4.1,

$$\|u\|_{W_{10/3}^{4,2}(\Omega^T)} \leq c(T),$$

so

$$\|\varepsilon_t\|_{W_{10/3}^{1,1/2}(\Omega^T)} \leq c(T). \quad (5.20)$$

In view of the imbedding

$$W_{10/3}^{1,1/2}(\Omega^T) \subset L_q(0, T; L_\infty(\Omega)) \quad \text{for} \quad q < 4,$$

(5.20) implies

$$\|\varepsilon_t\|_{L_{10}(\Omega^T)} + \|\varepsilon_t\|_{L_q(0,T;L_\infty(\Omega))} \leq c(T). \quad (5.21)$$

This estimate is crucial for obtaining  $L_\infty(\Omega^T)$ -norm bound and subsequently Hölder-norm bound on  $\theta$ . This is done in the next two steps which are the same as in [11].

**Step 3.7. Pointwise estimate on temperature.** Here we have to impose assumption (A5) on the separable form of  $F_1(\varepsilon, \theta)$ . Then we have

**Lemma 5.5:** (see [11], Lemma 5.1) Suppose (A5) is satisfied in addition to (A1)-(A4). Moreover, let

$$\theta_0 \in L_\infty(\Omega^T), \quad g \in L_1(0, T; L_\infty(\Omega)),$$

$$\theta \geq \theta_*(T) = \theta_* \exp(-\omega T),$$

$$\varepsilon \in L_\infty(\Omega^T), \quad \varepsilon_t \in L_2(0, T; L_\infty(\Omega)), \quad \text{with}$$

$$\|\varepsilon\|_{L_\infty(\Omega^T)} \leq c(T), \quad \|\varepsilon_t\|_{L_2(0, T; L_\infty(\Omega))} \leq c(T),$$

where  $c(T)$  is an increasing positive function. Then

$$\begin{aligned} \|\theta\|_{L_\infty(\Omega^T)} &\leq c \exp(c(T)T^{1/2}) \|\varepsilon_t\|_{L_2(0, T; L_\infty(\Omega))} \\ &\quad \cdot (\|\varepsilon_t\|_{L_2(0, T; L_\infty(\Omega))}^2 + \|\theta\|_{L_1(0, T; L_\infty(\Omega))} + \|\theta\|_{L_\infty(\Omega)}) \leq c(T). \end{aligned} \quad (5.22)$$

The proof of this lemma is based on multiplying equation (1.2)<sub>1</sub> by  $\theta^r$ ,  $r > 1$ , integrating over  $\Omega$  and introducing a specially constructed primitive of the function  $-\theta^{r+1}F_{1,\theta\theta}(\varepsilon, \theta)$  with respect to  $\theta$ . We point out that a similar idea was used in Lemma 5.3 where (1.2)<sub>1</sub> has been multiplied by  $\theta$  and a primitive of  $-\theta^2F_{1,\theta\theta}(\varepsilon, \theta)$  with respect to  $\theta$  has been constructed. As a result we obtain an estimate on  $\theta$  in  $L_\infty(0, T; L_r(\Omega))$ -norm which due to bound (5.21) on  $\varepsilon_t$  allows to pass to the limit with  $r \rightarrow \infty$  to conclude the assertion.

In view of (5.22), estimate (5.13) yields

$$\|\varepsilon\|_{W_p^{2,1}(\Omega^T)} \leq c(T) \quad \text{for } 1 < p < \infty. \quad (5.23)$$

**Step 3.8. Hölder continuity of temperature.** To prove Hölder continuity of  $\theta$  we apply DeGiorgi method in a way presented in [8]. Namely, we prove that  $\theta$  is an element of the space  $B_2(\Omega^T, M, \gamma, r, \delta, \kappa)$  where  $M, \gamma, r, \delta, \kappa$  are positive parameters (for definition of this space see [8], Chap. II. 7). The essential for the proof is  $L_\infty(\Omega^T)$ -norm estimate on  $\theta$  provided by Lemma 5.5 and  $L_p(\Omega^T)$ -norm estimate (5.23) on  $\varepsilon_t$ . We have the following

**Lemma 5.6:** (see [11], Lemma 6.1) Suppose that

$$\begin{aligned} |\varepsilon| &\leq c(T) \text{ in } \Omega^T, \quad \|\varepsilon_t\|_{L_p(\Omega^T)} \leq c(T), \quad 1 < p < \infty, \\ \|\theta\|_{W^{2,1}(\Omega^T)} &\leq c(T), \quad \|\theta\|_{L_\infty(\Omega^T)} \leq M = c(T), \\ |c_0(\varepsilon, \theta)| + |c_{0,\varepsilon}(\varepsilon, \theta)| + |c_{0,\theta}(\varepsilon, \theta)| &\leq c(T) \text{ in } \Omega^T, \\ g &\in L_p(\Omega^T), \quad 1 < p < \infty, \quad \theta_0 \in C^{\alpha_0}(\Omega), \quad 1 < \alpha_0 < 1. \end{aligned}$$

Furthermore, let  $k$  be a positive number such that

$$k > \sup_{\Omega} \theta_0(x) \text{ and } M - k < \delta \text{ with some } \delta > 0.$$

Then

$$\theta \in B_2(\Omega^T, M, \gamma, r, \delta, \varkappa), \quad (5.24)$$

where

$$r = q = \frac{10}{3}, \quad \varkappa \in \left(0, \frac{2}{3}\right), \quad \gamma = c(T).$$

By virtue of (5.24) we can apply the imbedding result of [8], Theorem II.7.1, to conclude that  $\theta$  is Hölder continuous in  $\Omega^T$ , and

$$\|\theta\|_{C^{\alpha, \alpha/2}(\Omega^T)} \leq c(T) \quad (5.25)$$

with Hölder exponent  $0 < \alpha < 1$  depending on  $M = c(T)$ ,  $\gamma = c(T)$ ,  $r$ ,  $\delta$  and  $\varkappa$ .

**Step 3.9. The final estimates.** In view of Hölder continuity of  $\varepsilon$  and  $\theta$  as well as bound (5.23) we can apply Lemmas 4.1 and 4.3 to conclude final estimates (2.10) and thereby prove property (iii) of the solution map. We have

**Lemma 5.7:** (see [11], Lemma 6.2). Suppose that  $\varepsilon$  and  $\theta$  are Hölder continuous in  $\Omega^T$ , and

$$\begin{aligned} |\varepsilon| + |\theta| &\leq c(T) \text{ in } \Omega^T, \\ \|\nabla \varepsilon\|_{L_\sigma(\Omega^T)} + \|\varepsilon_t\|_{L_\sigma(\Omega^T)} &\leq c(T) \text{ for } 1 < \sigma < \infty. \end{aligned}$$

Moreover, suppose that the data satisfy (A4). Then

$$\|\mathbf{u}\|_{W_p^{2,q}(\Omega^T)} \leq c(T), \quad \|\theta\|_{W_q^{2,1}(\Omega^T)} \leq c(T), \quad 5 < p, q < \infty. \quad (5.26)$$

Summarizing, we have shown that the solution map (5.1) satisfies assumptions (i)–(iv) of the Leray-Schauder theorem. Thus  $T(1, \cdot)$  has at least one fixed point in  $V(p, q)$  which is equivalent to a solution  $(\mathbf{u}, \theta) \in V(p, q)$  to problem (1.1), (1.2). Together with bounds (5.26) and (5.4) the proof is completed.



## References

1. R. Abeyaratne, J. K. Knowles, Implications of viscosity and strain-gradient effects for the kinetics of propagating phase boundaries in solids, *SIAM J Appl. Math.* 51, No 5 (1991) 1205–1221.
2. T. Aiki, Weak solutions for Falk's model of shape memory alloys, *Math. Meth. Appl. Sci.* 23 (2000) 299–319.
3. O. V. Besov, V. P. Il'in, S. M. Nikolski, Integral Representations of Functions and Imbedding Theorems, *Nauka*, Moscow, 1975 (in Russian).
4. M. Brokate, J. Sprekels, Hysteresis and Phase Transitions, *Applied Mathematical Sciences* 121, Springer, 1996.
5. S. D. Eidelman, Parabolic equations, in: Partial Differential Equations VI, *Encyclopaedia Math. Sci.* 63, Yu. V. Egorov and M. A. Shubin (Eds.), Springer, 1994, 205–316.
6. F. Falk, Elastic phase transitions and nonconvex energy functions, in: Free Boundary Problems: Theory and Applications, K.-H. Hoffmann and J. Sprekels (Eds.), vol. I *Pitman Research Notes Math. Ser.* 185 (1990), Longman, 45–59.
7. J. Nečas, Les Méthodes Directes en Théorie des Équations Elliptiques, *Mas-son*, Paris, 1967.
8. O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Ural'tseva, Linear and Quasi-linear Equations of Parabolic Type, *Nauka*, Moscow, 1967 (in Russian).
9. I. Pawłow, Three-dimensional model of thermomechanical evolution of shape memory materials, *Control Cybernet.* 29, No 1 (2000), 341–365.
10. I. Pawłow, W. M. Zajączkowski, Unique global solvability in two-dimensional non-linear thermoelasticity, *Math. Meth. Appl. Sci.* 28 (2005), 551–592.
11. I. Pawłow, W. M. Zajączkowski, Global existence to a three-dimensional non-linear thermoelasticity system arising in shape memory materials, *Math. Meth. Appl. Sci.* 28 (2005), 407–442.
12. I. Pawłow, W. M. Zajączkowski, On diffused-interface models of shape memory alloys, *Control Cybernet.* 32, No 3 (2003), 629–658.
13. I. Pawłow, A. Żochowski, Nonlinear thermoelastic system with viscosity and nonlocality, in: Proc. Free Boundary Problems: Theory and Applications I, N. Kenmochi (Ed.), *Gakuto Internat. Ser. Math. Sci. Appl.* 13, 2000, 251–265.
14. I. Pawłow, A. Żochowski, Existence and uniqueness for a three-dimensional thermoelastic system, *Dissertationes Mathematicae* 406 (2002), 46 pp.
15. M. Slemrod, Admissibility criteria for propagating phase boundaries in a van der Waals fluid, *Arch. Rational Mech. Anal.* 81 (1983), 301–315.
16. V. A. Solonnikov, On boundary value problems for linear parabolic systems of a general type, *Trudy Mat. Inst. Steklov.* 83 (1965) (in Russian).
17. J. Sprekels, S. Zheng, Global solutions to the equations of a Ginzburg-Landau theory for structural phase transitions in shape memory alloys, *Physica D* 39 (1989), 59–76.

18. J. Zimmer, Global existence for a nonlinear system in thermoviscoelasticity with nonconvex energy, *J. Math. Anal. Appl.* **292** (2004), 589–604.
19. S. Yoshikawa, Weak solutions for the Falk model system of shape memory alloys in energy class, *Preprint* 2004.
20. S. Yoshikawa, Unique global existence for a three-dimensional thermoelastic system of shape memory alloys, *Adv. Math. Sci. Appl.* (to appear).

and the 1990s, the population of *S. taylori* in the western Mediterranean has increased (Borner et al. 2008).

As a result of the increasing number of *S. taylori* specimens, the genetic diversity of this species has been investigated in the western Mediterranean (Borner et al. 2008) and in the eastern Mediterranean (Borner and Karam 2010). In the present study, we have investigated the genetic diversity of *S. taylori* in the eastern Mediterranean, with the aim of identifying the genetic structure of the population and its relationship with the population in the western Mediterranean.

For this purpose, we used a mitochondrial DNA marker, the control region, which is highly variable and suitable for population genetic analysis (Borner et al. 2008). The control region is a non-coding region of the mitochondrial DNA, which is highly variable and suitable for population genetic analysis (Borner et al. 2008). The control region is a non-coding region of the mitochondrial DNA, which is highly variable and suitable for population genetic analysis (Borner et al. 2008).

The results of the present study show that the genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified.

The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified.

The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified.

The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified.

The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified.

The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified.

The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified. The genetic diversity of *S. taylori* in the eastern Mediterranean is high, with 12 haplotypes identified.

