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Classical solvability of 1-D Cahn-Hilliard equation coupled with elasticity

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Lemma 4.1

Let assumptions of Lemma 3.1 be satisfied. Moreover, assume that

$$
\begin{aligned}
&x:\mathbb{R}\to\mathbb{R} \quad \text{is of class } C^3, \text{ satisfying (6);}\\
&\text{coefficients } A>0 \text{ and } B \text{ are constant;}\\
&u_0\in H^3(I)\cap H_0^1(I),\quad u_1\in H^2(I)\cap H_0^1(I),\quad b\in L_1(0,T;H^2(I)\cap H_0^1(I))\\
&\chi\in L_2(0,T;H^4(I)) \text{ with } \chi_x=0 \text{ on } S^T
\end{aligned}
\tag{60}
$$

Then there exists a constant $\delta\in(0,1)$ such that the solution u of problem (17) satisfies the estimate

$$\|u_{xxt}\|_{L_\infty(0,T;L_2(I))}+\|u_{xxx}\|_{L_\infty(0,T;L_2(I))}\leqslant\delta\|\chi_{xxxx}\|_{L_2(I^T)}+c(1/\delta,c_1,T)+c_3 \tag{61}$$

with constant c_1 introduced in (56) and

$$c_3=\|b\|_{L_1(0,T;H^2(I))}+\|u_0\|_{H^3(I)}+\|u_1\|_{H^2(I)} \tag{62}$$

Proof

An application of Lemma 2.1 to problem (17) gives the estimate

$$
\begin{aligned}
\|u_{xxt}\|_{L_\infty(0,T;L_2(I))}+\|u_{xxx}\|_{L_\infty(0,T;L_2(I))}\leqslant{}&\|Bz'(\chi)\chi_x\|_{L_1(0,T;L_2(I))}+\|(Bz'(\chi)\chi_x)_{xx}\|_{L_1(0,T;L_2(I))}\\
&+\|b\|_{L_1(0,T;H^2(I))}+\|u_0\|_{H^3(I)}+\|u_1\|_{H^2(I)}
\end{aligned}
\tag{63}
$$

In view of (59) the right-hand side of (63) is bounded by

$$\|(Bz'(\chi)\chi_x)_{xx}\|_{L_1(0,T;L_2(I))}+c(c_1,T)+c_3 \tag{64}$$

with constant c_3 given by (62).

We estimate now the first term in (64). Using the equality

$$(z'(\chi)\chi_x)_{xx}=z'''(\chi)\chi_x^3+3z''(\chi)\chi_x\chi_{xx}+z'(\chi)\chi_{xxx}$$

we have

$$
\begin{aligned}
&\|(Bz'(\chi)\chi_x)_{xx}\|_{L_1(0,T;L_2(I))}\\
&\leqslant c\left[\int_0^T\left(\int_I\chi_x^6\,\mathrm{d}x\right)^{1/2}\mathrm{d}t+\int_0^T\left(\int_I(\chi_x\chi_{xx})^2\,\mathrm{d}x\right)^{1/2}\mathrm{d}t+\int_0^T\left(\int_I\chi_{xxx}^2\,\mathrm{d}x\right)^{1/2}\mathrm{d}t\right]
\end{aligned}
\tag{65}
$$

Let us examine the three subsequent terms on the right-hand side of (65). For the first term, using (59), we have

$$
\begin{aligned}
\int_0^T\left(\int_I\chi_x^6\,\mathrm{d}x\right)^{1/2}\mathrm{d}t&\leqslant\sup_t\left(\int_I\chi_x^2\,\mathrm{d}x\right)^{1/2}\int_0^T\|\chi_x\|^2_{L_\infty(I)}\,\mathrm{d}t\\
&\leqslant c_1\int_0^T\|\chi_x\|^2_{L_\infty(I)}\,\mathrm{d}t=I_1
\end{aligned}
$$

Now, applying the interpolation inequality (see (47))

$$\|\chi_x\|_{L_\infty(I)}\leqslant c\|\chi_{xxxx}\|^{1/6}_{L_2(I)}\|\chi_x\|^{5/6}_{L_2(I)}+c\|\chi_x\|_{L_2(I)} \tag{66}$$

using (59) and the Young inequality, we get

$$\begin{aligned} I_1 &\leqslant c(c_1)\int_0^T \|\chi_{xxxx}\|_{L_2(I)}^{1/3}\,\mathrm{d}t + c(c_1,T) \\ &\leqslant c(c_1,T)(\|\chi_{xxxx}\|_{L_2(I^T)}^{1/3} + 1) \\ &\leqslant \delta\|\chi_{xxxx}\|_{L_2(I^T)} + c(1/\delta, c_1, T), \quad \delta > 0 \end{aligned} \tag{67}$$

The second term on the right-hand side of (65) is treated in the following way:

$$\begin{aligned} \int_0^T \left(\int_I (\chi_x\chi_{xx})^2\,\mathrm{d}x\right)^{1/2}\mathrm{d}t &\leqslant \left(\sup_t \int_I \chi_x^2\,\mathrm{d}x\right)^{1/2}\int_0^T \|\chi_{xx}\|_{L_\infty(I)}\,\mathrm{d}t \\ &\leqslant c_1\int_0^T \|\chi_{xx}\|_{L_\infty(I)}\,\mathrm{d}t = I_2 \end{aligned}$$

where in the last inequality we used (59). Now, applying the interpolation inequality

$$\|\chi_{xx}\|_{L_\infty(I)} \leqslant c\|\chi_{xxxx}\|_{L_2(I)}^{1/2}\|\chi_x\|_{L_2(I)}^{1/2} + c\|\chi_x\|_{L_2(I)} \tag{68}$$

using again (59) and the Young inequality, we find that

$$\begin{aligned} I_2 &\leqslant c(c_1)\int_0^T \|\chi_{xxxx}\|_{L_2(I)}^{1/2}\,\mathrm{d}t + c(c_1,T) \\ &\leqslant c(c_1,T)(\|\chi_{xxxx}\|_{L_2(I^T)}^{1/2} + 1) \\ &\leqslant \delta\|\chi_{xxxx}\|_{L_2(I^T)} + c(1/\delta, c_1, T), \quad \delta > 0 \end{aligned} \tag{69}$$

Finally, the third term on the right-hand side of (65) is estimated with the help of the interpolation theorem by

$$\begin{aligned} \int_0^T \left(\int_I \chi_{xxx}^2\,\mathrm{d}x\right)^{1/2}\mathrm{d}t &\leqslant \int_0^T (\delta\|\chi_{xxxx}\|_{L_2(I)} + c(1/\delta)\|\chi_x\|_{L_2(I)})\,\mathrm{d}t \\ &\leqslant \delta\|\chi_{xxxx}\|_{L_2(I^T)} + c(1/\delta, c_1, T), \quad \delta > 0 \end{aligned} \tag{70}$$

Combining estimates (67), (69) and (70) in (65) we conclude that

$$\|(Bz'(\chi)\chi_x)_{xx}\|_{L_1(0,T;L_2(I))} \leqslant \delta\|\chi_{xxxx}\|_{L_2(I^T)} + c(1/\delta, c_1, T) \tag{71}$$

Hence, using (71) in (64) we conclude the assertion. □

Making use of Lemma 4.1 we prove now

Lemma 4.2

Let the assumptions of Lemma 4.1 hold true, $\psi:\mathbb{R}\to\mathbb{R}$ satisfies (19) and $\chi_0 \in H^2(I)$. Then the solution χ of problem (17) satisfies the estimate

$$\|\chi\|_{W_2^{4,1}(I^T)} \leqslant c(c_1,T)(\|u_{xxx}\|_{L_\infty(0,T;L_2(I))} + 1) + c\|\chi_0\|_{H^2(I)} \tag{72}$$

Proof

By virtue of Lemma 2.2 the solution χ of (17) satisfies

$$\begin{aligned}\|\chi\|_{W_2^{4,1}(I^T)} &\leqslant c\|R\|_{L_2(I^T)} + c\|\chi_0\|_{H^2(I)} \\ &\leqslant c(c_1)(\|\chi_x^2\|_{L_2(I^T)} + \|\chi_{xx}\|_{L_2(I^T)} + \|\chi_x^2 u_x\|_{L_2(I^T)} \\ &\quad + \|\chi_{xx}u_x\|_{L_2(I^T)} + \|\chi_x u_{xx}\|_{L_2(I^T)} + \|u_{xxx}\|_{L_2(I^T)}) \\ &\quad + c\|\chi_0\|_{H^2(I)} \end{aligned} \tag{73}$$

where we used assumptions on ψ and z together with $L_\infty(I^T)$-norm bound (59) on χ.

We estimate the individual terms on the right-hand side of (73). For the first term we have

$$\begin{aligned}\|\chi_x^2\|_{L_2(I^T)} &= \left(\int_0^T \int_I \chi_x^4 \,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\ &\leqslant \sup_t \|\chi_x\|_{L_2(I)} \left(\int_0^T \|\chi_x\|^2_{L_\infty(I)}\,\mathrm{d}t\right)^{1/2} = J_1\end{aligned}$$

We apply now the interpolation inequality (66) together with estimate (59) and next the Young inequality to obtain

$$\begin{aligned}J_1 &\leqslant cc_1\left(\int_0^T (\|\chi_{xxx}\|^{1/3}_{L_2(I)}\|\chi_x\|^{5/3}_{L_2(I)} + \|\chi_x\|^2_{L_2(I)})\,\mathrm{d}t\right)^{1/2} \\ &\leqslant c(c_1,T)(\|\chi_{xxx}\|^{1/6}_{L_2(I^T)} + 1) \\ &\leqslant \delta\|\chi_{xxx}\|_{L_2(I^T)} + c(1/\delta, c_1, T)\end{aligned}$$

with some constant $\delta > 0$.

The second term on the right-hand side of (73) is estimated with the help of the interpolation theorem and estimate (59) by

$$\begin{aligned}\|\chi_{xx}\|_{L_2(I^T)} &\leqslant \delta\|\chi_{xxx}\|_{L_2(I^T)} + c(1/\delta)\|\chi_x\|_{L_2(I^T)} \\ &\leqslant \delta\|\chi_{xxx}\|_{L_2(I^T)} + c(1/\delta, c_1, T), \quad \delta > 0\end{aligned}$$

Let us examine the third term on the right-hand side of (73). We have

$$\begin{aligned}\|\chi_x^2 u_x\|_{L_2(I^T)} &= \left(\int_0^T \int_I \chi_x^4 u_x^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\ &\leqslant \left(\sup_t \|u_x\|_{L_\infty(I)}\right)\left(\sup_t \|\chi_x\|_{L_2(I)}\right)\left(\int_0^T \|\chi_x\|^2_{L_\infty(I)}\,\mathrm{d}t\right)^{1/2} = J_2\end{aligned}$$

We apply now the interpolation inequalities

$$\|u_x\|_{L_\infty(I)} \leqslant c\|u_{xxx}\|^{1/4}_{L_2(I)}\|u_x\|^{3/4}_{L_2(I)} + c\|u_x\|_{L_2(I)} \tag{74}$$

and (66) together with estimate (59) to get

$$J_2 \leqslant cc_1 \sup_t (\|u_{xxx}\|_{L_2(I)}^{1/4}\|u_x\|_{L_2(I)}^{3/4} + \|u_x\|_{L_2(I)}) \left(\int_0^T (\|\chi_{xxxx}\|_{L_2(I)}^{1/3}\|\chi_x\|_{L_2(I)}^{5/3} + \|\chi_x\|_{L_2(I)}^2)\,\mathrm{d}t\right)^{1/2}$$

$$\leqslant c(c_1, T)\left(\sup_t \|u_{xxx}\|_{L_2(I)}^{1/4} + 1\right)\left(\int_0^T \|\chi_{xxxx}\|_{L_2(I)}^{1/3}\,\mathrm{d}t + 1\right)^{1/2}$$

$$\leqslant c(c_1, T)\left(\sup_t \|u_{xxx}\|_{L_2(I)}^{1/4}\|\chi_{xxxx}\|_{L_2(I^T)}^{1/6} + \sup_t \|u_{xxx}\|_{L_2(I)}^{1/4} + \chi_{xxxx}\|_{L_2(I^T)}^{1/6} + 1\right)$$

$$\leqslant \delta\|\chi_{xxxx}\|_{L_2(I^T)} + c(1/\delta, c_1, T)\left(\sup_t \|u_{xxx}\|_{L_2(I)} + 1\right)$$

Next, for the fourth term on the right-hand side of (73), applying interpolation inequality (68), estimate (59) and the Young inequality, we obtain the following estimate:

$$\|\chi_{xx}u_x\|_{L_2(I^T)} = \left(\int_0^T \int_I \chi_{xx}^2 u_x^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}$$

$$\leqslant \left(\sup_t \int_I u_x^2\,\mathrm{d}x\right)^{1/2}\left(\int_0^T \|\chi_{xx}\|_{L_\infty(I)}^2\,\mathrm{d}t\right)^{1/2}$$

$$\leqslant cc_1\left(\int_0^T (\|\chi_{xxxx}\|_{L_2(I)}\|\chi_x\|_{L_2(I)} + \|\chi_x\|_{L_2(I)}^2)\,\mathrm{d}t\right)^{1/2}$$

$$\leqslant c(c_1)\left(\int_0^T (\|\chi_{xxxx}\|_{L_2(I)} + 1)\,\mathrm{d}t\right)^{1/2}$$

$$\leqslant c(c_1, T)(\|\chi_{xxxx}\|_{L_2(I^T)}^{1/2} + 1)$$

$$\leqslant \delta\|\chi_{xxxx}\|_{L_2(I^T)} + c(1/\delta, c_1, T)$$

The fifth term on the right-hand side of (73) is estimated with the help of the interpolation theorem and (59) as follows:

$$\|\chi_x u_{xx}\|_{L_2(I^T)} = \left(\int_0^T \int_I \chi_x^2 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}$$

$$\leqslant \sup_t \|u_{xx}\|_{L_\infty(I)}\left(\int_0^T \int_I \chi_x^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}$$

$$\leqslant c(c_1, T)\sup_t (\|u_{xxx}\|_{L_2(I)} + \|u_x\|_{L_2(I)})$$

$$\leqslant c(c_1, T)\sup_t \|u_{xxx}\|_{L_2(I)} + c(c_1, T)$$

Finally, the sixth term on the right-hand side of (73) we bound by

$$\|u_{xxx}\|_{L_2(I^T)} \leqslant c(T)\sup_t \|u_{xxx}\|_{L_2(I)}$$

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Combining the above estimates in (73) we see that

$$\|\chi\|_{W_2^{4,1}(I^T)} \leqslant c(c_1)[\delta\|\chi_{xxxx}\|_{L_2(I^T)} + c(1/\delta, c_1, T)\|u_{xxx}\|_{L_\infty(0,T;L_2(I))} + c(1/\delta, c_1, T)] + c\|\chi_0\|_{H^2(I)} \tag{75}$$

Hence, choosing δ sufficiently small, we obtain the assertion. □

From Lemmas 4.1 and 4.2 we can conclude immediately the following *a priori* estimates

Lemma 4.3
Let the assumptions of Lemma 4.2 hold true. Then the solution u, χ of system (17) and (17) satisfies the estimates

$$\begin{aligned} \|u_{xxt}\|_{L_\infty(0,T;L_2(I))} + \|u_{xxx}\|_{L_\infty(0,T;L_2(I))} &\leqslant c_2 \\ \|\chi\|_{W_2^{4,1}(I^T)} &\leqslant c_2 \end{aligned} \tag{76}$$

with constant c_2 given by

$$c_2 = c(c_1, T)(1 + c_3) + c\|\chi_0\|_{H^2(I)} \tag{77}$$

and c_1, c_3 defined in (56) and (62).

5. PROOF OF THEOREM 1.1

In this section, we prove the existence of solutions to problem (17) and (17). To this end we apply the Leray–Schauder fixed point theorem which we recall here for reader's convenience.

Theorem 5.1 (see Reference [19, Chapter 4, Section 10, Theorem 10.1])
Let $\mathscr{B}$ be a Banach space and $\bar{\mathscr{C}}$ be the closure of a connected, bounded, open subset $\mathscr{C}$ of $\mathscr{B}$. Assume that $\Phi : \mathscr{B} \times [0, 1] \to \mathscr{B}$ is a map with the following properties:

(i) The map $\Phi : \bar{\mathscr{C}} \times [0, 1] \to \mathscr{B}$ is completely continuous.
(ii) For all $\tau \in [0, 1]$ the boundary of $\mathscr{C}$ does not contain fixed points of Φ.
(iii) $\Phi(\cdot, 0)$ has precisely one fixed point in $\mathscr{B}$.

Then $\Phi(\cdot, 1)$ has at least one fixed point in $\mathscr{B}$.

We construct the map

$$\chi = \Phi(\bar{\chi}, \tau) \tag{78}$$

with a parameter $\tau \in [0, 1]$ by means of the following system of problems:

$$\begin{aligned} u_{tt} - Au_{xx} &= \tau Bz'(\bar{\chi})\bar{\chi}_x + b \quad \text{in } I^T \\ u|_{t=0} = u_0,\ & u_t|_{t=0} = u_1 \quad \text{in } I \\ u &= 0 \quad \text{on } S^T \end{aligned} \tag{79}$$

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and

$$
\begin{aligned}
\chi_t + \Gamma\chi_{xxxx} = \tau\{&\psi'''(\bar\chi)\bar\chi_x^2 + \psi''(\bar\chi)\bar\chi_{xx} + z'''(\bar\chi)\bar\chi_x^2(Bu_x + Dz(\bar\chi) + E)\\
&+ z''(\bar\chi)\bar\chi_{xx}(Bu_x + Dz(\bar\chi) + E) + 2z''(\bar\chi)\bar\chi_x(Bu_{xx} + Dz'(\bar\chi)\bar\chi_x)\\
&+ z'(\bar\chi)(Bu_{xxx} + Dz''(\bar\chi)\bar\chi_x^2 + Dz'(\bar\chi)\bar\chi_{xx})\} \quad \text{in } I^T\\
\chi|_{t=0} = \chi_0 \quad &\text{in } I\\
\chi_x = 0, \quad \chi_{xxx} = 0 \quad &\text{on } S^T
\end{aligned}
\tag{80}
$$

Our goal is to show that the map Φ satisfies assumptions of Theorem 5.1. Firstly, we examine problem (79). To this end let us introduce the space

$$
\mathfrak{M}_1(I^T) = \{\chi : \chi \in L_\infty(0,T;W_\infty^1(I)) \cap L_2(0,T;H^3(I)),\ \chi_x|_{S^T} = 0\}
$$

The basic *a priori* estimates for (79) are given by

Lemma 5.1
Assume that z satisfies (20), $u_0 \in H^3(I) \cap H_0^1(I)$, $u_1 \in H^2(I) \cap H_0^1(I)$, $b \in L_1(0,T;H^2(I) \cap H_0^1(I))$, and $\bar\chi \in \mathfrak{M}_1(I^T)$. Then there exists a unique solution u of problem (79) such that

$$
u \in L_\infty(0,T;H^3(I) \cap H_0^1(I)) \cap W_\infty^1(0,T;H^2(I) \cap H_0^1(I)) \tag{81}
$$

satisfying estimate

$$
\|u_t\|_{L_\infty(0,T;H^2(I))} + \|u\|_{L_\infty(0,T;H^3(I))} \leqslant \varphi_1(\|\bar\chi\|_{\mathfrak{M}_1(I^T)}, d_1) \tag{82}
$$

where

$$
d_1 = \|b\|_{L_1(0,T;H^2(I))} + \|u_0\|_{H^3(I)} + \|u_1\|_{H^2(I)}
$$

and $\varphi_1(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing positive function.

Proof
An application of Lemma 2.1 yields (81) and the estimate

$$
\begin{aligned}
&\|u_t\|_{L_\infty(0,T;H^2(I))} + \|u\|_{L_\infty(0,T;H^3(I))}\\
&\quad \leqslant \|\tau Bz'(\bar\chi)\bar\chi_x\|_{L_1(0,T;H^2(I))} + \|b\|_{L_1(0,T;H^2(I))} + \|u_0\|_{H^3(I)} + \|u_1\|_{H^2(I)}
\end{aligned}
\tag{83}
$$

Recalling assumptions on z, the first term on the right-hand side of (83) is bounded by

$$
\begin{aligned}
&\|\tau Bz'(\bar\chi)\bar\chi_x\|_{L_1(0,T;H^2(I))}\\
&\quad \leqslant c\int_0^T \|z'(\bar\chi)\bar\chi_x\|_{L_2(I)}\,dt + c\int_0^T \|z'''(\bar\chi)\bar\chi_x^3 + 3z'(\bar\chi)\bar\chi_x\bar\chi_{xx} + z'(\bar\chi)\bar\chi_{xxx}\|_{L_2(I)}\,dt\\
&\quad \leqslant c(T)(\|\bar\chi_x\|_{L_\infty(I^T)} + \|\bar\chi_x\|_{L_\infty(I^T)}^3) + c(1 + \|\bar\chi_x\|_{L_\infty(I^T)})\|\bar\chi\|_{L_1(0,T;H^3(I))}
\end{aligned}
\tag{84}
$$

From (83) and (84) we conclude (82) what completes the proof. □

In the second step, we examine the continuity of the map

$$\Phi_1(\cdot,\tau):\mathfrak{M}_1\to L_\infty(0,T;H^3(I))\cap W^1_\infty(0,T;H^2(I)),\quad \tau=[0,1]$$

that gives a solution u of (79) for a given $\bar{\chi}$. Let u_1 and u_2 be two solutions of (79) corresponding to $\bar{\chi}_1\in\mathfrak{M}_1$ and $\bar{\chi}_2\in\mathfrak{M}_1$, respectively. Subtracting the corresponding equations and denoting

$$U=u_1-u_2,\quad \bar{H}=\bar{\chi}_1-\bar{\chi}_2$$

we have

$$\begin{aligned} U_{tt}-AU_{xx}&=\tau B[(z'(\bar{\chi}_1)-z'(\bar{\chi}_2))\bar{\chi}_{1x}+z'(\bar{\chi}_2)\bar{H}_x]\quad \text{in } I^T\\ U|_{t=0}&=0,\quad U_t|_{t=0}=0\quad \text{in } I\\ U&=0\quad \text{on } S^T \end{aligned}\tag{85}$$

The continuity of the map Φ_1 is asserted by

Lemma 5.2

Assume that z satisfies (20) and

$$\|\bar{\chi}_i\|_{\mathfrak{M}_1}\leqslant\bar{c}_1,\quad i=1,2\tag{86}$$

Then there exists an increasing positive function $\bar{\varphi}_1:\mathbb{R}_+\to\mathbb{R}_+$ such that the solution U of problem (85) satisfies

$$\|U_t\|_{L_\infty(0,T;H^2(I))}+\|U\|_{L_\infty(0,T;H^3(I))}\leqslant\bar{\varphi}_1(\bar{c}_1)\|\bar{H}\|_{\mathfrak{M}_1}\tag{87}$$

Proof

An application of Lemma 2.1 to problem (85) yields the estimate

$$\|U_t\|_{L_\infty(0,T;H^2(I))}+\|U\|_{L_\infty(0,T;H^3(I))}\leqslant\|\tau B[(z'(\bar{\chi}_1)-z'(\bar{\chi}_2))\bar{\chi}_{1x}+z'(\bar{\chi}_2)\bar{H}_x]\|_{L_1(0,T;H^2(I))}\tag{88}$$

Let $Z^{(i)}=z^{(i)}(\bar{\chi}_1)-z^{(i)}(\bar{\chi}_2)$, $i=0,1,2,3$, $z^{(i)}(\chi)=\partial^{(i)}_\chi z$. Then

$$\begin{aligned} [(z'(\bar{\chi}_1)-z'(\bar{\chi}_2))\bar{\chi}_{1x}+z'(\bar{\chi}_2)\bar{H}_x]_{,xx}&=Z^{(3)}\bar{\chi}^3_{1x}+z^{(3)}(\bar{\chi}_2)(\bar{\chi}_{1x}+\bar{\chi}_{2x})\bar{\chi}_{1x}\bar{H}_x+Z^{(2)}\bar{\chi}_{1xx}\bar{\chi}_{1x}\\ &\quad+z^{(2)}(\bar{\chi}_2)\bar{\chi}_{1x}\bar{H}_{xx}+2Z^{(2)}\bar{\chi}_{1x}\bar{\chi}_{1xx}+2z^{(2)}(\bar{\chi}_2)\bar{\chi}_{1xx}\bar{H}_x\\ &\quad+Z^{(1)}\bar{\chi}_{1xxx}+z^{(3)}(\bar{\chi}_2)\bar{\chi}^2_{2x}\bar{H}_x+z^{(2)}(\bar{\chi}_2)\bar{\chi}_{2xx}\bar{H}_x\\ &\quad+2z^{(2)}(\bar{\chi}_2)\bar{\chi}_{2x}\bar{H}_{xx}+z^{(1)}(\bar{\chi}_2)\bar{H}_{xxx}\equiv I \end{aligned}$$

Taking into account that $z^{(3)}$ is Lipschitz continuous and $\bar{\chi}_i,\bar{\chi}_{ix}\in L_\infty(I^T)$, $i=1,2$, we obtain

$$\begin{aligned} \|I\|_{L_1(0,T;L_2(\Omega))}&\leqslant\varphi(\|\bar{\chi}_1\|_{\mathfrak{M}_1},\|\bar{\chi}_2\|_{\mathfrak{M}_1})\left(\sum_{i=1}^{3}\|\partial^i_x\bar{H}\|_{L_1(0,T;L_2(I))}+\|\bar{H}\bar{\chi}_{1xx}\|_{L_1(0,T;L_2(I))}\right.\\ &\quad\left.+\|\bar{H}\bar{\chi}_{1xxx}\|_{L_1(0,T;L_2(I))}+\|\bar{H}_x\bar{\chi}_{1xx}\|_{L_1(0,T;L_2(I))}+\|\bar{H}_x\bar{\chi}_{2xx}\|_{L_1(0,T;L_2(I))}\right)\\ &\equiv K \end{aligned}$$

where φ is an increasing positive function.

In view of the other properties of $\mathfrak{M}_1(I^T)$ it follows that

$$K \leqslant \varphi(\|\bar{\chi}_1\|_{\mathfrak{M}_1}, \|\bar{\chi}_2\|_{\mathfrak{M}_1})\|\bar{H}\|_{\mathfrak{M}_1} \tag{89}$$

Clearly, we also have

$$\|(z'(\bar{\chi}_1) - z'(\bar{\chi}_2))\bar{\chi}_{1x} + z'(\bar{\chi}_2)\bar{H}_x\|_{L_1(0,T;L_2(I))} \leqslant \varphi(\|\bar{\chi}_1\|_{\mathfrak{M}_1}, \|\bar{\chi}_2\|_{\mathfrak{M}_1})\|\bar{H}\|_{\mathfrak{M}_1} \tag{90}$$

Combining (89) and (90) in (88) we conclude the assertion. □

Let us turn now to problem (80). We introduce the space

$$\mathfrak{M}_2(I^T) = L_4(0,T; W_4^1(I)) \cap L_2(0,T; H^2(I))$$

The basic *a priori* estimate for a solution of (80) is given by

Lemma 5.3
Assume ψ and z satisfy (19) and (20), $\chi_0 \in H^2(I)$ with $\chi_{0x} = 0$ on S. Moreover, let $\bar{\chi} \in \mathfrak{M}_1(I^T) \cap \mathfrak{M}_2(I^T)$.
Then there exists an increasing positive function $\varphi_2 : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that the solution χ of (80) satisfies

$$\|\chi\|_{W_2^{4,1}(I^T)} \leqslant \varphi_2(\|\bar{\chi}\|_{\mathfrak{M}_1(I^T)\cap\mathfrak{M}_2(I^T)}, d_1, d_2) \tag{91}$$

where constant d_1 is defined in (82) and

$$d_2 = \|\chi_0\|_{H^2(I)}$$

Proof
By virtue of Lemma 2.2 the solution of problem (80) satisfies

$$\begin{aligned}\|\chi\|_{W_2^{4,1}(I^T)} &\leqslant \varphi(\|\bar{\chi}\|_{L_\infty(I^T)})\Bigg[\int_{I^T}(\bar{\chi}_x^4 + \bar{\chi}_{xx}^2 + \bar{\chi}_x^4 u_x^2 \\ &\quad + \bar{\chi}_{xx}^2 u_x^2 + \bar{\chi}_x^2 u_{xx}^2 + \bar{\chi}_x^2 + u_{xxx}^2 + 1)\,\mathrm{d}x\,\mathrm{d}t\Bigg]^{1/2} + \|\chi_0\|_{H^2(I)} \\ &\leqslant \varphi(\|\bar{\chi}\|_{L_\infty(I^T)})(\|\bar{\chi}_x\|_{L_4(I^T)}^2 + \|\bar{\chi}_x\|_{L_2(I^T)} + \|\bar{\chi}_{xx}\|_{L_2(I^T)} + 1) \\ &\quad \times \|u\|_{L_\infty(0,T;H^3(I))} + \|\chi_0\|_{H^2(I)}\end{aligned} \tag{92}$$

where φ is an increasing positive function. From (92) and (82) we obtain (91). This concludes the proof. □

Let

$$\mathfrak{M}(I^T) := \mathfrak{M}_1(I^T) \cap \mathfrak{M}_2(I^T) = \mathfrak{M}_1(I^T)$$

We examine the continuity of the map

$$\Phi_2(\cdot,\cdot,\tau) : \mathfrak{M}(I^T) \times L_\infty(0,T; H^3(I)) \cap W_\infty^1(0,T; H^2(I)) \to W_2^{4,1}(I^T), \quad \tau \in [0,1]$$

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that gives a solution χ of (80) for a given $\bar{\chi}$ and $u=\Phi_1(\bar{\chi},\tau)$. Let χ_1, χ_2 be the solutions of (80) corresponding, respectively, to

$$(\bar{\chi}_1,u_1),\quad (\bar{\chi}_2,u_2)\in\mathfrak{M}(I^T)\times L_\infty(0,T;H^3(I))\cap W^1_\infty(0,T;H^2(I))$$

where $u_1=\Phi_1(\bar{\chi}_1,\tau)$, $u_2=\Phi_1(\bar{\chi}_2,\tau)$.

Subtracting the corresponding equations and denoting

$$H=\chi_1-\chi_1,\quad \bar{H}=\bar{\chi}_1-\bar{\chi}_2,\quad U=u_1-u_2$$

we obtain the following problem:

$$\begin{aligned}
H_t+\Gamma H_{xxxx}=\tau\{&(\psi'''(\bar{\chi}_1)-\psi'''(\bar{\chi}_2))\bar{\chi}^2_{1x}+(\psi''(\bar{\chi}_1)-\psi''(\bar{\chi}_2))\bar{\chi}_{1xx}\\
&+(z'''(\bar{\chi}_1)-z'''(\bar{\chi}_2))[B\bar{\chi}^2_{1x}u_{1x}+Dz(\bar{\chi}_1)\bar{\chi}^2_{1x}+E\bar{\chi}^2_{1x}]\\
&+(z''(\bar{\chi}_1)-z''(\bar{\chi}_2))[B\bar{\chi}_{1xx}u_{1x}+Dz(\bar{\chi}_1)\bar{\chi}_{1xx}+E\bar{\chi}_{1xx}+2B\bar{\chi}_{1x}u_{1xx}\\
&+3Dz'(\bar{\chi}_1)\bar{\chi}^2_{1x}]\\
&+(z'(\bar{\chi}_1)-z'(\bar{\chi}_2))[3Dz''(\bar{\chi}_2)\bar{\chi}^2_{1x}+D(z'(\bar{\chi}_1)+z'(\bar{\chi}_2))\bar{\chi}_{1xx}+Bu_{1xxx}]\\
&+(z(\bar{\chi}_1)-z(\bar{\chi}_2))[Dz'''(\bar{\chi}_2)\bar{\chi}^2_{1x}+Dz''(\bar{\chi}_2)\bar{\chi}_{1xx}]\\
&+\psi'''(\bar{\chi}_2)(\bar{\chi}_{1x}+\bar{\chi}_{2x})\bar{H}_x+\psi''(\bar{\chi}_2)\bar{H}_{xx}\\
&+[(Bz'''(\bar{\chi}_2)u_{1x}+Dz'''(\bar{\chi}_2)z(\bar{\chi}_2)+Ez'''(\bar{\chi}_2)\\
&+3Dz''(\bar{\chi}_2)z'(\bar{\chi}_2))(\bar{\chi}_{1x}+\bar{\chi}_{2x})+2Bz''(\bar{\chi}_2)u_{1xx}]\bar{H}_x\\
&+[Bz''(\bar{\chi}_2)u_{1x}+Dz''(\bar{\chi}_2)z(\bar{\chi}_2)+Ez''(\bar{\chi}_2)+D(z'(\bar{\chi}_2))^2]\bar{H}_{xx}\\
&+[Bz'''(\bar{\chi}_2)\bar{\chi}^2_{2x}+Bz''(\bar{\chi}_2)\bar{\chi}_{2xx}]U_x+2Bz''(\bar{\chi}_2)\bar{\chi}_{2x}U_{xx}\\
&+Bz'(\bar{\chi}_2)U_{xxx}\}\equiv\bar{R}\quad\text{in } I^T\\
H|_{t=0}=0\quad&\text{in } I\\
H_x=0,\quad H_{xxx}&=0\quad\text{on } S^T
\end{aligned}\tag{93}$$

Lemma 5.4

Assume that ψ and z satisfy (19) and (20), $u_0\in H^3(I)\cap H^1_0(I)$, $u_1\in H^2(I)\cap H^1_0(I)$, $b\in L_1(0,T;H^2(I)\cap H^1_0(I))$. Moreover, let

$$\|\bar{\chi}_i\|_{\mathfrak{M}(I^T)}\leqslant\bar{c}_1,\quad i=1,2\tag{94}$$

Then the solution H of problem (93) satisfies

$$\|H\|_{W^{4,1}_2(I^T)}\leqslant\bar{\varphi}_2(\bar{c}_1,d_1)(\|\bar{H}\|_{\mathfrak{M}(I^T)}+\|U\|_{L_\infty(0,T;H^3(I))})\tag{95}$$

with an increasing positive function $\bar{\varphi}_2$ and constant d_1 defined in (82).

Proof

By virtue of assumptions on ψ, z and estimates (94) and (82) the right-hand side of $(93)_1$ is bounded by

$$\|\bar{R}\|_{L_2(\Omega^T)} \leq \bar{\varphi}_2(\bar{c}_1, d_1)(\|\bar{H}\|_{\mathfrak{M}(I^T)} + \|U\|_{L_\infty(0,T;H^3(I))})$$

Hence, by Lemma 2.2 we conclude the assertion. □

Let us consider now the composed map

$$\Phi(\cdot,\tau) = \Phi_2(\cdot, \Phi_1(\cdot,\tau),\tau) : \mathfrak{M}(I^T) \to W_2^{4,1}(I^T) \subset \mathfrak{M}(I^T), \quad \tau \in [0,1]$$

that gives a solution χ of system (79) and (80) for a given $\bar{\chi}$. We prove

Lemma 5.5

Let assumptions (19) and (20) for ψ and z be satisfied, and

$$\chi_0 \in H^2(I) \text{ with } \chi_{0x} = 0 \text{ on } S, \quad u_0 \in H^3(I) \cap H_0^1(I)$$

$$u_1 \in H^2(I) \cap H_0^1(I), \quad b \in L_2(0,T; H^2(I) \cap H_0^1(I))$$

Then the map $\Phi(\cdot,1)$ has at least one fixed point in $\mathfrak{M}$.

Proof

We show that assumptions (i)–(iii) of the Leray–Schauder fixed point theorem are satisfied.

Firstly, by Lemmas 5.4 and 5.2, the map $\Phi(\cdot,\tau)$, for any fixed $\tau \in [0,1]$, is continuous with respect to τ. Secondly, since the imbedding $W_2^{4,1}(I^T) \subset \mathfrak{M}(I^T)$ is compact (see Reference [18, Chapter 6, Section 26]), the map $\Phi(\cdot,\tau)$ is compact, i.e. it transforms bounded sets in $\mathfrak{M}$ into compact sets. Moreover, it is straightforward to check that $\Phi(\bar{\chi},\cdot)$ is equicontinuous with respect to $\tau \in [0,1]$ uniformly with respect to $\bar{\chi} \in \mathfrak{M}$.

This shows that $\Phi : \mathfrak{M}(I^T) \times [0,1] \to \mathfrak{M}(I^T)$ is completely continuous, i.e. (i) is satisfied. Condition (ii) is also satisfied by virtue of *a priori* estimates (76) for a fixed point of the map $\Phi(\cdot,1)$. Finally, for $\tau = 0$ problem (79) and (80) has the unique solution, so that condition (iii) is satisfied as well. Hence, $\Phi(\cdot,1)$ has at least one fixed point in $\mathfrak{M}$. □

Clearly, the fixed point of the map $\Phi(\cdot,1)$ is equivalent to a solution (u,χ) of problem (17) and (17). Moreover, *a priori* estimates (59) and (76) imply (23). This concludes the proof. □

6. PROOF OF THEOREM 1.2

Let (u_1,χ_1) and (u_2,χ_2) be two solutions of problem (17) and (17) corresponding to the same data. Subtracting the corresponding equations and denoting

$$U = u_1 - u_2, \quad H = \chi_1 - \chi_2$$

we obtain the following system for (U,H):

$$\begin{aligned} &U_{tt} - AU_{xx} = B[(z'(\chi_1) - z'(\chi_2))\chi_{1x} + z'(\chi_2)H_x] \quad \text{in } I^T \\ &U|_{t=0} = 0, \quad U_t|_{t=0} = 0 \quad \text{in } I \\ &U = 0 \quad \text{on } S^T \end{aligned} \tag{96}$$

$$
\begin{aligned}
H_t + \Gamma H_{xxxx} &= [(\psi'(\chi_1) - \psi'(\chi_2)) + (z'(\chi_1) - z'(\chi_2))(Bu_{1x} + Dz(\chi_1) + E) \\
&\quad + z'(\chi_2)(BU_x + D(z(\chi_1) - z(\chi_2)))]_{xx} \quad \text{in } I^T \\
H|_{t=0} &= 0 \quad \text{in } I \\
H_x = 0, &\quad H_{xxx} = 0 \quad \text{on } S^T
\end{aligned} \tag{97}
$$

We note that according to Equation (15), boundary conditions $U_{xx}=0$ (see (14)), $H_x=0$ and $H_{xxx}=0$ on S^T imply that

$$
\begin{aligned}
&[(\psi'(\chi_1) - \psi'(\chi_2)) + (z'(\chi_1) - z'(\chi_2))(Bu_{1x} + Dz(\chi_1) + E) \\
&\quad + z'(\chi_2)(BU_x + D(z(\chi_1) - z(\chi_2)))]_x = 0 \quad \text{on } S^T
\end{aligned} \tag{98}
$$

Multiplying $(96)_1$ by U_t, integrating over I and by parts, in view of assumption on z, we obtain

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_I U_t^2\,\mathrm{d}x + A\int_I U_x^2\,\mathrm{d}x\right) \leqslant c\int_I (|H||\chi_{1x}| + |H_x|)|U_t|\,\mathrm{d}x
$$

Hence,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|U_t\|_{L_2(I)}^2 + \|U_x\|_{L_2(I)}^2) \\
&\quad \leqslant c(\|\chi_{1x}\|_{L_\infty(I)})\|H\|_{H^1(I)}\|U_t\|_{L_2(I)} \\
&\quad \leqslant \delta_1\|H_{xx}\|_{L_2(I)}^2 + c(1/\delta_1)\|H\|_{L_2(I)}^2 + c\|U_t\|_{L_2(I)}^2, \quad \delta_1 > 0
\end{aligned} \tag{99}
$$

where we used (24) and the interpolation inequality.

In turn, multiplying $(97)_1$ by H, integrating twice by parts with respect to x and using boundary conditions $(97)_3$ and (98) on S^T, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I H^2\,\mathrm{d}x + \Gamma\int_I H_{xx}^2\,\mathrm{d}x \\
&\quad = \int_I [(\psi'(\chi_1) - \psi'(\chi_2)) + (z'(\chi_1) - z'(\chi_2))(Bu_{1x} + Dz(\chi_1) + E) \\
&\qquad + z'(\chi_2)(BU_x + D(z(\chi_1) - z(\chi_2)))]H_{xx}\,\mathrm{d}x \\
&\quad \leqslant c(\|\chi_1\|_{L_\infty(I)}, \|u_{1x}\|_{L_\infty(I)})\int_I (|H| + |U_x|)|H_{xx}|\,\mathrm{d}x \\
&\quad \leqslant \delta_2\|H_{xx}\|_{L_2(I)}^2 + c(1/\delta_2, \|\chi_1\|_{L_\infty(I)}, \|u_{1x}\|_{L_\infty(I)}) \\
&\qquad \times(\|H\|_{L_2(I)}^2 + \|U_x\|_{L_2(I)}^2), \quad \delta_2 > 0
\end{aligned} \tag{100}
$$

where we applied assumptions on ψ and z, and the Young inequality.

Consequently, choosing δ_2 small, in view of (24) we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|H\|^2_{L_2(I)} + \|H_{xx}\|^2_{L_2(I)} \leqslant c(\|H\|^2_{L_2(I)} + \|U_x\|^2_{L_2(I)}) \tag{101}$$

Finally, adding (99) and (101), and choosing δ_1 sufficiently small, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|H\|^2_{L_2(I)} + \|U_t\|^2_{L_2(I)} + \|U_x\|^2_{L_2(I)}) \leqslant c(\|H\|^2_{L_2(I)} + \|U_t\|^2_{L_2(I)} + \|U_x\|^2_{L_2(I)}) \tag{102}$$

Hence, by Gronwall's inequality, $H=0$ and $U=0$ in I^T. Thereby the proof is completed. □

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Classical solvability of 1-D Cahn–Hilliard equation coupled with elasticity

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SUMMARY

In this paper, we prove the classical solvability of a nonlinear 1-D system of hyperbolic–parabolic type arising as a model of phase separation in deformable binary alloys. The system is governed by the nonstationary elasticity equation coupled with the Cahn–Hilliard equation. The existence proof is based on the application of the Leray–Schauder fixed point theorem and standard energy methods. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: phase separation; Cahn–Hilliard equation; nonstationary elasticity; Leray–Schauder theorem; classical solvability

1. INTRODUCTION

This paper is concerned with the classical solvability of a one-dimensional (1-D) initial-boundary-value problem for hyperbolic–parabolic system arising as a model of phase separation in deformable binary alloys. Such alloys are nonsimple materials characterized by an evolving microstructure driven by thermomechanical effects. In view of exceptional properties they find many applications in modern technologies (see References [1,2]). The system under consideration combines the linear momentum balance, represented by the nonstationary elasticity equation, coupled with the mass balance described by the Cahn–Hilliard equation. Such problem is a special simple version of 3-D model of phase separation in elastic solids

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proposed by Gurtin [3]. It refers to the 1-D setting and a homogeneous situation in which all material parameters except the eigenstrains are assumed to be the same for both phases.

In recent years, phase separation process driven by thermomechanical effects attracted a lot of modelling, mathematical and numerical interest, see e.g. References [1–13].

The mathematical results in the above-mentioned papers were concerned with the existence of weak solutions for various variants of phase separation model. In most papers, a quasi-stationary approximation of the elasticity system was assumed, i.e. the inertial term in the linear momentum balance was neglected. Such assumption is justified provided the time scale of mechanical effects is much faster than that of diffusion.

The phase separation model with nonstationary elasticity was considered in References [10,13], and is studied in the present paper as well. From the mathematical point of view the nonstationary elasticity leads to a hyperbolic system instead of the elliptic one and thereby makes the analysis more complicated. Apart from the mathematical interest on its own the obtained results can be applied to phase separation processes with comparable mechanical and diffusive time scales. We underline that the previous studies on the Cahn–Hilliard system coupled with elasticity were concerned with the existence and properties of weak solutions. The classical solvability of the single Cahn–Hilliard equation in 1-D and 3-D case was studied by Elliott and Zheng [14].

In the present paper, we extend the classical solvability result to the 1-D Cahn–Hilliard equation coupled with nonstationary elasticity.

The existence proof is based on the Leray–Schauder fixed point theorem and the classical existence results for linear hyperbolic and parabolic problems. Moreover, we use standard energy estimates and Sobolev's imbeddings which in 1-D case provide sufficiently high regularity of solutions. We point out that in three space dimensions, the problem shows features that cannot be found in the 1-D setting. In particular, the equations are then coupled not only through the right-hand sides but also through the boundary conditions. For that reason the arguments used in the present paper do not extend to such a case. We shall address the 3-D problem in a separate paper.

We describe now the problem of our concern. Let us consider an elastic bar made from a binary $a-b$ alloy. Assume that the bar occupies the interval $[0,L]$ in the reference configuration. If the bar is exposed to thermo-mechanical loads, e.g. instantaneous cooling below a critical temperature, the phase separation process develops reflected by the motion of phase interfaces and drastic changes of the morphology in microscale.

Like the Landau theory of phase transitions Gurtin's model uses an order parameter χ to characterize the material phase. In case of a binary $a-b$ alloy the order parameter is related to the volumetric fraction of one of the two phases, characterized by different crystalline structures of the components. Here we shall identify $\chi=-1$ with the phase a and $\chi=1$ with the phase b.

The bar is assumed to be deformable and its longitudinal motion be described by a mapping

$$y(x,t)=x+u(x,t)$$

where x denotes position, t the time, $u(x,t)$ the displacement and $y(x,t)$ the placement.

In addition to χ and u, the independent variable in the model is the chemical potential difference between the components μ (shortly referred to as chemical potential).

The model expressed in terms of u, χ and μ, with some prescribed initial and boundary conditions, has the form

$$u_{tt}-(W_{,\varepsilon}(\varepsilon,\chi))_x=b \quad \text{in } I^T=I\times(0,T)$$
$$u|_{t=0}=u_0, \quad u_t|_{t=0}=u_1 \quad \text{in } I \tag{1}$$
$$u=0 \quad \text{on } S^T=S\times(0,T)$$

$$\chi_t-M\mu_{xx}=0 \quad \text{in } I^T$$
$$\chi|_{t=0}=\chi_0 \quad \text{in } I \tag{2}$$
$$\mu_x=0 \quad \text{on } S^T$$

$$\mu=-\Gamma\chi_{xx}+\psi'(\chi)+W_{,\chi}(\varepsilon,\chi) \quad \text{in } I^T$$
$$\chi_x=0 \quad \text{on } S^T \tag{3}$$

Here $I=(0,L)$ is the interval with the boundary $S=\{0\}\cup\{L\}$, $T>0$ is an arbitrary fixed time, $u:I^T\to\mathbb{R}$ is the displacement, $\chi:I^T\to\mathbb{R}$ is the order parameter, $\mu:I^T\to\mathbb{R}$ is the chemical potential, and $\varepsilon=u_x$ is the strain. Further, $W(\varepsilon,\chi)$ is the elastic energy given by

$$W(\varepsilon,\chi)=\tfrac{1}{2}A(\varepsilon-\bar{\varepsilon}(\chi))^2 \tag{4}$$

where $A>0$ denotes the elasticity constant assumed to be the same for both phases. The quantity $\bar{\varepsilon}(\chi)$ denotes the eigenstrain defined by

$$\bar{\varepsilon}(\chi)=(1-z(\chi))\bar{\varepsilon}_a+z(\chi)\bar{\varepsilon}_b=\bar{\varepsilon}_a+z(\chi)\bar{\bar{\varepsilon}} \tag{5}$$

with $\bar{\varepsilon}_a$ and $\bar{\varepsilon}_b$ denoting constant eigenstrains of phases a and b, $\bar{\bar{\varepsilon}}=\bar{\varepsilon}_b-\bar{\varepsilon}_a$, and $z(\chi)$ being a sufficiently smooth interpolation function with values in the interval $[0,1]$, satisfying

$$z(\chi)=0 \ \text{for } \chi\leqslant-1 \quad \text{and} \quad z(\chi)=1 \ \text{for } \chi\geqslant 1 \tag{6}$$

Moreover, $\psi(\chi)$ is a double-well potential with two minima at $\chi=-1$ and 1.

Generally, ψ is a polynomial of even degree $2p+2$, $p\geqslant 1$, with strictly positive dominant coefficient. A well-known example is the polynomial of fourth order

$$\psi(\chi)=\tfrac{1}{2}(1-\chi^2)^2 \tag{7}$$

The constants $M>0$ and $\Gamma>0$ represent the mobility and the gradient energy coefficients. We assume them to be the same for both phases.

Finally, b represents the body force and u_0, u_1, χ_0 are initial conditions for the displacement, the velocity and the order parameter.

We point out that in general the coefficients A, M and Γ can be different in each phase. Then their dependence on the order parameter is usually accounted for like in (5). Since such dependences introduce too many complexities and related mathematical difficulties (especially the case $A(\chi)$) their investigation requires separate studies.

The free energy density underlying system (1)–(1) has the Landau–Ginzburg–Cahn–Hilliard form accounting for elastic effects

$$f(\varepsilon,\chi,\chi_x)=W(\varepsilon,\chi)+\psi(\chi)+\tfrac{1}{2}\Gamma\chi_x^2 \tag{8}$$

The three terms on the right-hand side of (8) represent, respectively, the elastic energy, the exchange (separation) energy and the interfacial energy. We note that for W given by (4)–(6) and $\psi(\chi)\geqslant -c$, free energy (8) satisfies the structural bound

$$f(\varepsilon,\chi,\chi_x)\geqslant c_f(\varepsilon^2+\chi_x^2)-c_f' \tag{9}$$

with some constants $c_f>0$ and $c_f'\geqslant 0$. This bound plays the essential role in deriving energy estimates for (1)–(1).

Later on for simplicity we assume that $M=1$. Then, in view of (4) and (5), problem (1)–(1) reduces to the form

$$\begin{aligned} &u_{tt}-Au_{xx}=Bz'(\chi)\chi_x+b \quad \text{in } I^T\\ &u|_{t=0}=u_0,\quad u_t|_{t=0}=u_1 \quad \text{in } I\\ &u=0 \quad \text{on } S^T \end{aligned} \tag{10}$$

$$\begin{aligned} &\chi_t-\mu_{xx}=0 \quad \text{in } I^T\\ &\chi|_{t=0}=\chi_0 \quad \text{in } I\\ &\mu_x=0 \quad \text{on } S^T \end{aligned} \tag{11}$$

$$\begin{aligned} &\mu=-\Gamma\chi_{xx}+\psi'(\chi)+z'(\chi)(Bu_x+Dz(\chi)+E) \quad \text{in } I^T\\ &\chi_x=0 \quad \text{on } S^T \end{aligned} \tag{12}$$

where B,D,E are constants given by

$$B=-A\bar{\varepsilon},\quad D=-B\bar{\bar{\varepsilon}},\quad E=-B\bar{\varepsilon}_a,\quad \bar{\bar{\varepsilon}}=\bar{\varepsilon}_b-\bar{\varepsilon}_a$$

For mathematical analysis it is convenient to transform (10)–(10) into the Cahn–Hilliard form in which the variable μ does not appear explicitly, that is the problem is expressed in terms of u and χ. The first boundary condition for χ is given by $(12)_2$. To determine the second one let us assume that

$$b=0 \quad \text{on } S^T \tag{13}$$

Then, in view of Equation $(10)_1$ and boundary conditions $(10)_3$, $(12)_2$, it follows that

$$u_{xx}=0 \quad \text{on } S^T \tag{14}$$

Further, making use of the equality

$$\mu_x=-\Gamma\chi_{xxx}+\psi''(\chi)\chi_x+z''(\chi)\chi_x(Bu_x+Dz(\chi)+E)+z'(\chi)(Bu_{xx}+Dz'(\chi)\chi_x) \tag{15}$$

we see that conditions $(11)_3$, $(12)_2$ and (14) imply

$$\chi_{xxx}=0 \quad \text{on } S^T \tag{16}$$

Now, inserting μ from $(12)_1$ into $(11)_1$ and taking into account conditions (14) and (16), we convert system (10)–(10) into the following equivalent form:

$$\begin{aligned} &u_{tt}-Au_{xx}=Bz'(\chi)\chi_x+b \quad \text{in } I^T \\ &u|_{t=0}=u_0, \quad u_t|_{t=0}=u_1 \quad \text{in } I \\ &u=0 \quad \text{on } S^T \end{aligned} \tag{17}$$

$$\begin{aligned} \chi_t+\Gamma\chi_{xxxx}&=\psi'''(\chi)\chi_x^2+\psi''(\chi)\chi_{xx}+z'''(\chi)\chi_x^2(Bu_x+Dz(\chi)+E) \\ &\quad +z''(\chi)\chi_{xx}(Bu_x+Dz(\chi)+E)+2z''(\chi)\chi_x(Bu_{xx}+Dz'(\chi)\chi_x) \\ &\quad +z'(\chi)(Bu_{xxx}+Dz''(\chi)\chi_x^2+Dz'(\chi)\chi_{xx}) \\ &\equiv R \quad \text{in } I^T \\ \chi|_{t=0}&=\chi_0 \quad \text{in } I \\ \chi_x&=0, \quad \chi_{xxx}=0 \quad \text{on } S^T \end{aligned} \tag{18}$$

The main results of the paper are stated in the following existence and uniqueness theorems.

Theorem 1.1 (*Existence*)
Assume that

$$\psi:\mathbb{R}\to\mathbb{R} \text{ is a polynomial of even degree } 2p+2,\ p\geqslant 1 \text{ with strictly positive dominant coefficient} \tag{19}$$

$$z:\mathbb{R}\to[0,1] \text{ is of class } C^3 \text{with } z''' \text{ Lipschitz continuous, satisfying (6)} \tag{20}$$

$$\text{coefficients } A>0,\ \Gamma>0,\ B,D,E \text{ are constant} \tag{21}$$

$$\begin{aligned} &u_0\in H^3(I)\cap H_0^1(I), \quad u_1\in H^2(I)\cap H_0^1(I) \\ &\chi_0\in H^2(I) \quad \text{with } \chi_{0x}=0 \text{ on } S \\ &b\in L_1(0,T;H^2(I)\cap H_0^1(I)) \end{aligned} \tag{22}$$

Then for any $T>0$ there exists a solution (u,χ) of problem (17) and (17) such that

$$\begin{aligned} &u\in L_\infty(0,T;H^3(I)\cap H_0^1(I))\cap W_\infty^1(0,T;H^2(I)\cap H_0^1(I)) \\ &\chi\in W_2^{4,1}(I^T) \end{aligned}$$

satisfying *a priori* estimates

$$\begin{aligned} \|u\|_{L_\infty(0,T;H^1(I))} + \|u_t\|_{L_\infty(0,T;L_2(I))} + \|\chi\|_{L_\infty(0,T;H^1(I))\cap L_\infty(I^T)} &\leqslant c_1 \\ \|u\|_{L_\infty(0,T;H^3(I))} + \|u_t\|_{L_\infty(0,T;H^2(I))} + \|\chi\|_{W_2^{4,1}(I^T)} &\leqslant c_2 \end{aligned} \tag{23}$$

with positive constants c_1, c_2 given by

$$c_1 = c\left(c_0, c_f, M, L, \int_I \chi_0(x)\,\mathrm{d}x\right)$$

$$c_2 = c(c_1, T)(1 + \|b\|_{L_1(0,T;H^2(I))} + \|u_0\|_{H^3(I)} + \|u_1\|_{H^2(I)}) + \|\chi_0\|_{H^2(I)}$$

and

$$c_0 = \left(\frac{1}{2}\|u_1\|^2_{L_2(I)} + \|W(u_{0x},\chi_0)\|_{L_1(I)} + \|\psi(\chi_0)\|_{L_1(I)}\frac{\Gamma}{2}\|\chi_{0x}\|^2_{L_2(I)} + \|b\|^2_{L_1(0,T;L_2(I))} + c_f'\right)^{1/2}$$

where c_f, c_f' are defined in (9).

Remark 1.1
In view of $L_\infty(I^T)$-norm estimate $(23)_1$ on χ it follows that with an appropriate choice of the data the values of χ can remain within the closed interval $[-1,1]$ for $t \in [0,T]$.

Remark 1.2
Estimates (23) imply that the chemical potential μ given by (10) satisfies $\mu \in L_2(0,T;H^2(I))$ and

$$\|\mu\|_{L_2(0,T;H^2(I))} \leqslant c_2$$

Theorem 1.2 (*Uniqueness*)
Let assumptions (19) and (20) hold true, and (u,χ) be a solution to problem (17) and (17) such that

$$u \in L_\infty(0,T;W^1_\infty(I)), \quad \chi \in L_\infty(0,T;W^1_\infty(I))$$

with

$$\|u\|_{L_\infty(0,T;W^1_\infty(I))} + \|\chi\|_{L_\infty(0,T;W^1_\infty(I))} \leqslant c \tag{24}$$

Then the solution (u,χ) is unique.

The paper is organized as follows. In Section 3, we derive energy estimates. The crucial *a priori* estimates are established in Section 4. The proofs of Theorems 1.1 and 1.2 are presented in Sections 5 and 6.

Throughout the paper, we use the following notations: $I=(0,L) \subset \mathbb{R}$, $I^t = I \times (0,t)$, $t \in [0,T]$; $\chi_t, \chi_x, \chi_{xx}, \ldots,$ denote, respectively, time derivative $D_t\chi$ and space derivatives $D_x^j\chi$, $j=1,2,\ldots$;

$$F_{,\varepsilon}(\varepsilon,\chi) = \frac{\partial F(\varepsilon,\chi)}{\partial \varepsilon}, \quad F_{,\chi} = \frac{\partial F(\varepsilon,\chi)}{\partial \chi}, \quad \psi'(\chi) = \frac{\mathrm{d}\psi(\chi)}{\mathrm{d}\chi}$$

We use the standard Sobolev spaces notation as in Reference [15]. In particular, $W_2^{4,1}(I^T)$ denotes the Sobolev space with the norm

$$\|\chi\|_{W_2^{4,1}(I^T)} = \sum_{j=0}^{4} \|D_x^j \chi\|_{L_2(I^T)} + \|D_t \chi\|_{L_2(I^T)}$$

For simplicity we write

$$H^l(I) = W_2^l(I), \quad l \in \mathbb{N}$$

Further, $W_p^1(0,T;H^l(I))$, $l \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$, denotes the space with the norm

$$\|\chi\|_{W_p^1(0,T;H^l(I))} = \|\chi\|_{L_p(0,T;H^l(I))} + \|D_t\chi\|_{L_p(0,T;H^l(I))}$$

By c we denote a generic positive constant different in various instances; whenever it is of interest its dependence on parameters is indicated. Moreover, by δ we denote a generic, sufficiently small, positive constant.

2. AUXILIARY RESULTS

In this section, we present some auxiliary existence results for linear hyperbolic and parabolic problems that will be used in the subsequent sections. First let us consider the following problem:

$$\begin{aligned} &u_{tt} - u_{xx} = f \quad \text{in } I^T \\ &u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1 \quad \text{in } I \\ &u = 0 \quad \text{on } S^T \end{aligned} \tag{25}$$

where u_0, u_1 are given initial data satisfying compatibility conditions

$$u_0|_S = 0, \quad u_1|_S = 0 \tag{26}$$

and the right-hand side f is such that

$$f|_{S^T} = 0 \tag{27}$$

The following lemma represents a particular version of the abstract existence result for evolution equations of the second order in time that can be found e.g. in Reference [16, Volume I, Chapter 3] and Reference [17, Chapter II, Theorem 4.1, 4.2].

Lemma 2.1
Assume that

$$f \in L_1(0,T;H^2(I) \cap H_0^1(I)), \quad u_0 \in H^3(I) \cap H_0^1(I), \quad u_1 \in H^2(I) \cap H_0^1(I) \tag{28}$$

Then problem (25)–(27) has a unique weak solution u such that

$$u \in L_\infty(0,T;H^3(I) \cap H_0^1(I)), \quad u_t \in L_\infty(0,T;H^2(I) \cap H_0^1(I))$$

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satisfying estimate

$$\begin{aligned}&\|u_t\|_{L_\infty(0,T;L_2(I))}+\|u_x\|_{L_\infty(0,T;L_2(I))}\\&\quad+\|u_{xxt}\|_{L_\infty(0,T;L_2(I))}+\|u_{xxx}\|_{L_\infty(0,T;L_2(I))}\leqslant c\end{aligned}\tag{29}$$

with constant c given by

$$c=\|f\|_{L_1(0,T;L_2(I))}+\|f_{xx}\|_{L_1(0,T;L_2(I))}+\|u_{0x}\|_{L_2(I)}+\|u_{0xxx}\|_{L_2(I)}+\|u_1\|_{L_2(I)}+\|u_{1xx}\|_{L_2(I)}\tag{30}$$

Proof

We use the Faedo–Galerkin method. Let $H=L_2(I)$, $V=H_0^1(I)$ and $a(\cdot,\cdot)$ be a bilinear form on V given by

$$a(u,v)=(u_x,v_x)$$

where $(\cdot,\cdot)$ and $\|\cdot\|$ denote the scalar product and the norm in $L_2(I)$.

Further, assume that the basis $\{w_j\}_{j\in\mathbb{N}}$ of V consists of the eigenfunctions for

$$\begin{aligned}-w_{jxx}&=\lambda_j w_j\quad\text{in } I\\ w_j&=0\quad\text{on } S,\ j\in\mathbb{N}\end{aligned}\tag{31}$$

Such family $w_j\in C_0^\infty(I)$ is orthonormal in H and orthogonal for $a(\cdot,\cdot)$ in V:

$$\begin{aligned}(w_j,w_k)&=\delta_{jk}\\ a(w_j,w_k)&=\lambda_j\delta_{jk}\quad\forall j,k\end{aligned}\tag{32}$$

Let V_m denote the finite-dimensional subspace of V spanned by $w_1,\dots,w_m$.

The approximate solution to problem (25)–(27) is defined by

$$u^m(t)=\sum_{i=1}^m g_i^m(t)w_i\tag{33}$$

with $g_j^m(t)$ being determined so that

$$\begin{aligned}&(u^m_{tt},w_j)+(u^m_{xxtt},w_{jxx})+a(u^m,w_j)+a(u^m_{xx},w_{jxx})\\&\quad=(f,w_j)+(f_{xx},w_{jxx}),\quad j=1,\dots,m\\&u^m(0)=u_0^m,\quad u_t^m(0)=u_1^m\end{aligned}\tag{34}$$

where u_0^m (resp. u_1^m) is the projection in $H^3(I)\cap V$ (resp. $H^2(I)\cap V$) of u_0 (resp. u_1) onto V_m, satisfying

$$\begin{aligned}&u_0^m\to u_0\ \text{strongly in } H^3(I)\\&u_1^m\to u_1\ \text{strongly in } H^2(I)\ \text{for } m\to\infty\end{aligned}\tag{35}$$

Clearly, by standard arguments system of ordinary differential equations (34) has a unique solution $\{g_i^m(t)\}_{i=1,\dots,m}$ such that $g_i^m, g_{it}^m\in C([0,T])$ and $g_{itt}^m\in L_1(0,T)$.

To derive *a priori* estimates for u^m we multiply $(34)_1$ by $g^m_{jt}(t)$ and sum over j from 1 to m. As a result we obtain the identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u^m_t\|^2+\|u^m_{xxt}\|^2+\|u^m_x\|^2+\|u^m_{xxx}\|^2)=(f,u^m_t)+(f_{xx},u^m_{xxt}) \tag{36}$$

Hence, by the Hölder inequality,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u^m_t\|^2+\|u^m_{xxt}\|^2+\|u^m_x\|^2+\|u^m_{xxx}\|^2) \leqslant (\|f\|+\|f_{xx}\|)(\|u^m_t\|^2+\|u^m_{xxt}\|^2+\|u^m_x\|^2+\|u^m_{xxx}\|^2)^{1/2}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u^m_t\|^2+\|u^m_{xxt}\|^2+\|u^m_x\|^2+\|u^m_{xxx}\|^2)^{1/2}\leqslant\|f\|+\|f_{xx}\| \tag{37}$$

Integrating (37) with respect to $t\in(0,T]$ yields

$$(\|u^m_t\|^2+\|u^m_{xxt}\|^2+\|u^m_x\|^2+\|u^m_{xxx}\|^2)^{1/2}\leqslant\int_0^t(\|f\|+\|f_{xx}\|)\,\mathrm{d}t' + (\|u^m_1\|^2+\|u^m_{1xx}\|^2+\|u^m_{0x}\|^2+\|u^m_{0xxx}\|)^{1/2} \tag{38}$$

In view of (35) inequality (38) implies the uniform in m estimate

$$\|u^m_t\|_{L_\infty(0,T;L_2(I))}+\|u^m_{xxt}\|_{L_\infty(0,T;L_2(I))}+\|u^m_x\|_{L_\infty(0,T;L_2(I))}+\|u^m_{xxx}\|_{L_\infty(0,T;L_2(I))}\leqslant c \tag{39}$$

with constant c given by (30).

From (39) it follows that there exists a function u with

$$u\in L_\infty(0,T;H^3(I)\cap V),\quad u_t\in L_\infty(0,T;H^2(I)\cap V) \tag{40}$$

and a subsequence of u_m, still denoted by the same index, such that

$$\begin{aligned} u^m &\to u \ \text{weakly-*} \quad \text{in } L_\infty(0,T;H^3(I))\\ u^m_t &\to u_t \ \text{weakly-*} \quad \text{in } L_\infty(0,T;H^2(I)) \end{aligned} \tag{41}$$

Using standard arguments (see e.g. Reference [16, Volume I, Chapter 3.8]) we can pass to the limit with $m\to\infty$ in (33) and (34) to conclude the function u is a solution of the problem in the following sense:

$$\int_0^T[a(u,\eta)+a(u_{xx},\eta_{xx})-(u_t,\eta_t)-(u_{xxt},\eta_{xxt})]\,\mathrm{d}t = \int_0^T[(f,\eta)+(f_{xx},\eta_{xx})]\,\mathrm{d}t+(u_1,\eta(0))+(u_{1xx},\eta_{xx}(0)) \tag{42}$$

for all test functions $\eta\in C([0,T];H^3(I))\cap V)$ such that

$$\eta_t\in C([0,T];H^2(I)\cap V)\quad\text{and}\quad\eta(T)=0,\quad \eta_{xx}(T)=0$$

Moreover, from (39) and (41) it follows that the solution constructed above satisfies estimate (29) with constant c given by (30). This completes the proof. □

Remark 2.1
In standard approach [16,17] problem (25) is approximated by

$$\begin{gathered}(u^m_{tt}, w_j) + a(u^m, w_j) = (f, w_j), \quad j = 1, \ldots, m \\ u^m(0) = u^m_0, \quad u^m_t(0) = u^m_1\end{gathered} \tag{43}$$

which yields the existence of the solution $u \in L_\infty(0, T; H^1_0(I))$ with $u_t \in L_\infty(0, T; L_2(I))$. Since our purpose is to prove the existence of more regular solution we add to (43) the identity

$$(u^m_{xxtt}, w_{jxx}) + a(u^m_{xx}, w_{jxx}) = (f_{xx}, w_{jxx}), \quad j = 1, \ldots, m \tag{44}$$

This identity results from $(43)_1$ by multiplying its both sides by λ^2_j, using properties (31) and (32) of the basis $\{w_j\}_{j \in \mathbb{N}}$ and taking into account the identity

$$(f, w_j) + (f_{xx}, w_{jxx}) = (f^m, w_j) + (f^m_{xx}, w_{jxx})$$

where $f^m = \sum_{l=1}^m (f, w_l) w_l$.

The second result presented below concerns the existence and uniqueness of solutions to the following initial-boundary value problem for the fourth order parabolic equation

$$\begin{gathered}\chi_t + \chi_{xxxx} = g \quad \text{in } I^T \\ \chi|_{t=0} = \chi_0 \quad \text{in } I \\ \chi_x = 0, \quad \chi_{xxx} = 0 \quad \text{on } S^T\end{gathered} \tag{45}$$

According to Lions–Magenes [16, Volume II, Chapter 4, Theorem 4.3], the following result holds true.

Lemma 2.2
Suppose that $\chi_0 \in H^2(I)$ satisfies the compatibility condition $\chi_{0x} = 0$ on S and $g \in L_2(I^T)$. Then problem (45) has a unique solution $\chi \in W^{4,1}_2(I^T)$ such that, with some constant $c > 0$,

$$\|\chi\|_{W^{4,1}_2(I^T)} \leqslant c(\|g\|_{L_2(I^T)} + \|\chi_0\|_{H^2(I)}) \tag{46}$$

Finally, we recall the Gagliardo–Nirenberg interpolation inequality which we shall frequently use in our analysis. In 1-D case it takes the form (see e.g. Reference [18, Chapter III, Section 15])

$$\|D^r f\|_{L_p(I)} \leqslant c\|f\|^{1-\theta}_{L_{p_1}(I)} \|D^l f\|^\theta_{L_{p_2}(I)} + c\|f\|_{L_{p_1}(I)} \tag{47}$$

where

$$1 \leqslant p_1, p_2, p \leqslant \infty, \quad 0 \leqslant r < l$$

$$\frac{1}{p} - r = (1-\theta)\frac{1}{p_1} + \theta\left(\frac{1}{p_2} - l\right), \quad \frac{r}{l} \leqslant \theta \leqslant 1$$

3. ENERGY ESTIMATES

In this section we derive energy estimates for problem (17) and (17). To this purpose it is convenient to use its primary form (1)–(1).

Lemma 3.1
Let $f(\varepsilon,\chi,\chi_x)$ be given by (8) and satisfies structural bound (9). Moreover, let the data be such that

$$\begin{aligned} &W(u_{0x},\chi_0),\quad \psi(\chi_0)\in L_1(I),\quad \chi_{0x}\in L_2(I),\quad u_1\in L_2(I)\\ &b\in L_1(0,T;L_2(I)) \end{aligned}\tag{48}$$

Then a solution (u,χ,μ) of problem (1)–(1) satisfies estimate

$$\tfrac{1}{2}\|u_t\|_{L_\infty(0,T;L_2(I))}+c_f^{1/2}(\|\varepsilon\|_{L_\infty(0,T;L_2(I))}+\|\chi_x\|_{L_\infty(0,T;L_2(I))})+M^{1/2}\|\mu_x\|_{L_2(I^T)}\leq c_0\tag{49}$$

with a positive constant c_0 independent of T, given by

$$c_0=\left(\frac{1}{2}\|u_1\|^2_{L_2(I)}+\|W(u_{0x},\chi_0)\|_{L_1(I)}+\|\psi(\chi_0)\|_{L_1(I)}+\frac{\Gamma}{2}\|\chi_{0x}\|^2_{L_2(I)}+\|b\|^2_{L_1(0,T;L_2(I))}+c_f'\right)^{1/2}$$

Proof
Assume that solutions of (1)–(1) are sufficiently regular. Multiplying $(1)_1$ by u_t, integrating over I and by parts, and using $(1)_3$ we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I u_t^2\,\mathrm{d}x+\int_I W_{,\varepsilon}(\varepsilon,\chi)\varepsilon_t\,\mathrm{d}x=\int_I bu_t\,\mathrm{d}x\tag{50}$$

Further, multiplying $(2)_1$ by μ, integrating by parts and using $(2)_3$ yields

$$\int_I \chi_t\mu\,\mathrm{d}x+M\int_I\mu_x^2\,\mathrm{d}x=0\tag{51}$$

Finally, multiplying $(3)_1$ by $-\chi_t$, integrating by parts and using $(3)_2$ we obtain

$$-\int_I\mu\chi_t\,\mathrm{d}x+\Gamma\int_I\chi_x\chi_{xt}\,\mathrm{d}x+\int_I\psi'(\chi)\chi_t\,\mathrm{d}x+\int_I W_{,\chi}(\varepsilon,\chi)\chi_t\,\mathrm{d}x=0\tag{52}$$

Summing up (50)–(52) we arrive at the energy identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I u_t^2\,\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\int_I\left[W(\varepsilon,\chi)+\psi(\chi)+\frac{1}{2}\Gamma\chi_x^2\right]\mathrm{d}x+M\int_I\mu_x^2\,\mathrm{d}x=\int_I bu_t\,\mathrm{d}x\tag{53}$$

Integrating (53) over $(0,t)$ for $t\in(0,T)$ gives

$$\begin{aligned}&\frac{1}{2}\int_I u_t^2\,\mathrm{d}x+\int_I f(\varepsilon,\chi,\chi_x)\,\mathrm{d}x+M\int_{I^t}\mu_x^2\,\mathrm{d}x\,\mathrm{d}t'\\ &\quad=\frac{1}{2}\int_I u_1^2\,\mathrm{d}x+\int_I f(u_{0x},\chi_0,\chi_{0x})\,\mathrm{d}x+\int_{I^t} bu_{t'}\,\mathrm{d}x\,\mathrm{d}t'\end{aligned}\tag{54}$$

where f is defined by (8). By virtue of condition (9) the left-hand side of (54) is bounded from below by

$$\tfrac{1}{2}\|u_t\|^2_{L_2(I)} + c_f\|\varepsilon\|^2_{L_2(I)} + c_f\|\chi_x\|^2_{L_2(I)} + M\|\mu_x\|^2_{L_2(\Omega^t)} - c'_f$$

Finally, using (48) and the estimate

$$\left|\int_{I^t} b u_{t'}\,\mathrm{d}x\,\mathrm{d}t'\right| \leqslant \|u_{t'}\|_{L_\infty(0,t;L_2(I))}\|b\|_{L_1(0,t;L_2(I))}$$
$$\leqslant \frac{1}{4}\|u_{t'}\|^2_{L_\infty(0,t;L_2(I))} + \|b\|^2_{L_1(0,t;L_2(I))}$$

we arrive at the assertion. □

We note some important implications of energy estimate (49). In view of the conservation property of system (2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I \chi(t)\,\mathrm{d}x = 0$$

the mean value of χ is preserved

$$\int_I \chi(t)\,\mathrm{d}x = \int_I \chi_0(x)\,\mathrm{d}x \quad \text{for all } t\in[0,T] \tag{55}$$

Hence, by the Poincaré inequality,

$$\|\chi\|_{L_\infty(0,T;H^1(I))} \leqslant c_1 \tag{56}$$

where

$$c_1 = c\left(\varrho_0, c_f, M, L, \int_I \chi_0(x)\,\mathrm{d}x\right)$$

Consequently, by the Sobolev imbedding,

$$\|\chi\|_{L_\infty(I^T)} \leqslant c_1 \tag{57}$$

Moreover, since $u=0$ on S^T, by the Poincaré–Friedrichs inequality it follows from (49) that

$$\|u\|_{L_\infty(0,T;H^1(I))} \leqslant c_1 \tag{58}$$

4. *A PRIORI* ESTIMATES

According to energy estimates (49), (56)–(58) we have

$$\|u_t\|_{L_\infty(0,T;L_2(I))} + \|u\|_{L_\infty(0,T;H^1(I))} + \|\chi\|_{L_\infty(0,T;H^1(I))\cap L_\infty(I^T)} \leqslant c_1 \tag{59}$$

with constant c_1 given in (56). Making use of (59) we derive now the crucial *a priori* estimates.

