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Thermoelastic wheel - rail contact problem with temperature dependent friction coefficient

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Abstract

The paper deals with the numerical solution of wheel - rail rolling contact problems with the temperature field and wear as additional components. We shall consider the contact of a wheel with an elastic rail resting on a rigid foundation. It is assumed that the friction between the bodies is described by the Coulomb law. Moreover we assume a frictional heat generation and heat transfer across the contact surface as well as Archard's law of wear in contact zone. The friction forces and the heat flux depend on the friction coefficient. This coefficient is assumed to depend on temperature.
In the paper quasistatic approach to solve this contact problem is employed. This approach is based on an assumption that for the observer moving with the rolling body the displacement of the supporting foundation is independent on time. The system is described by the elliptic variational inequality governing the displacement and the parabolic equation governing the heat flow. In order to solve numerically this system we will decouple it into mechanical and thermal parts. Finite element method is used as a discretization method. Using duality theory and the regularized relation between tangent and normal components of the contact stress we formulate the mechanical part of this problem as an optimization problem with respect to the normal contact stress. Pschenichnyj method combined with Newton method are used to solve numerically this discretized optimization problem. Next for calculated displacement field the thermal part of the system is solved using Newton method. Numerical examples showing the influence of the temperature dependent friction coefficient on the contact local traction and the length of the contact zone are provided.

1 Introduction

This paper deals with the numerical solution of the rolling contact problems taking into account the temperature field and wear as additional components of the rolling contact problem. Temperature field and wear in the wheel-rail system have influence on the phenomenon which occur in the contact area [1, 15]. We shall consider the contact of a rigid wheel with an elastic rail lying on a rigid foundation. The friction between the bodies described by the Coulomb law [15, 16, 26] is assumed. Frictional heat generation and heat transfer across the contact surface are also assumed.
The wear process at the interface depends on kinematics, material and geometry of the contacting bodies as well as on the enviroment (see [3, 5, 8, 10, 13, 22, 26]). The wear may be caused by adhesion, abrasion, corrosion and surface fatigue. On a macro - scale the existence of the wear process can be identified as wear debris. This debris is assumed to disappear immediately at the point were it is formed. In the model the wear is identified as an increase in the gap between bodies. Moreover the dissipation energy is being changed due to wear. We employ the Archard's

law of wear where the wear rate is proportional to the normal contact pressure and the sliding velocity.
The elastic rolling contact problem was considered by many authors [3, 4, 9, 13, 14, 15, 16, 19, 26]. Among others, one of the first rolling contact problem model was described in [9] where the contact zone and the normal contact stress are assumed to be known. This model was developed and employed to calculate numerical solution of the wheel-rail wear problem in [3]. In [19] this contact problem was described by hyperbolic variational inequality and solved numerically using incremental finite element method. The numerical algorithm proposed in [19] is very general and very slow convergent. The effects of heat generation and heat transfer involving contact has been analysed in literature for many years (see [1, 6, 7, 12, 21, 23, 25, 26]). The thermomechanical interface models taking into account micro-geometrical shape of surfaces were introduced in [15]. Finite element formulations including thermomechanical coupling for contact problems has been presented in [7, 12, 21, 23, 25]. In [7, 12] the Green function approach has been used used to solve the thermoelastic contact problems numerically. Papers were wear is included in the rolling contact model are less numerous. Reviewing paper [5] contains bibliography of papers dealing with contact and wear. In [3] Hertz contact model with Archard law of wear were employed to solve numerically wheel - rail contact problem. Coulomb friction model including heat generation and wear combined with Green function approach were used in [7, 8, 22] to solve numerically the rolling contact problem. Models of contact with friction, heat generation and wear are introduced and discussed in [10, 13, 22, 26].
In literature this contact problem is usually considered assuming the constant friction coefficient. Numerous experiments [18, 24, 2] indicate that this coefficient is depending, among others, on the sliding velocity or the temperature. The friction coefficient first abruptly decreases and than monotonously increases with the sliding velocity. The friction coefficient is also temperature dependent through the temperature dependence of mechanical parameters of the wheel - rail contact [18]. The existence of solutions for viscoelastic contact problem has been shown in [11] provided the friction coefficient is Lipschitz continuous function of sliding velocity or temperature.
In this paper we solve numerically this thermoelastic wheel - rail contact problem assuming piecewise linear dependence of the friction coefficient on the temperature. Following [4, 14] we use a quasistatic approach to solve this contact problem. This approach is based on assumption that for the observer moving with the rolling wheel the displacement of the rail is independent of time. Moreover we shall assume that the length of the rail is much bigger than the diameter of the wheel. We shall confine ourselves to the case of small velocities of the wheel, i.e. we do not consider the vibration of the wheel. Under this assumption the rolling contact problem is described by an elliptic variational inequality instead of hyperbolic variational inequality. The thermal field is described by the parabolic equation. After brief introduction of the thermoelastic model of the rolling contact problem in the framework of two-dimensional linear elasticity theory [7, 9, 15, 16, 21] the general coupled parabolic - hyperbolic system describing this physical problem is formulated. Under the mentioned earlier assumptions we obtain quasistatic formulation of this contact problem in the form of the parabolic - elliptic system. To solve numerically this system we will decouple it into mechanical and thermal parts [21]. Finite element method is used as a discretization method. First for given temperature field we solve the mechanical part. In order to solve the mechanical part of this system we introduce regularization of the friction conditions. Moreover we replace solving the elliptic inequality by solving an auxiliary optimization problem to find the normal contact tractions only. This approach is based on duality theory [16]. Having obtained the normal contact tractions we can calculate the displacement and stress fields in the whole domain by back substitution. Pschenicznyj linearization method is used to solve this auxiliary optimization problem [17]. Newton method is employed to calculate tangent contact stress from regularized friction conditions. In the second step for the calculated displacement field we solve the thermal part of the system using the Newton method. The applications are for wheel-rail systems. The attention will be paid on the influence of the

temperature which is generated in the contact zone on the contact local tractions and the length of the contact. The results are discussed.

2 Contact Problem Formulation

Consider deformations of an elastic strip lying on a rigid foundation (Fig. 1). The strip has constant height h and occupies domain $\Omega \in R^2$ with the boundary Γ. A wheel rolls along the upper surface Γ_C of the strip. The wheel has radius r_0, rotating speed ω and linear velocity V. The axis of the wheel is moving along a straight line at a constant altitude h_0 where $h_0 < h + r_0$, i.e. the wheel is pressed in the elastic strip. It is assumed that the head and tail ends of the strip are clamped, i.e. we assume that the lenght of the strip is much bigger than the radius of the wheel. Moreover it is assumed that there is no mass forces in the strip. We denote by $u = (u_1, u_2)$, $u = u(x,t)$, $x \in \Omega$, $t \in (0,T)$, $T > 0$, a displacement of the strip and by $\theta = \theta(x,t)$ the absolute temperature of the strip.

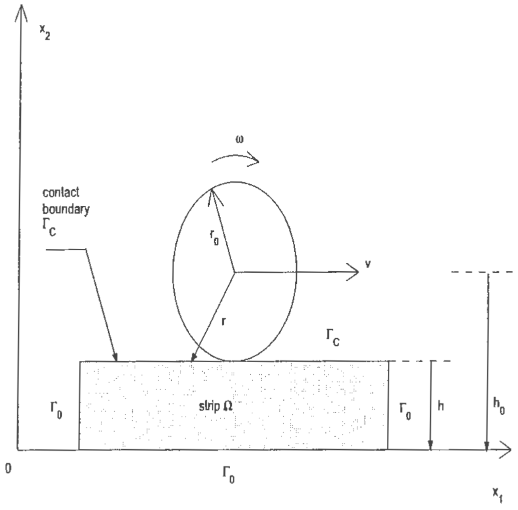

Figure 1: Wheel rolling over the rail.

In an equlibrium state the displacement u and the temperature θ of the strip satisfiy the system of equations [1, 7, 16, 21]:

$$\rho \frac{\partial^2 u}{\partial t^2} = A^\star D A u - \alpha(3\lambda + 2\gamma)\nabla\theta \quad \text{in } \Omega \times (0,T) \tag{1}$$

$$\rho c_p \frac{d\theta}{dt} = \bar{\kappa}\Delta\theta \quad \text{in } \Omega \times (0,T) \tag{2}$$

with initial and boundary conditions :

$$u = 0 \quad \text{on } \Gamma_0 \times (0,T) \tag{3}$$

$$B^\star D A u = F \quad \text{on } \Gamma_C \times (0,T) \tag{4}$$

$$u(0) = \bar{u}_0 \quad u'(0) = \bar{u}_1 \quad \text{in } \Omega \tag{5}$$

$$\theta(0) = \bar{\theta} \quad \text{in } \Omega \tag{6}$$

$$\frac{\partial \theta}{\partial n}(x,t) = q(t) \quad \text{on } \Gamma \tag{7}$$

where $u(0) = u(x,0)$, $u' = du/dt$, $\bar{u}_0$, $\bar{u}_1$, $\bar{\theta}$, $q(t)$ are given functions, ρ is a mass density of the strip material, α is a coefficient of thermal expansion, $\bar{\kappa}$ is a thermal conductivity coefficient, c_p is a heat capacity coefficient, $\Gamma_0 = \Gamma \setminus \Gamma_C$, the operators A, B and D are defined as follows [20]

$$A = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix}, \quad B = \begin{bmatrix} n_1 & 0 \\ 0 & n_2 \\ n_2 & n_1 \end{bmatrix}, \quad D = \begin{bmatrix} \lambda + 2\gamma & \lambda & 0 \\ \lambda & \lambda + 2\gamma & 0 \\ 0 & 0 & \gamma \end{bmatrix} \tag{8}$$

where $n = (n_1, n_2)$ is outward normal versor to the boundary Γ of the domain Ω, λ and γ are Lame coefficients [15, 16], A^* denotes a transpose of A. By $\sigma = (\sigma_{11}, \sigma_{22}, \sigma_{12})$ and F we denote the stress tensor in domain Ω and surface traction vector on the boundary Γ respectively. The surface traction vector $F = (F_1, F_2)$ on the boundary Γ_C is a priori unknown and is given by conditions of contact and friction. Under the assumptions that the strip displacement is small the contact conditions take a form [7, 9, 15, 16]:

$$u_2 + g_r + w \leq 0, \quad F_2 \leq 0, \quad (u_2 + g_r + w)F_2 = 0 \quad \text{on } \Gamma_C \times (0,T) \tag{9}$$

$$g_r = r - r_0$$

$$| F_1 | \leq \mu | F_2 |, \quad F_1 \frac{du_1}{dt} \leq 0, \quad (| F_1 | - \mu | F_2 |)\frac{du_1}{dt} = 0 \quad \text{on } \Gamma_C \times (0,T) \tag{10}$$

where $\mu = \mu(\theta)$ is a friction coefficient dependent on temperature θ and r is the distance between the center of the wheel and a point $x \in \Gamma_C$ lying on the boundary Γ_C of the strip Ω. Under suitable assumptions $g_r = h - h_0 + \sqrt{r_0^2 - (u_1 + x_1)^2}$. $w = w(x,t)$ denotes the distance between the bodies due to wear [10, 22] and satisfies the Archard law [5, 10],

$$\frac{dw}{dt} = kVF_2 \tag{11}$$

$w = w(x,t)$ is an internal state variable to model the wear process taking place at the contact interface [10]. k is a wear constant. The wear process can be identified as wear debris, i.e. the removal of material particles from the contacting surfaces. The wear process between contacting surfaces may be caused by adhesion, abrasion, corrosion or surface fatigue [5, 10]. In the considered model the wear is described as an increase in the gap in the normal direction between the contacting bodies.

3 Friction coefficient dependent on temperature

The relation between the friction coefficient and temperature is subject of intensive research. Among others in [18] an analytical approximation of the friction coeeficient is developed. In this model the friction coefficient is described by a rational function dependent on tensile strength and the Young modulus of the rail steel as well as other parametars. The values of these parameters are calculated by the least square method. In this model there is no explicit dependence on the temperature. The friction coefficient can be change only due to changes in the strength σ_0 or Young modulus E of the steel caused by temperature variations. Strength of the material is most heavily temperature dependent. Assuming that the decrease in strength at increased temperatures is due to heat-activated plastic deformation processes, the theoretical dependence of the steel friction coefficient displayed in Fig. 2 may be concluded. Analytically

$$\mu = 0.15 + 1020\frac{\sigma_0}{E} - 2 \cdot 10^7 (\frac{\sigma_0}{E})^2 - 1.6 \cdot 10^6 (\frac{\sigma_0}{E})^2 \ln(2.5 \cdot 10^5 + 159\frac{E}{\sigma_0}),$$

where $E = 206GPa$ and

$$\sigma_0 = 925.7\text{MPa}, \quad \text{if} \quad \theta \leq 663^\circ C,$$

and

$$\sigma_0 = 5.936\theta \ln(0.0023e^{\frac{3074}{\theta}} + \sqrt{(0.0025e^{\frac{3074}{\theta}})^2 + 1}, \quad \text{if} \quad \theta > 663^\circ C.$$

It is known that the increase in the average temperature at the Hertzian contact does not exceed 150 K, and thus the absolute temperature reaches 450 K at maximum. This suggests that temperature cannot considerably alter the friction coefficient. However, peak temperatures at microcontacts can achieve much higher values (about 1000 K) and significantly affect the friction coefficient. Similar dependence is reported in [24] where the hard carbon films are considered. On the other hand in [2], where automobile brake materials are considered, the friction coefficient is assumed to be strongly dependent on the temperature in the range -100 [deg] C - 200 [deg] C and attaining several local minima and maxima.

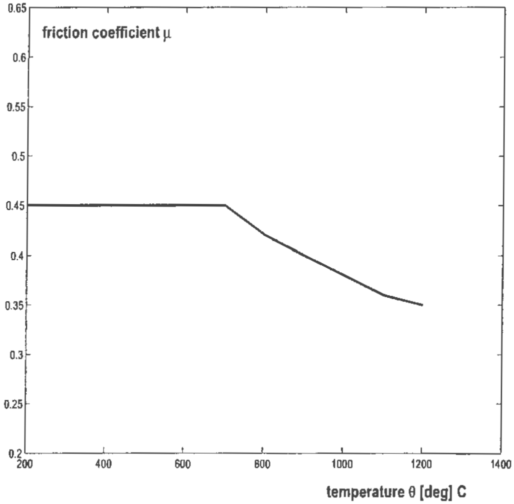

Figure 2: Theoretical dependence of the rail steel friction coefficient on the absolute temperature [18].

Following experimental investigation in [18, 2] we employ the following temperature dependence friction coefficient (see Fig. 3):

$$\mu = \mu_0, \quad \text{if} \quad \theta \leq \theta_0, \tag{12}$$

and

$$\mu = -\frac{\mu_0 - \mu_F}{\theta_F - \theta_0}\theta + \frac{\mu_0\theta_F - \mu_F\theta_0}{\theta_F - \theta_0}, \quad \text{if} \quad \theta > \theta_0, \tag{13}$$

where μ_0, $0 < \mu_F < \mu_0$ and $0 < \theta_0 < \theta_F$ are given positive constants. Note, the function given by (12) - (13) is Lipschitz continuous.

3.1 Quasistatic formulation

Let be given an observer moving with the wheel with the constant linear velocity V. We shall assume :

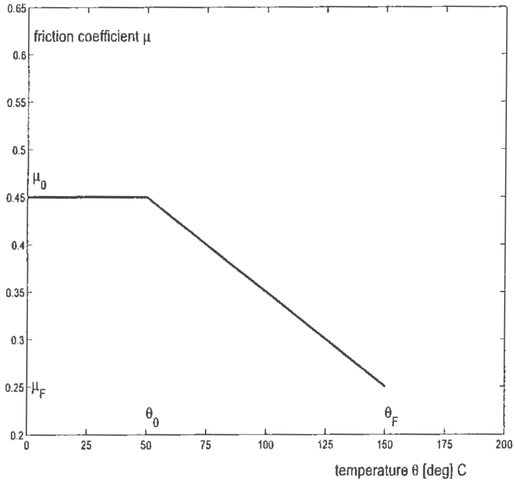

Figure 3: Experimental based dependence of the friction coefficient on temperature.

(i) the lenght of the strip is much bigger than the radius of the wheel
(ii) for the observer moving with a wheel the displacement of the strip does not depend on time
(iii) the velocity of the wheel is small enough, i.e. vibrations of wheel can not appear

If the running velocity is constant the temperature very soon approach steady-state values. We assume in the contact area the heat is generated due to friction and the heat flow rate is transformed completely into heat. Moreover we assume the wear debris disappear immediately at the point where it is formed influencing the contact conditions by increasing the gap between the contacting bodies only. Since the wear debris will be warm due to conduction from heated contacted bodies as well as due to wear processes the disappearing wear debris will carry away also the heat energy [10].
Let us introduce the new cartesian coordinate system $O'x_1'x_2'$ hooked in the middle of the wheel. The systems $O'x_1'x_2'$ and Ox_1x_2 are related by :

$$\begin{aligned} x_1' &= x_1 - Vt \\ x_2' &= x_2 \end{aligned} \tag{14}$$

Since by assumptions (i)-(iii) $u(x_1', x_2')$ does not depend on time we obtain :

$$\frac{du}{dt}(x_1', x_2') = \frac{du}{dt}(x_1 - Vt, x_2, t) = 0 \tag{15}$$

It implies

$$\frac{du}{dt} = -V\frac{du}{dx_1} \quad \text{and} \quad \frac{d^2u}{dt^2} = V^2\frac{d^2u}{dx_1^2} \tag{16}$$

Using the same argument we obtain :

$$\frac{d\theta}{dt} = -V\frac{d\theta}{dx_1}, \quad \frac{dw}{dt} = -V\frac{dw}{dx_1} \tag{17}$$

Let Ω denotes now the moving part of the strip seen by the observer. Taking into account (14) quasistatic approximation of the problem (1)-(3) takes the form :

Find u and θ satisfying

$$A^\star DAu - \rho V^2 u_{11} - \alpha(3\lambda + 2\gamma)\nabla\theta = 0 \quad \text{in } \Omega \tag{18}$$

$$-V\frac{\partial\theta}{\partial x_1} = \kappa\frac{\partial^2\theta}{\partial x_2^2} \quad \text{in } \Omega \tag{19}$$

$$u = 0 \quad \text{on } \Gamma_0 \tag{20}$$

$$B^\star DAu = F \quad \text{on } \Gamma_C \tag{21}$$

$$\begin{array}{lll} u_2 + g_r + w \le 0, & F_2 \le 0, & (u_2 + g_r + w)F_2 = 0 \quad \text{on } \Gamma_C \\ | F_1 | \le \mu(\theta) | F_2 | & F_1 u_{1,1} \le 0, & (| F_1 | - \mu(\theta) | F_2 |)u_{1,1} = 0 \quad \text{on } \Gamma_C \end{array} \tag{22}$$

$$-\bar\kappa\frac{\partial\theta}{\partial x_2} = \bar\alpha[\frac{\theta}{r}F_2(x) + (1 - \frac{k\rho c_p\theta}{\mu(\theta)}\mu(\theta)VF_2(x)] \quad \text{on } \Gamma_C \tag{23}$$

$$\frac{dw}{dx_1} = -kF_2 \tag{24}$$

where $u_{i,j} = \frac{\partial u_i}{\partial x_j}$, $u_{i,jk} = \frac{\partial^2 u_i}{\partial x_j \partial x_k}$, $i,j,k = 1,2$, $u_{ij} = (u_{m,ij})_{m=1,2}$, $i,j = 1,2$, $\kappa = \bar\kappa/\rho c_p$ is the thermal diffusivity coefficient, $\bar\alpha$ represents the fraction of frictional heat flow rate entering the rail, r is thermal resistance constant [10]. There are also given initial conditions (5) - (6). We assume in (22) the heat flows through the contact surface only, therefore $\theta = 0$ on Γ_0.

3.2 Variational formulation

Let us introduce a space and a set :

$$Z = \{z \in [H^1(\Omega)]^2 \; : \; z = 0 \text{ on } \Gamma_0\}, \qquad K = \{z \in Z \; : \; z_2 + g_r + w \le 0 \text{ on } \Gamma_C\} \tag{25}$$

w is supposed to be enough regular function, i.e., for each $t \in (0,T)$ $w \in H^{3/2}(\Gamma_C)$. Let $a(.,.) : Z \times Z \to R$ denote bilinear form defined by

$$a(z,v) = \int_\Omega (Au)^\star DAv dx \tag{26}$$

We denote by $(z,v) = \int_\Omega z^\star v dx$ the scalar product in $L^2(\Omega)$.
We formulate quasistatic problem (18)-(22) in the variational form. Let us denote by a' the bilinrear form depending on $\theta \in Z$:

$$a'(\theta; z, v) \; : \; Z \times Z \to R$$

$$a'(\theta; z, v) = a(z,v) - \rho V^2 \Sigma_{i=1}^2 (z_{i,1}, v_{i,1}) - \alpha(3\lambda + 2\gamma)(\theta, v_{1,1} + v_{2,2}) \tag{27}$$

Note, that the coercivicity of the form (27) is assured for small velocities only. Problem (18) – (22) is equivalent to the following variational problem [16] :

Find $u \in K$ and $\theta \in Z$ satisfying :

$$a'(\theta; u, \phi - u) + \int_{\Gamma_C} F_1(u_1 - Vu_{1,1})d\Gamma + \int_{\Gamma_C} \mu(\theta) | F_2 | (| \phi_1 | - V | u_{1,1} |)d\Gamma \ge 0 \quad \forall \phi \in K \tag{28}$$

$$\int_\Omega \{\kappa\frac{\partial\theta}{\partial x_2}\frac{\partial\phi}{\partial x_2} + V\theta\frac{\partial\phi}{\partial x_1}\}dx = \int_{\Gamma_C} \{-\frac{\kappa}{\bar\kappa}\bar\alpha[\frac{\theta}{r}F_2(x) + (1 - \frac{k\rho c\theta}{\mu(\theta)})\mu(\theta)VF_2\phi] + V\theta n_1\phi\}ds \quad \forall\phi \in Z \tag{29}$$

Using the same arguments as in [16] we can show that problems (18)-(22) and (28)-(29) are equivalent.

3.3 Regularization

In order to assure the existence [9, 15, 16] of solutions to the problem (28)-(29) we have to regularize it i.e. we will consider problem (28)-(29) as problem with the prescribed friction. Let $\varepsilon > 0$ be a regularization parameter. We propose the following formula relating tangential and normal tractions on the contact boundary Γ_C [15] :

$$F_1 = F_1(\varepsilon, F_2, u_1) = -\mu(\theta) \mid F_2 \mid \arctan(\frac{Vu_{1,1}}{\varepsilon}) \tag{30}$$

$$l_\varepsilon = F_1(\varepsilon, F_2, u_1)$$

The regularized problem (28)-(29) takes the form :

$$\text{For given } \varepsilon > 0, \quad \text{find } u \in K \text{ and } \theta \in Z \text{ satisfying}$$

$$a'(\theta; u, \phi - u) + \int_{\Gamma_C} l_\varepsilon(u_1 - Vu_{1,1})d\Gamma + \int_{\Gamma_C} \mu(\theta) \mid F_2 \mid (\mid \phi_1 \mid -V \mid u_{1,1} \mid)d\Gamma \geq 0 \;\; \forall\phi \in K \tag{31}$$

$$\int_\Omega \{\kappa \frac{\partial\theta}{\partial x_2}\frac{\partial\phi}{\partial x_2} + V\theta\frac{\partial\phi}{\partial x_1}\}dx = \int_{\Gamma_C} \{-\frac{\kappa}{\bar{\kappa}}\bar{\alpha}[\frac{\theta}{r}F_2(x) + (1 - \frac{k\rho c\theta}{\mu(\theta)})\mu(\theta)VF_2\phi] + V\theta n_1\phi\}ds \quad \forall\phi \in Z \tag{32}$$

It can be shown that for any $\varepsilon > 0$ as well as enough small velocity V problem (31) - (32) has a unique solution $\bar{u} \in K$ and $\bar{\theta} \in Z$.

4 The Solution Algorithm

Problem (31) - (32) is a coupled thermoelastic problem since the contact traction will depend on the thermal distortion of the bodies and wear process. On the other hand, the amount of heat generated due to friction will depend on the contact traction. The main solution strategies for coupled problems are global solution algorithms where the differential systems for the different variables are solved together or operator splitting methods. In this paper we employ operator split algorithm.

The conceptual algorithm for solving (31) - (32) is as follows:

Step 1 : Choose $\theta = \theta^0$ and $w = w^0$. Choose $\eta \in (0, 1)$. Set $k = 0$.
Step 2 : For given θ^k and w^k find $u^k \in K$ and F_2^k satisfying system (31).
Step 3 : For given $u^k \in K$ and F_2^k find w^{k+1} as well as θ^{k+1} satisfying equations (11), (32) respectively.
Step 4 : If $\| \theta^{k+1} - \theta^k \| \leq \eta$, Stop. Otherwise : set $k = k + 1$, go to Step 2.

The convergence of the operator split algorithm using Fixed Point Theorem was shown in [1]. Let us present in details the algorithms for solving disrete mechanical and thermal subproblems.

4.1 Solution of the mechanical subproblem

In order to solve numerically problem (31) we have to discretize it. For the sake of simplicity we shall assume that $\Omega \in R^2$ is a polygonal domain.

Let h be a discretization parameter tending to 0. By $s_1, s_2, \ldots, s_{n(h)}$ we denote a regular partition [6, 21] of the boundary Γ_C. $h = \min\{h_i : i = 1, \ldots, n(h)\}$, $h_i = s_i - s_{i-1}$. By $\mathcal{T}_h$ we denote a regular family [6] of triangulations of domain Ω. We shall assume that the nodes of $\mathcal{T}_h$ on Γ_C coincide with the nodes s_j, $j = 1, \ldots, n(h)$.

To any finite dimensional approximation Ω_h of the domain Ω we attribute a finite dimensional subspace Z_h given by :

$$Z_h = \{z_h : z_h \in [C(\Omega)]^2 \; : z_h \in [P_1(E_i)]^2 \;\text{ for all }\; E_i \in \mathcal{T}_h,\; i = 1, \ldots, I\} \cap Z \tag{33}$$

Here $C(\Omega)$ denotes the space of continuous functions on Ω and $P_k(E_i),\ k=0,1$ denotes the set of all polynomials of degree less or equal to k over a triangle $E_i,\ i=1,\dots,I$. We introduce also the set Λ_h :

$$\Lambda_h = \{F_{2h}\ :\ F_{2h} \in P_0([s_{j-1}, s_j]),\ j=2,\dots,n(h),\ F_{2h} \le 0 \text{ on } \Gamma_C\} \tag{34}$$

The finite dimensional problem corresponding to the problem (31) has the form :

For given $\theta_h \in Z_h$ and w_h, find $F_{2h} \in \Lambda_h$ minimizing the functional :

$$\Pi(F_{2h}) = \frac{1}{2}(F_{2h}, [G(\theta_h, F_{2h})]_2) + (F_{2h}, [G(\theta_h, l_{\epsilon h})]_2) + (l_{\epsilon h}, [G(\theta_h, F_{2h}]_1) + \frac{1}{2}(l_{\epsilon h}, [G(\theta_h, l_{\epsilon h})]_1) \tag{35}$$

over the set Λ_h .

where $l_{\epsilon h}$ in (35) is given by (30) with F_2 replaced by F_{2h} and G denotes the Green operator depending on θ_h.
From numerical point of view problem (35) is difficult to solve because the explicit form of the Green operator is unknown. To avoid this difficulty we calculate for given θ_h the stiffeness matrix S_h resulting from discretization of the bilinear form (27) on the domain Ω_h. The inverse S_h^{-1} of the matrix S_h approximates the Green operator G [16]. The finite dimensional problem (35) can be reformulated as follows :

For given $\theta_h \in Z_h$ and w_h, find $F_{2h} \in \Lambda_h$ minimizing the functional :

$$\begin{aligned}\Pi(F_{2h}) = \ & \tfrac{1}{2}(F_{2h}, [S_h^{-1}(\theta_h, F_{2h})]_2) + (F_{2h}, [S_h^{-1}(\theta_h, l_{\epsilon h})]_2) + (l_{\epsilon h}, S_h^{-1}(\theta_h, F_{2h})]_1) + \\ & \tfrac{1}{2}(l_{\epsilon h}, [S_h^{-1}(\theta_h, l_{\epsilon h})]_1)\end{aligned} \tag{36}$$

over the set Λ_h.

The finite dimensional problem (36) approximating the problem (27) obtained by direct mathematical discretization to the problem (27) is still difficult to solve. However we can use mechanical meaning of the problem (27) [15, 16] and obtain its discrete form much easier.
Consider first the structure Ω_h obtained from the initial one by assuming only the kinematical constraints on the boundary Γ. Then we solve this structure for unite load $F_{2h}(s_1) = 1$ on the first node s_1 of the boundary Γ_C and we obtain the corresponding normal displacement of all $n(h)$ nodes of the boundary Γ_C. These displacements constitute the first column of a matrix M. This procedure is repeated for each node $s_j, j=2,\dots,n(h)$ of the boundary Γ_C and the others columns of the matrix M are constituted. The normal displacements of the nodes $s_j,\ j=1,\dots,n(h)$ of the boundary Γ_C for the same structure under the given load l_ϵ constitutes a vector $m=m_\epsilon$. Then the unknown normal traction vector $F_{2h} = \alpha = \{\alpha_j\}_{j=1}^{n(h)}$ on the boundary Γ_C can be found as a solution to the following quadratic programming problem :

Find $\alpha \le 0$ minimizing the functional :

$$\Pi(\alpha) = \frac{1}{2}\alpha^* M\alpha - m_\epsilon^*\alpha \tag{37}$$

Having obtained the optimal solution $\hat{\alpha}$ to the problem (37) we can calculate the displacement and stress fields of the whole structure by back substitution.
To solve the discretized problem (36) we use the following algorithm :

Step 1 : Calculate matrix : M. Choose $\epsilon \in (0,1)$, $\eta \in (0,1)$, α^0. Set $k=0$.
Step 2 : Calculate l_ϵ^k from (30) with $F_2 = \alpha^k$ and G= M.
Step 3 : Calculate vector $m_\epsilon^k = m(l_\epsilon^k)$.
Step 4 : Find $\alpha^{k+1} = \text{argmin}\ \{\frac{1}{2}\alpha^* M\alpha + m^k\alpha_\epsilon^k \mid \alpha \le 0\}$
Step 5 : If $\| \alpha^{k+1} - \alpha^k \| \le \eta$, Stop. Otherwise : set $k=k+1$, go to Step 2.

The convergence of this algorithm was considered in [17]. Step 2 in the above algorithm is realized using Newton method [7, 8, 15, 16, 17]. The quadratic programming problem (37) in Step 4 is solved using Pschenichnyj algorithm [17]. This algorithm for solving the optimization problem :

$$\text{Find } \hat{\alpha} = \text{ argmin } \{f(\alpha) \mid g(\alpha) \leq 0\}$$

has the following form :

Step 1 : Choose α^0, $\kappa \in (0,1)$. Set : $k = 0$.
Step 2 : At a point α^k find vector $p = p(\alpha^k)$ solving
the following auxiliary optimization problem :
$\min \{\bigtriangledown f(\alpha^k)p + \frac{1}{2} \parallel p \parallel^2 \mid \bigtriangledown g(\alpha^k)p + g(\alpha^k) \leq 0\}$
Set : $p^k = p$.
Step 3 : Find τ such that :
$\phi(\alpha^k + \tau p^k) \leq \phi(\alpha^k) - \bar{\eta}\tau \parallel p^k \parallel^2$
where $\phi(\alpha) = f(\alpha) + Ng(\alpha)$, $\bar{\eta} \in (0,1)$ and $N > 0$ are given numbers.
Set : $\tau^k = \tau$.
Step 4 : Set : $\alpha^{k+1} = \alpha^k + \tau^k p^k$. If $\parallel \bigtriangledown f(\alpha^{k+1}) \parallel \leq \kappa$, Stop.
Otherwise : set $k = k + 1$ and go to Step 2.

The convergence of the Pschenichnyj algorithm is discussed in [17].

4.2 Solution of the thermal subproblem

The finite dimensional problem corresponding to problem (32) has the form :

For given $F_{2h} \in L^2(\Omega_h)$ find w_h and $\theta_h \in Z_h$ satisfying the equations

$$\frac{dw_h}{dx_1} = -kF_{2h} \tag{38}$$

$$\int_{\Omega_h} \{\kappa \frac{\partial \theta_h}{\partial x_2} \frac{\partial \phi}{\partial x_2} + V\theta_h \frac{\partial \phi}{\partial x_1}\} dx = \int_{\Gamma_{Ch}} \{-\frac{\kappa}{\bar{\kappa}} \bar{\alpha} [\frac{\theta_h}{r} F_{2h}(x) + (1 - \frac{k\rho c\theta_h}{\mu(\theta)}) \mu(\theta) V F_{2h} \phi] + V\theta_h n_1 \phi\} ds \quad \forall \phi \in Z \tag{39}$$

Let us introduce the notation :

$$\theta_h = \Sigma_{i=1}^n q_i \phi_i, \quad w_h = \Sigma_{i=1}^n \beta_i \phi_i, \qquad P = \{\int_{\Gamma_{Ch}} [\mu(\theta) V F_{2h} \phi_i] ds\}_{i=1}^n$$

$$A = \{\int_{\Omega_h} [\kappa \frac{\partial \theta_h}{\partial x_2} \frac{\partial \phi_j}{\partial x_2} + V\theta_h \frac{\partial \phi_j}{\partial x_1}] dx + \int_{\Gamma_{Ch}} [\frac{\kappa}{\bar{\kappa}} \bar{\alpha} (\frac{\theta_h}{r} F_{2h}(x) - k\rho c\theta_h V F_{2h} \phi_j) + V\theta_h n_1 \phi_j] ds\}_{i,j=1}^n \tag{40}$$

In matrix form system (39) takes the form :

For given α, find q satisfying

$$Aq = P(\alpha, \beta) \tag{41}$$

The equation (41) is solved using Choleski algorithm.

5 Numerical Results

Problem (31)-(32) was solved numerically using the described in the previous section algorithms. Polygonal domain Ω given by

$$\Omega = \{(x_1, x_2) \in R^2 \; : \; x_1 \in (-2, 2), \; x_2 \in (0, 1)\} \tag{42}$$

was divided into 192 triangles. The contact boundary Γ_C is modeled by 13 nodes. The Lame constants were $\lambda = 11.66 \cdot 10^{10}$ [N/m^2], $\gamma = 8.2 \cdot 10^{10}$ [N/m^2], the density $\rho = 7.8 \cdot 10^3$ [kg/m^3], the velocity $V = 10$ [m/s] , radius of the wheel $r = 0.46$ [m]. The penetration of the wheel was taken as $\delta = 0.1 \cdot 10^{-3}$ [m]. The heat capacity $c = 460$ J/kgK, thermal diffusitivy coefficient $\kappa = 1,4410^{-5}\ m^2/s$, thermal expansion coefficient $\gamma = 1210^{-6}$. The thermal resistance coefficient $r = 1000$ KNs/J, the wear constant $k = 0.510^{-6}\ MPa^{-1}$. $\varepsilon = 0.001$. $\bar{u}_0$ and $\bar{u}_1$ in (5) as well as $\bar{\theta}$ in (6) are equal to 0. The friction coefficient μ is given by (12) - (13), with $\mu_0 = 0.45$, $\mu_F = 0.25$, $\theta_0 = 50°$ C, and a) $\theta_F = 150°$ C, b) $\theta_F = 125°$ C, c) $\theta_F = 100°$ C.
In all cases a) - c) the normal traction F_2 has its peak around the middle of the contact area. Tangent traction F_1 has different shapes in front and behind of the rolling wheel. The wear gap is slightly smaller than in the constant friction coefficient case. When θ_F is decreasing the friction coeeficient is more rapidly decreasing. Therefore it was observed that the maximal temperature increment at a contact point on the surface of the rail (see Fig. 4) is lower up to $5 - 20°$ C. The temperature is rapidly decreasing while entering into the rail however the zone of higher than air temperatures is longer than in the constant friction coefficient case. As far as it concerns the tangential temperature distribution (see Fig. 5) it was observed that essentially the distribution is similar as in the constant friction coefficient case however the decrease of temperature behind the rolling wheel looks slower.
The proposed algorithm converges very quickely. Its speed of convergence depends on the choice of the regularization parameter ε value. For ε very small we obtain much more accurate results than for big values of ε at a cost of increase in computational time.

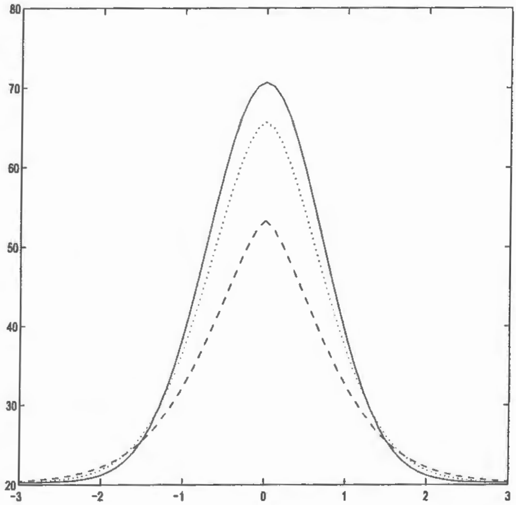

Figure 4: Temperature normal distribution at a contact point.

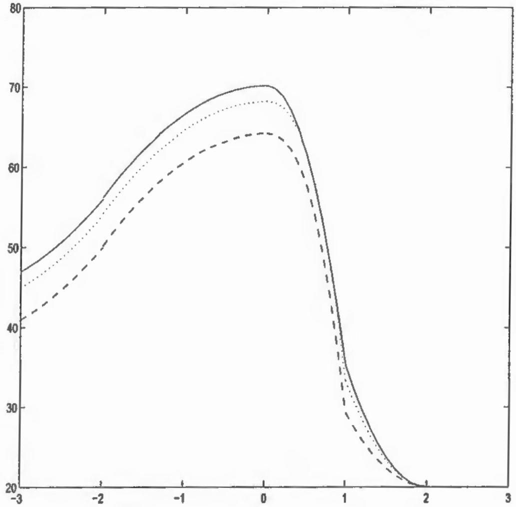

Figure 5: Temperature tangential distribution at a contact point.

6 Conclusions

The thermoelastic rolling contact problem where the friction coefficient is dependent on the temperature was solved numerically using the quasistatic approach. The piecewise linear dependence of the friction coefficient on the temperature is assumed. Such selection of the friction coefficient ensures the existence of solutions to the rollling contact problem.

The obtained numerical results are in accordance with physical reasoning [7]. Using the quasistatic approach we can observe also dynamic phenomena of the rolling wheel. Since we confine to the elastic contact model the computations indicate that the changes in the highest temperature value and the distribution of temperature are moderate with the comparison to the constant friction coefficient case.

Note, that the dependence of the friction coefficient on the temperature may be strongly nonlinear, nonconvex and nondifferentiable [2]. Moreover, as it is indicated in [18], plastic deformation and very high temperatures achieving almost 1000° C may occure in the contact zone. Therefore to investigate the influence of the temperature dependent friction coefficients on the contact phenomenon, the computations should be carried out for elasto - plastic contact model with the friction coefficient depending nonlineary on the temperature.

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