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Two-Factor Utility Approach to Valuation of Catastrophe Bonds

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Abstract

A new approach to valuation of bonds under the default risk conditions, based on the concept of the investors' two-factor utility function is proposed. The first factor describes the expected average return from the risky investments, while the second - the worst case return. As a class of risky securities the so called catastrophe bonds are considered. It is assumed that depending on the structure of the security contract, the investor who buys the bond issued by a local authority governing the risky region - will lose his interest payments and/or the principal value, if a catastrophic event occurs. For the purpose of the valuation procedure, the new notions of the security safety level, the safety index, as well as a two-rule decision model are successively introduced. The subjective scale as a measure of the degree of individuals' risk aversion is proposed. The idea of objective and subjective risk components is investigated. The methodology proposed is illustrated by a computational example.

Keywords: pricing procedure, catastrophe bonds, two-factor utility, expected return, worst case return, risk aversion, subjective scale, safety level.

1. Introduction

The problems of evaluation of risky investments, such as industrial projects, investments in securities, insurance contracts etc., require application of a new methodology including utility and risk assessment. This general class of problems can be illustrated by evaluation of risky bonds. Unlike the Treasury bonds, the municipal and corporate bonds involve a risk that the agreement on the coupons and/or the principal value payments will not be met. Such a situation can happen e.g. due to a random catastrophic event, like flood, hurricane, earthquake, drought, forest fire etc.; which may damage the municipal/corporate budget and make impossible meeting the financial obligations.

Such risky securities are called *catastrophe bonds* and belong to a more general class of the Insurance-Linked Securities (ILS); see Stripple (1998). The idea of insurance-linked securities is related to the notion of securitizing of catastrophe risk. During the second part of last decade, an intensive development of these new types of financial instruments for handling various kinds of catastrophic risks took place. These new instruments entail the absorption of the risks directly on the capital market and have two forms: options and catastrophe bonds.

As it was pointed out by Stripple (1998), the securitization of catastrophic risk means that „the risk is packaged into a standarized form (e.g. as a bond) and sold on the capital market".

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Hence, the risk is „secured" on the market. Depending on the structure of the security contract, the investor who buys the bond issued by the insurer will lose his interest payments and/or the principal value, if there is a catastrophe of a defined magnitude of loss. On the other hand, if there is no catastrophe, the investor will get the money back along with an attractive rate of interest. For example, in 1997 the U.S. Treasury 30-year bonds paid about 6% of interest, while the average catastrophe bond paid 10 to 12 percent. The first example of such a governmental catastrophe security is the bond issued by the California Earthquake Authority; see Cholnoky, Zief, et al. (1998).

In general, there are two main reasons for the investors' interest in catastrophe bonds. First, they pay an additional yield as compared to other types of bonds, although at a greater risk of loss. Further, the performance of a catastrophe bond market is not significantly linked to the performance of the other financial markets. Therefore, the risk attached to the catastrophe bonds is for the most part uncorrelated with traditional market risks and these bonds can serve as means of diversifying an investor's portfolio. So, from the investors perspective, the risk premium attached to a catastrophe bond may increase their return without adding to the investment portfolio additional variance or risk; see Stripple (1998), Doherty (1997). The above conclusion follows directly from the classical Markowitz portfolio theory; Elton, Gruber (1995).

The securitization of catastrophic risk can be advantageous not only for the investors buying the risky financial instruments. It is generally believed that by issuing the catastrophe bonds (or more generally - the insurance-linked securities), the government or a local authority can utilize securitization in order to more efficiently and equitably cope with the financial cost of natural disasters. One main advantage for governments is that the financial uncertainty to the public budget is reduced, which enhances the governmental management and control over the risks. The possible future costs of a catastrophe are transformed into a predictable budgetary item, related to fixed expenses on the bond coupon and principal payments. If a catastrophe occurred, the bond issuer would stop the payments and would have in such a way the financial means for covering (at least - to some extent) the potential losses. Also, the issue of a catastrophe bond (being a source of a long-term credit) provides the financial means, which can be allocated to the regional infrastructure improvement and to all other *ex ante* preparations against the future possible losses, caused by a catastrophic event.

Further on, as it was mentioned by Stripple (1998), the catastrophe securities transactions on international capital markets can substantially increase the geographical spread of the cost of a catastrophe. Up to present, in most cases, „... the natural disasters are borne by the country and people affected. International catastrophe aid amounts only to a small part of the financial relief ". For instance, from this point of view, the securitizing of flood risk in Poland can be an attractive mean of risk management in this area. It can provide a useful and desirable instrument for the government in preparing for natural catastrophes. The case study of the Polish Flood 1997, undertook by IIASA and described by Stripple (1998), has confirmed this conclusion.

The results given in this paper are an extension of ones presented by the author (et al.) in recent years; see Kulikowski, Jakubowski (1999, 2000), Jakubowski (2002 a,b).

2. The rate of return and risk of catastrophe bond

We will define the catastrophe bond as the multi-coupon bond with the default risk attached. The above means that depending on whether the catastrophe occurs or not, the agreements related to the coupon and the principal value payments will or will not be met. In other words, the investor who bought the bond is receiving the coupon payments in successive years up to the date when the possible catastrophe takes place. If no catastrophe occurs during the bond maturity period, all the obligations from the side of the bond issuer will be fulfilled, i.e. the investor will receive all the coupon payments as well as the principal payment at the end of maturity term. It is of course assumed that the term „catastrophe" as well as the magnitude of potential losses is strictly determined in the bond contract.

Obviously, if the catastrophe bond is subscribed, the expected rate of return will have to be attractive, i.e. it has to be much more higher than the Treasury bond rate. How much higher is an open question depending on the risk of the catastrophe, the volume of anticipated damage and other factors; see Stripple (1998). In assessing risk, investors must have a good idea of the probability of the catastrophe occurrence.

In the present paper, we will consider as an example of a catastrophe a flood which can occur in a region. We will also assume that the flood occurrences or non-occurrences in the successive years are statistically independent random events with the prescribed probabilities, estimated per one year. Let us denote: α - the probability of flood occurrence in a given year.

We assume that for the considered region the probability α is estimated by a meteorological institution in the following form:

$\alpha = 0.002$ (p.a.) - the „500-year water", $\alpha = 0.01$ (p.a.) - the „100-year water",
$\alpha = 0.02$ (p.a.) - the „50-year water", $\alpha = 0.05$ (p.a.) - the „20-year water",
$\alpha = 0.10$ (p.a.) - the „10-year water", etc.

We begin our problem description with the example of the 3-year fixed-coupon bond with coupon periods equal to 1 year. Thus, considering that the flood events are statistically independent, we have

$p_1 = \alpha$ - the probability of flood occurrence in the first year;

$p_2 = (1-\alpha)\alpha$ - the probability of flood in the second year, conditioned upon the flood non-occurrence in the first year;

$p_3 = (1-\alpha)^2\alpha$ - the probability of flood in the third year, conditioned upon the flood non-occurrence in the first two years; and

$p_4 = (1-\alpha)^3$ - the probability of no flood occurrence in the whole period of the first 3 years.

From the above formulae, one can easily verify that we have

$$p_1 + p_2 + p_3 + p_4 = \alpha + (1-\alpha)\alpha + (1-\alpha)^2\alpha + (1-\alpha)^3 = 1.$$

Thus, the considered four discrete random events related to the flood occurrence or non-occurrence in the successive years $t = 1,2,3$, span the whole probability space of events. For instance: for $\alpha = 0.10$ (i.e. 10-year water), we have

$$p_1 = 0.1000,\ p_2 = 0.0900,\ p_3 = 0.0810,\ p_4 = 0.7290.$$

For the catastrophe bond we introduce the following notation:

$t = 1,2,3$ - the coupon periods (in years), C - the coupon payments,

N - the bond principal (face) value, P_* - the price of the bond at time $t = 0$.

We shall also adopt the following assumptions concerning the financial market.

(i) The term structure of interest rates is characterized by the flat *yield curve*, i.e.

$r_{o1} = \ldots = r_{ot} = \ldots = r_{on} = r_f$, where r_{ot} - the spot interest rates (p.a.) determined for the years $t = 1,\ldots,n$; r_f - the default risk-free market interest rate determined as *yield-to-maturity* (YTM) of the one-year Treasury Bill.

(ii) The only investment risk analysed in this paper is the bond default-risk. Thus, the interest rate risk, which is also a very important factor concerned with the investment decisions - will not be taken into account. We will assume that one-year market interest rate r_f is given and it will remain constant over the time horizon analysed. Thus, if the value of r_f changed, the whole bond pricing procedure would have to be repeated.

(iii) The considered bond market is in an equilibrium state, i.e. the market price P_o of every default risk-free coupon bond is equal to its present value PV, called the bond *intrinsic value*, i.e.

$$P_o = PV = \frac{C}{(1+r_f)} + \frac{C}{(1+r_f)^2} + \ldots + \frac{C}{(1+r_f)^{n-1}} + \frac{C+N}{(1+r_f)^n}. \quad (1)$$

From the description given above it follows that the price P_* of catastrophe bond should be significantly lower than the present value P_o of the risk-free coupon bond having the same cash flows; i.e. $P_* < P_o$. In other words, there should exist a *price discount*

$$D_* = \frac{P_* - P_o}{P_o} \times 100\ [\%],\quad (D_* < 0)$$

in order to compensate for the default risk attached to the catastrophe bond. The value of this discount will depend upon the probability of flood α, reflecting the risk considered.

It should be pointed out that the catastrophe bond under consideration will be - in general case - the bond with „the incomplete" cash flows in comparison to the default risk-free bond having the same coupons C and the principal value N. This fact is illustrated in Fig. 1.

Let us take the following notation:

R^* - the rate of return (over the 3-year horizon) from the investment in catastrophe bond. We assume that the coupon payments C received from the investment in catastrophe bond are reinvested up to the end of the 3-year time horizon – using the (annual) risk-free interest rate r_f. Thus, from the cash flow structure presented in Fig. 1 it follows that the rate of return R^* is a discrete random variable having - with probabilities $p_1,\ldots,p_4$ - the realizations $R_1^*,\ldots,R_4^*$ given by the formula

$$R^* = \begin{cases} R_1^* = \dfrac{0-P_*}{P_*} = -1 = -100\% & ; \quad \text{w.p.} \quad p_1 = \alpha \\ R_2^* = \dfrac{C(1+r_f)^2 - P_*}{P_*} & ; \quad \text{w.p.} \quad p_2 = (1-\alpha)\alpha \\ R_3^* = \dfrac{C(1+r_f)^2 + C(1+r_f) - P_*}{P_*} & ; \quad \text{w.p.} \quad p_3 = (1-\alpha)^2\alpha \\ R_4^* = \dfrac{C(1+r_f)^2 + C(1+r_f) + C + N - P_*}{P_*} & ; \quad \text{w.p.} \quad p_4 = (1-\alpha)^3 \end{cases} \qquad (2)$$

where „w.p." means „with probability" (for shortness).

Equation (2) forms the probability distribution function of the discrete random variable R^*. It should be pointed out that the value of price P_* is „ex ante" not known; this is the subject of our valuation model we will develop in the successive sections. Having the price P_* determined (and therefore the values of rates of return $R_1^*,\ldots,R_4^*$ - known), we will say that *the scenario* of possible investment outcomes has been established.

We will transform the equation (2) into the following form: let us denote

$x = \dfrac{N}{P_*}$; $\quad i = \dfrac{C}{N}$ - the *nominal interest rate* of the bond;

$R_1 = r_f, \quad R_2 = (1+r_f)^2 - 1, \quad R_3 = (1+r_f)^3 - 1$

– the risk-free rates of return over 1, 2 and 3-year investment periods, respectively; and

$\Re_2 = (1+R_2)\,i, \quad \Re_3 = (1+R_1)i + (1+R_2)i, \quad \Re_4 = (1+R_1)i + (1+R_2)i + (1+i).$

We assume that all the values as given above are known, except for the value P_* (and therefore x). Taking into account the above notation, after some calculations, the formula (2) can be expressed as:

$$R^* = \begin{cases} R_1^* = -1 & ; \quad \text{w.p.} \quad p_1 = \alpha \\ R_2^* = \Re_2 x - 1 & ; \quad \text{w.p.} \quad p_2 = (1-\alpha)\alpha \\ R_3^* = \Re_3 x - 1 & ; \quad \text{w.p.} \quad p_3 = (1-\alpha)^2\alpha \\ R_4^* = \Re_4 x - 1 & ; \quad \text{w.p.} \quad p_4 = (1-\alpha)^3 \end{cases} \qquad (3)$$

For further derivations, the most important thing is to determine the expected value and the variance of the random rate of return R^*. Let us denote: R_e - the expected value of R^*, σ - the standard deviation of R^* (σ^2 - the variance). Introducing additional notation

$$a \stackrel{\Delta}{=} p_2\Re_2 + p_3\Re_3 + p_4\Re_4, \tag{4}$$

$$b \stackrel{\Delta}{=} [p_1 a^2 + p_2(\Re_2 - a)^2 + p_3(\Re_3 - a)^2 + p_4(\Re_4 - a)^2]^{1/2}, \tag{5}$$

from equation (3), after some transformations, we obtain

$$R_e \stackrel{\Delta}{=} \sum_{i=1}^{4} p_i R_i^* = (p_2\Re_2 + p_3\Re_3 + p_4\Re_4)x - 1 = ax - 1, \tag{6}$$

$$\begin{aligned} \sigma &\stackrel{\Delta}{=} \left[\sum_{i=1}^{4} p_i (R_i^* - R_e)^2\right]^{1/2} = \\ &= [p_1 a^2 + p_2(\Re_2 - a)^2 + p_3(\Re_3 - a)^2 + p_4(\Re_4 - a)^2]^{1/2} x = bx. \end{aligned} \tag{7}$$

Thus, the expected return R_e from the investment into the catastrophe bond, as well as the standard deviation σ of this return - have been expressed as linear functions of the unknown variable $x = N / P_*$, what will significantly simplify the considerations given in the succeeding sections. Thanks to this form of R_e and σ, some analytical solutions to our valuation problem will also be possible (for some special cases).

From (6)-(7) it also follows that

$R_e > -1$ (as $ax > 0$), and

$$\sigma = \frac{b}{a}(1 + R_e);$$

i.e. the standard deviation σ is a linear function of the expected return R_e - in case of the catastrophe bond considered.

A generalization of the results given by (3)-(7) to the case of n-year coupon-paying bond will be given in Section 9.

3. The safety level as a measure of investment risk

For the purpose of the valuation model considered we will use the concept of the so called *safety level* of a security (or a portfolio of securities). The idea was originated by Kataoka within his „safety first" portfolio model (Elton, Gruber, 1995) and then modified by Kulikowski (1998 a,b), who elaborated the „two-factor utility approach" to portfolio optimization problems.

Let us define

R_{K} - the security safety level, i.e. the rate of return from the investment into the security in „the worst case";

p_{K} - the probability of „the worst case" - a fixed small positive number, e.g. $p_K = 0.05$.

The formal definition of the *safety level* R_κ is

$$P(R^* \leq R_\kappa) \stackrel{\Delta}{=} p_\kappa, \tag{8}$$

where $P(\cdot)$ - the probability, R^* - the rate of return (a random variable of a given probability distribution).

In other words, considering that p_κ is small, we can define the value of R_κ as the rate of return that „almost never" be crossed downwards by the real (random) rate return R^*. For example, for $p_\kappa = 0.05$, the probability that R^* will be greater or equal to R_κ is $(1-p_\kappa) = 0.95$.

From the definition (8) it follows that for a given security, i.e. for a given probability distribution of the rate of return R^*, the value of safety level R_κ is strictly determined by the value of the „worst case" probability p_κ. Thus, it is intuitively obvious that - for a fixed in advance value of p_κ - the greater value of R_κ means the lower level of risk attached to the security. And conversely - when two different securities are compared, the security with the lower safety level R_κ is more risky.

Of course, the security safety level R_κ depends upon the prescribed value of „the worst case" probability p_κ. The value p_κ can be considered as some measure of the investor's degree of risk aversion and it could be different for various groups of investors. From the above it follows that the security *safety level* R_κ is some measure of the investment risk but this measure depends upon not only the distribution characteristics of the security itself but also it is influenced by the investors' attitude to risk, expressed by the value of p_κ.

Now, we can modify the described approach to the investment risk as follows.

Introducing the notation:

κ - the investor's risk attitude coefficient, being a subjective measure of the investor's „degree of risk aversion"; $\kappa \geq 0$, we define the security *safety level* as

$$R_\kappa \stackrel{\Delta}{=} R_e - \kappa\sigma, \tag{9}$$

where R_e - the expected value of the rate of return R^*, and σ - the standard deviation.

From the definition (9) it follows that the investor's risk coefficient κ can be interpreted as some „cost" of bearing the investment risk represented by σ. Moreover, from (8) and (9) we have

$$P(R^* \leq R_\kappa) = P(R^* \leq R_e - \kappa\sigma) = p_\kappa. \tag{10}$$

Let us assume that all the risky securities are characterized by the same probability distribution of their rates of return R^*; e.g. let the distribution be Gaussian. Thus, from (10) it follows that the investor's risk coefficient κ is strictly determined by the value of „the worst case" probability p_κ. To show it more legibly, let us denote

$$X = \frac{R^* - R_e}{\sigma} \quad \text{- the standardized value of the random variable } R^*. \tag{11}$$

Thus, $R^* = R_e + \sigma X$, and from (10) we have

$$P(R^* \le R_e - \kappa\sigma) = P(R_e + \sigma X \le R_e - \kappa\sigma) = P(X \le -\kappa) = F(-\kappa) = p_k, \tag{12}$$

where $F(.)$ - the distribution function of a random variable. For continuous random variables, $F(.)$ is a strictly increasing, continuous and differentiable function.

We have obtained the following result: the smaller the value assigned by the investor to the probability p_κ of „the worst case", the greater is his value of the risk coefficient κ. Reciprocally, the greater value of p_κ implies the smaller value of κ. Thus, a „risk averse" investor (i.e. with a small value of p_κ) will be characterized by the greater value of his risk coefficient κ than an investor whose attitude to risk is very often called „the risk seeking".

In order to make specific the above conclusion, let us assume that the distributions of all random rates of return R^* are Gaussian. In this case, from (12) - using the statistical tables - we can determine the values of coefficient κ corresponding to some prescribed values of the probabilities p_κ, i.e.

$$p_\kappa = 0.309 \approx 1/3, \qquad \kappa = 0.5 \text{ („risk seeking")}, \tag{13}$$

$$p_\kappa = 0.159 \approx 1/6, \qquad \kappa = 1.0 \text{ („risk tolerance")}, \tag{14}$$

$$p_\kappa = 0.067 \approx 1/15, \qquad \kappa = 1.5 \text{ („risk aversion")}. \tag{15}$$

In the results given above, we have also determined (to some extent arbitrarily) the qualitative scale of the investor's attitude to risk. Namely, we have assumed that the individual who admits the level of return R^* less than the safety level R_κ not more often than 1 out of 3 times (i.e. $p_\kappa \approx 1/3$ and $\kappa = 0.5$) - is characterized by „risk seeking", the individual demanding $p_\kappa \approx 1/6$ (i.e. $\kappa = 1.0$) - is „risk tolerant", and the individual calling for $p_\kappa \approx 1/15$ (i.e. $\kappa = 1.5$) - is „risk averse".

It should be mentioned that in the real life, the matter of determining the concrete, numerical values of p_κ and thus - κ, is usually much more complicated. The values of risk coefficients κ may depend on many individual characteristics, such as the age and/or the wealth level. From the above it follows that the coefficient κ of the investor may not be stable in time. For more extensive discussion of this problem see Kulikowski (1998 a,b).

3.1. The acceptance rule

We will introduce the notion of acceptance rule. Namely, we assume that an investor can accept the security for his investment purposes only if *the security safety level* R_κ is nonnegative, i.e.

$$R_\kappa = R_e - \kappa\sigma \ge 0; \quad \text{where} \quad \kappa \ge 0. \tag{16}$$

Thus, from the formula (16) it follows that the expected return R_e from the investment should be „large enough" or the standard deviation σ (i.e. the risk) should be „sufficiently small" for this investment to be accepted for further considerations. For instance, if $\kappa = 1$, the acceptance rule simply states that the inequality $R_e \ge \sigma$ has to be satisfied.

3.2. The security safety index

Assuming that the acceptance rule is satisfied for a given security, we introduce the notion of the security safety (or assurance) index S, i.e.

$$S \stackrel{\Delta}{=} \frac{R_\kappa}{R_e} = 1 - \kappa \frac{\sigma}{R_e}; \quad \text{where} \quad S \in [0,1]. \tag{17}$$

Let us observe that the above definition is well justified only in the case, when the acceptance rule holds, i.e. when $R_e \geq \kappa\sigma$; otherwise the index S would have a negative value.

From the definition (17) it follows that the safety index S characterizes the degree of investor's confidence attached to the risky asset. For the risk-free asset (e.g. the Treasury bill) we have $\sigma = 0$ and thus $S = 1$. Observe also that the value of index S increases (up to $S = 1$) along with an increase of the return/risk ratio R_e / σ; and conversely - S decreases (down to $S = 0$), when the security *variation coefficient* defined by σ / R_e increases.

It should however be pointed out that the value of index S depends not only upon the security risk measured by the σ / R_e ratio, but also on the investor's κ coefficient, characterizing his attitude to the risk factor attached to all the investments. Therefore, the same security can have different safety indices S assigned, depending on the individuals' risk characteristics.

4. Two-factor utility function

We will shortly present and interpret the two-factor utility approach to investment problems. As it was pointed out by Kulikowski (1998), the approach stems from the belief that in order to properly describe the investor's behaviour one should take into account at least two factors: the expected return R_e and „the worst case" return R_κ. It should be noticed that from the formal point of view, the risk measure, in the single-factor utility investment models, enters into the constraints. In the two-factor approach it is incorporated - in the form of the security safety level R_κ - into the structure of the utility function.

For the purpose of further considerations, we will assume that the investor's utility function is of the *Cobb-Douglas form*; i.e.

$$U = U(R_e, R_\kappa) = \gamma R_e^{1-\beta} R_\kappa^{\beta}, \tag{18}$$

where $\gamma > 0$, $\beta \in [0,1]$ - given constants; and $R_\kappa = R_e - \kappa\sigma$ - the security safety level.

It follows from the above that the utility function (18) is assumed to be homogeneous of degree one; such functions are called *constant return to scale* (CRS). The utility function $U = U(R_e, R_\kappa)$ can be easily transformed into the form $U = U(R_e, S)$, where S - the security safety index.

Namely, from (18) we have $$U = \gamma R_e^{1-\beta} R_\kappa^{\beta} = \gamma R_e \left(\frac{R_\kappa}{R_e} \right)^{\beta}; \tag{19}$$

and from (17), (19), we finally obtain $$U = U(R_e, S) = \gamma R_e S^{\beta}, \quad \gamma > 0, \quad \beta \in [0,1]. \tag{20}$$

The derived equation (20) for the investors' utility function will have an essential meaning for our further analysis.

We can now define the *elasticity coefficient* ε_S associated with the utility function $U = U(R_e, S)$; i.e.

taking into account $\frac{\partial U}{\partial S} = \gamma \beta R_e S^{\beta-1}$, after some transformations, we obtain

$$\varepsilon_S \overset{\Delta}{=} \frac{dU}{U} / \frac{dS}{S} = \frac{\partial U}{\partial S}\frac{S}{U} = \beta. \tag{21}$$

Thus, considering (21), we have obtained:

$\beta = \frac{dU}{U} / \frac{dS}{S}$, which allows for the following interpretation of parameter $\beta \in [0,1]$.

Parameter β is equal to the elasticity ε_S of the investor's utility function U with respect to the utility factor S. This means that the value of β can be estimated as the percentage increment $(dU / U) \times 100$ of the investor's utility caused by the increment of the security safety index S by one percent. Of course, in general, the method of estimating the parameter β, suggested above, could be extremely difficult for investors. Nevertheless, one important conclusion results from the interpretation given above. It is that the parameter β represents the investor's sensitivity to risk, in the meaning that the greater the β the more of investor's attention is paid to the safety factor S. Thus, the greater investor's parameter β means the higher degree of this investor's *risk aversion*.

5. The preference rule for the catastrophe bond case

Assuming that the rule of acceptance (i.e. $R_\kappa \geq 0$) has been for a given investment verified, we turn to the preference rule which constitutes the second part of the approach considered.

In the process of valuation of catastrophe bonds, the procedure is as follows. We consider possible investments into two alternative securities:

(i) The risk-free Treasury bond with the rate of return R_f (over the whole investment period) and $\sigma = 0$; thus $S = 1$ and from (20), $U_f = U(R_f, 1) = \gamma R_f$.

(ii) The catastrophe bond with the expected rate of return R_e and the standard deviation σ given by equations (6), (7); thus, we can determine

$$S = 1 - \kappa \frac{\sigma}{R_e}; \qquad \text{and} \qquad U = U(R_e, S) = \gamma R_e S^\beta .$$

The investment (ii) is preferred to the investment (i), when $U \geq U_f$, i.e. $\gamma R_e S^\beta \geq \gamma R_f$, and thus (*the preference rule*):

$$R_e \geq \frac{R_f}{S^\beta}. \tag{22}$$

According to (22), the investment in the catastrophe bond can be preferred, if the expected rate of return R_e resulting from this decision is not less than the risk-free rate of return R_f divided by S^β. So, if the catastrophe bond safety index S decreases, it must be compensated (in order for the bond to be preferred) by a large enough increase of the expected return R_e.

It should be also noticed that from (22), taking into account that $S^\beta \le 1$, we have

$$R_e \ge R_f , \tag{23}$$

which forms a *necessary condition* for preference of the catastrophe bond over the risk-free investment.

5.1. The separation curve

By expressing the inequality (22) in the form of equality we will find some limit condition, when the investments in a catastrophe bond and in the risk-free bond are equivalent. Thus, from (22) we have

$$R_e = R_f S^{-\beta}; \quad \text{where} \quad S = 1 - \kappa \frac{\sigma}{R_e}. \tag{24}$$

So,
$$R_e = R_f \left(1 - \kappa \frac{\sigma}{R_e}\right)^{-\beta}. \tag{25}$$

Solving the equation (25) with respect to σ, and denoting the resulting value by σ_g, we obtain

$$\sigma_g = \sigma_g(R_e) = \frac{R_e}{\kappa}\left[1 - \left(\frac{R_f}{R_e}\right)^{1/\beta}\right], \tag{26}$$

where $R_e \ge R_f$ (see (23)), and $\sigma_g(R_e)$ - the threshold value of standard deviation σ as a function of the expected rate of return R_e.

Equation (26) defines a *separation curve* on the plane $(0, R_e, \sigma)$ in the meaning that:

- for every R_e and $\sigma < \sigma_g(R_e)$, the investment in the catastrophe bond is preferred to the investment in the risk-free bond;
- for every R_e and $\sigma > \sigma_g(R_e)$, the opposite case takes place, i.e. the investment in the risk-free bond is more preferred.
- for every R_e and $\sigma = \sigma_g(R_e)$, the two investments are equivalent in view of the investor's utility function.

It follows from the above that considering the equation (26) and setting the values $R_e = ax - 1$ and $\sigma = \sigma_g = bx$ defined for a catastrophe bond by (6) and (7), we can solve the equation (26) with respect to the variable $x = N / P_*$. Thus, we can find some threshold value for the price $(P_*)_g$ of the catastrophe bond, having such a property that for every price $P_* < (P_*)_g$, the investment in the catastrophe bond is more preferred than the risk-free investment.

In other words, the resulting threshold value of $(P_*)_g$ is the solution of the considered catastrophe bond valuation problem. For a graphical illustration of the separation curves for different values of coefficient β as well as for the further discussion see Jakubowski (2002b).

6. Proposal of an evaluation of the investors' risk parameters

From the approach presented in the previous sections it follows that we have assumed two subjective parameters characterizing investors' attitude to the risk factor. These are the $\kappa > 0$ and $\beta \in [0,1]$ parameters. Recalling the definition (17) of the security safety index S, i.e.

$$S = 1 - \kappa \frac{\sigma}{R_e}, \tag{27}$$

we can observe that index S depends upon a subjective coefficient κ characterizing the degree of the individual's risk aversion and the objective measure σ / R_e characterizing the security risk. Assuming that the security volatility (σ / R_e) is given, we can see that when the coefficient κ increases, the security safety index S decreases what directly follows from the increasing of the investor's *risk aversion*.

On the other hand, recalling the *preference rule* (22), i.e.

$$R_e \geq \frac{R_f}{S^\beta}, \tag{28}$$

and noting that S^β is the decreasing function of β for a given $S \in [0,1]$, we can formulate the following conclusion. The greater value of β implies the greater value of the risky security expected return R_e - necessary for this security to be more preferred than the risk-free investment. Thus, the greater parameter β means the higher level of the degree of investor's *risk aversion*.

It follows from the above that the parameters β and κ, as influencing in the same direction the individual's degree of risk aversion, should be in some way interconnected. Thus, taking into account some characteristic values of $\kappa = 0.5$, 1.0 and 1.5, analysed in Section 3 - see formulae (13) - (15), and considering that the values of β are changing in the interval [0,1], we can arbitrarily propose the following relation between β and κ investors' risk parameters:

$$\kappa = 0.5 + \beta. \tag{29}$$

Thus, for some prescribed values of β ranging from 0 to 1, we obtain some subjective scale of the individual's attitude to risk:

$$\beta = \begin{cases} 0.00; & \kappa = 0.50 \quad - \text{risk seeking} \qquad (p_\kappa \approx 1/3) \\ 0.25; & \kappa = 0.75 \quad - \text{weak risk seeking} \\ 0.50; & \kappa = 1.00 \quad - \text{risk tolerance} \qquad (p_\kappa \approx 1/6) \\ 0.75; & \kappa = 1.25 \quad - \text{weak risk aversion} \\ 1.00; & \kappa = 1.50 \quad - \text{risk aversion} \qquad (p_\kappa \approx 1/15) \end{cases} \tag{30}$$

The proposed subjective scale as given by (30), implies the specific forms of the security safety indices, the investors' utility functions as well as the forms of preference rules; given by (22).

7. Deriving the valuation equations

We will present the final derivations concerning the catastrophe bond valuation. Let us denote by

$$R_f \overset{\Delta}{=} R_3 = (1+r_f)^3 - 1 \text{ - the risk-free rate of return over the period of } n = 3 \text{ years;}$$

and, for the general case of the bond with n coupon periods,

$$R_f \overset{\Delta}{=} R_n = (1+r_f)^n - 1.$$

Also, in order to simplify the notation, when speaking about the threshold values, e.g. σ_g, $(P_*)_g$, the index „ g " will be skipped.

7.1. The rule of acceptance

First, let us specify the *rule of acceptance* for the analysed bond; from (16) we have

$$R_\kappa = R_e - \kappa\sigma \geq 0; \quad \text{where} \quad \kappa \geq 0. \tag{31}$$

Recalling the equations (6) and (7), for the catastrophe bond, we have

$$R_e = ax - 1, \quad \sigma = bx, \tag{32}$$

where $x = N / P_*$; and coefficients a, b are known and determined by the formulae (4), (5). Thus, from (31) and (32), the following inequality must hold

$$(a - \kappa b)x - 1 \geq 0. \tag{33}$$

The necessary condition: For (33) to be satisfied, it is necessary that $a - \kappa b > 0$, so

$$\kappa < \kappa_{\max} = \frac{a}{b}. \tag{34}$$

We have obtained that in order for the acceptance rule to be satisfied, the investor's risk coefficient κ should not be „too large".

The sufficient condition: Assuming that the condition (34) holds, from (33) we have

$$x \geq \frac{1}{a - \kappa b}, \quad \text{and setting} \quad x = \frac{N}{P_*},$$

$$P_* \leq (a - \kappa b)N. \tag{35}$$

The above inequality is the sufficient condition for the bond to be accepted for further analysis. As it follows from (35), the price P_* of such a bond should be „sufficiently small".

7.2. The rule of preference

Now, let us consider the preference rule for the analysed case of catastrophe bond.

The necessary condition: In Section 5 we have found out that for the preference rule to be satisfied it is necessary that

$$R_e \geq R_f ; \tag{36}$$

considering $R_e = ax - 1$, we obtain $x \geq \frac{1+R_f}{a}$, and setting $x = \frac{N}{P_*}$,

$$P_* \leq \frac{a}{1+R_f} N . \tag{37}$$

Combining the sufficient condition for the acceptance rule given by (35) with the necessary condition (37) formulated for the preference rule, we obtain that for the catastrophe bond to be considered for a possible investment, it is necessary that its price P_* is constrained by the following upper bound:

$$P_* \leq \min \left\{ (a - \kappa b)N; \ \frac{a}{1+R_f} N \right\} . \tag{38}$$

The expression (38) constitutes some introductory condition, which should be checked out before entering into further analysis.

The sufficient condition: For $\beta \in (0,1]$, assuming that the necessary condition (36) holds, the general form of the equation of *separation curve* is given by (26), i.e.

$$\sigma = \frac{R_e}{\kappa} \left[1 - \left(\frac{R_f}{R_e} \right)^{1/\beta} \right]; \quad \text{where} \quad R_e \geq R_f . \tag{39}$$

From (32) and (39), after some transformations, we obtain

$$(ax-1)^{1-\beta} \left[(a - \kappa b)x - 1 \right]^{\beta} = R_f , \qquad \text{where} \quad x = \frac{N}{P_*} . \tag{40}$$

It is now obvious that solving the equation (40) with respect to the unknown variable x, and taking into account that $P_* = N / x$, we will find the threshold value for the price P_* of the catastrophe bond. In general, the equation (40) is a nonlinear equation in one variable; so some numerical procedure (e.g. *Newton's* scheme) should be applied in this case. Thus, the analysed valuation problem has been solved.

Assuming some typical values for the elasticity coefficient β and „putting all the pieces together", we will present the equation (40) in the more legible forms; giving in some cases, also its analytical solutions. The set of solutions will also be completed with the case of $\beta = 0$. Thus, we have:

7.3. The valuation equations

For $\beta = 0$: from (38), we have

$$P_* = \min\left\{(a-\kappa b)N;\quad \frac{a}{1+R_f}N\right\}. \tag{41}$$

For $\beta = 1/4 = 0.25$:

$$(ax-1)^3\,[(a-\kappa b)x-1] = R_f^4; \quad \text{and} \quad P_* = N/x. \tag{42}$$

For $\beta = 1/3 = 0.33$:

$$(ax-1)^2\,[(a-\kappa b)x-1] = R_f^3; \quad \text{and} \quad P_* = N/x. \tag{43}$$

For $\beta = 1/2 = 0.50$:

$(ax-1)\,[(a-\kappa b)x-1] = R_f^2$; thus (the analytical solution)

$$x = \frac{(2a-\kappa b)+[(\kappa b)^2+4a(a-\kappa b)R_f^2]^{1/2}}{2a(a-\kappa b)}, \quad \text{and} \tag{44}$$

$$P_* = \frac{N}{x} = \frac{2a\,(a-\kappa b)}{(2a-\kappa b)+[(\kappa b)^2+4a\,(a-\kappa b)\,R_f^2]^{1/2}}\,N.$$

For $\beta = 2/3 = 0.67$:

$$(ax-1)\,[(a-\kappa b)x-1]^2 = R_f^3; \quad \text{and} \quad P_* = N/x. \tag{45}$$

For $\beta = 3/4 = 0.75$:

$$(ax-1)\,[(a-\kappa b)x-1]^3 = R_f^4; \quad \text{and} \quad P_* = N/x. \tag{46}$$

For $\beta = 1$: $\quad (a-\kappa b)x-1 = R_f; \quad \text{and} \quad P_* = \frac{N}{x} = \frac{a-\kappa b}{1+R_f}N.$ (47)

One can observe some „symmetry" in the form of equations presented above. It should be pointed out that the equations (41)-(47) have the same form also in the general case of n-coupon catastrophe bond. This follows directly from the fact that the formulae $R_e = ax-1$, $\sigma_e = bx$, are linear functions of x. One thing will be only different - this is the values of the coefficients a and b that will have a more general form; see Jakubowski (2002b).

8. The examples of catastrophe bond valuation

Let us consider now the series of illustrative computational examples related to the valuation process investigated in the previous sections. For the purpose of analysis, we assume that the catastrophe security is a 3-year coupon paying bond characterized by the following cash flows (see Fig. 1):

$C = 10$ PLN (p.a.) - the coupons, $N = 100$ PLN - the principal payment,

where PLN – denotes Polish zloty.

Let the market (default) risk-free interest rate be $r_f = 10\%$ (p.a.), and thus - equal to the bond nominal interest rate i ; i.e.

$$i = \frac{C}{N} = 10\% = r_f .$$

In such a case, from the formula (1) it can be easily shown that the present value P_o of the risk-free bond (with the same cash flows as those assumed for the catastrophe bond) is equal to the bond principal value N ; i.e. $P_o = N = 100$ PLN.

Taking into account the notation introduced in Section 2, we have:

$$R_1 = r_f = 0.1000; \; R_2 = (1+r_f)^2 - 1 = 0.2100; \; R_3 = (1+r_f)^3 - 1 = 0.3310; \qquad (48)$$

Thus, for the value of risk-free rate of return R_f over the whole investment horizon of 3 years, we have

$$R_f = R_3 = 0.3310 = 31.10\% . \qquad (50)$$

We also have

$$\Re_2 = (1+R_2)i = 0.1210; \quad \Re_3 = (1+R_1)i + (1+R_2)i = 0.2310; \qquad (51)$$

$$\Re_4 = (1+R_1)i + (1+R_2)i + (1+i) = 1.3310.$$

We will consider a few scenarios of the investment outcomes depending on the value of the estimated (assumed) probability α of a flood occurrence. First, let us consider the „20-year water", i.e. $\alpha = 0.05$ (p.a.). For the values of scenario probabilities $p_1, \ldots, p_4$, we obtain

$$\begin{array}{ll} p_1 = \alpha = 0.0500, & p_2 = (1-\alpha)\alpha = 0.0475, \\ p_3 = (1-\alpha)^2\alpha = 0.0451, & p_4 = (1-\alpha)^3 = 0.8574. \end{array} \qquad (52)$$

Thus, from (51), (52) the coefficients a and b given by eqs. (4) and (5), are equal

$$a = 1.1574, \qquad b = 0.4273. \qquad (53)$$

and we can determine the formulae for expected return R_e and σ ; i.e. from (6) and (7)

$$R_e = ax - 1 = 1.1574x - 1; \qquad \sigma = bx = 0.4273x; \qquad (54)$$

where $x = N / P_* = 100 / P_*$.

According to the conclusions given in Section 6, we will consider three values of the investors' risk coefficient κ, i.e.

$$\begin{aligned} &\kappa = 0.5 \quad (\text{and } \beta = 0.0) \quad \text{for the risk seeking;} \\ &\kappa = 1.0 \quad (\text{and } \beta = 0.5) \quad \text{for the risk tolerance;} \\ &\kappa = 1.5 \quad (\text{and } \beta = 1.0) \quad \text{for the risk aversion.} \end{aligned} \tag{55}$$

The cases $\kappa = 0.5$ and $\kappa = 1.5$ are the extreme ones, while the case $\kappa = 1.0$ (called „risk tolerance") is some intermediate case related to the degree of investors' risk aversion.

Now, we verify whether the *necessary condition* of the acceptance rule is satisfied; from (34) and (53) we obtain

$$\kappa \leq \kappa_{\max} = \frac{a}{b} = \frac{1.1574}{0.4273} = 2.7086\,. \tag{56}$$

Thus, for the assumed values of $\kappa = 0.5,\ 1.0,\ 1.5$, the condition (56) holds. It should be at this point noticed that, although we obtained rather a large value of $\kappa_{\max} = 2.6996$, this value - for a given rates of return R_1, R_2, R_3 - is highly influenced by the value of probability of flood occurrence α. It can be easily verified that for $\alpha = 0.10$ (i.e. for „10-year water"), we would obtain, from the analogous calculations, $\kappa_{\max} = 1.8337$. So, for the „very risk averse" investor characterized by e.g. $\kappa = 2.0$, the catastrophe bond should not be considered as a possible investment, because its safety level R_κ would have in this case a negative value. Moreover, it would take place for every value of $x = N / P_*$ and thus – for every price P_*; see inequality (33).

The *necessary conditions* of the rule of preference are in the considered case as follows; from (38), (50), (53) and (55) we obtain:

$$\begin{aligned} \text{For}\quad &\kappa = 0.5, \qquad P_* \leq 86.95 \text{ PLN}, \\ &\kappa = 1.0, \qquad P_* \leq 73.01 \text{ PLN}, \\ &\kappa = 1.5, \qquad P_* \leq 51.65 \text{ PLN}. \end{aligned} \tag{57}$$

In further calculations we will concentrate on the case of the „risk tolerant" investor, characterized by the parameters $\beta = 0.5$ and $\kappa = 1.0$.

From the *sufficient condition* of preference rule represented for this case by eq. (44), taking into account the values of R_f, a, b given by (50) and (53), we obtain

$$x = 1.5568, \quad \text{and} \quad P_* = \frac{N}{x} = \frac{100}{1.5568} = 64 \text{ PLN}\,. \tag{58}$$

Thus, the threshold value of the price $P_* = 64.23$ PLN is the solution of the catastrophe bond valuation problem, in the case under consideration.

Moreover, using (58) and (54) we obtain that the values of expected return R_e and the standard deviation σ are equal to

$$R_e = ax - 1 = 0.8018 = 80.18\% \quad \text{(per 3 years)}, \tag{59}$$

$$\sigma = bx = 0.6651 = 66.51\% \quad \text{(per 3 years)}. \tag{60}$$

The bond *safety level* R_κ is equal to

$$R_\kappa \stackrel{\Delta}{=} R_e - \kappa\sigma = 80.18\% - 1.0 \times 66.51\% = 13.67\% \quad \text{(per 3 years)}. \tag{61}$$

The bond *safety index* S is equal to

$$S \stackrel{\Delta}{=} \frac{R_\kappa}{R_e} = 1 - \kappa \frac{\sigma}{R_e} = 1 - 1.0 \times \frac{66.51}{80.18} = 0.1705\,. \tag{62}$$

The *price discount* of the analysed catastrophe bound (as compared to the risk-free bond) is equal to

$$D_* \stackrel{\Delta}{=} \frac{P_* - P_0}{P_0} \times 100 = \frac{64.23 - 100.00}{100.00} \times 100 = -35.77\%\,. \tag{63}$$

For the catastrophe bond, we will also define some measure of the risk premium; i.e.

– *The expected risk premium*:

$$\Pi \stackrel{\Delta}{=} R_e - R_f = 80.18\% - 33.10\% = 47.08\% \quad \text{(per 3 years)}. \tag{64}$$

The expected risk premium Π defined by eq. (64) is often called „the expected excess return"; or "the default premium"; see Litzenberger, Beaglehole, et al. (1996), Canabarro, Finkemeier, et al. (1998).

Closing our exemplary calculations for ($\beta = 0.5, \kappa = 1.0$) case, we will present *the scenario* of possible investment outcomes, resulting from the determined bond price $P_* = 64.23$ PLN. From the general formula (3), considering the values of $\Re_2$, $\Re_3$, $\Re_4$ and x as well as p_1, p_2, p_3, p_4, given by (51), (58) and (52), respectively, we obtain

$$R^* = \begin{cases} R_1^* = -1 \\ R_2^* = \Re_2 x - 1 \\ R_3^* = \Re_3 x - 1 \\ R_4^* = \Re_4 x - 1 \end{cases} = \begin{cases} -100.0\% & ; \text{ w.p. } p_1 = 0.0500 \\ -81.2\% & ; \text{ w.p. } p_2 = 0.0475 \\ -64.0\% & ; \text{ w.p. } p_3 = 0.0451 \\ +107.2\% & ; \text{ w.p. } p_4 = 0.8574 \end{cases} \tag{65}$$

From the methodology presented in this paper follows that the scenario (65) of possible outcomes resulting from the investment in the catastrophe bond is equivalent - in the meaning of the utility function adopted - to the investment in the risk-free 3-year bond, yielding the return $R_f = 33.1\%$ with the probability 1. The results obtained seem to be intuitively reasonable.

On Figure 2 we present the results of analogous calculations carried out for different assumptions on the probability of flood occurrence, i.e. for $\alpha = 0.01$, $\alpha = 0.05$ and $\alpha = 0.10$. The appropriate scenarios of possible investment outcomes are shown and compared to the risk-free investment.

9. The generalization of results

The approach presented in the previous sections concerned mainly the 3-year catastrophe bond valuation. The generalization of results to the case of n-coupon bond is straightforward. Especially, the basic set of equations (2)-(7) determined in Section 2 for the 3-years investment horizon, can be easily extended to the general case of the bond with n-years maturity period. The counterparts of these equations are as follows.

The scenario of possible investment outcomes in the case of the n-year catastrophe bond is

$$R^* = \begin{cases} R_1^* = \dfrac{0-P_*}{P_*} = -1 = -100\% & ;\quad \text{w.p.} \quad p_1 = \alpha \\ R_2^* = \dfrac{C(1+r_f)^{n-1} - P_*}{P_*} & ;\quad \text{w.p.} \quad p_2 = (1-\alpha)\alpha \\ R_3^* = \dfrac{C(1+r_f)^{n-1} + C(1+r_f)^{n-2} - P_*}{P_*} & ;\quad \text{w.p.} \quad p_3 = (1-\alpha)^2\alpha \\ \vdots & \qquad\qquad\quad \vdots \\ R_n^* = \dfrac{C(1+r_f)^{n-1} + ... + C(1+r_f) + C - P_*}{P_*} & ;\quad \text{w.p.} \quad p_n = (1-\alpha)^{n-1}\alpha \\ R_{n+1}^* = \dfrac{C(1+r_f)^{n-1} + ... + C(1+r_f) + (C+N) - P_*}{P_*} & ;\quad \text{w.p.} \quad p_{n+1} = (1-\alpha)^n \end{cases} \tag{66}$$

and $\sum_{i=1}^{n+1} p_i = 1$.

All the notations given above as well as considered in the further parts of this section are analogous to those from the Section 2. In particular, p_{n+1} means the probability of flood non-occurrence in the whole period of n-years, and R^* is the rate of return over n-years - a discrete random variable with the realizations given by $R_1^*,...,R_{n+1}^*$, under the fixed scenario probabilities $p_1,...,p_{n+1}$.

Moreover, as previously, it is assumed that all the coupon payments C are being reinvested under the (default) risk-free market interest rate r_f - up to the end of investment horizon of n years.

Similarly to the derivations given in Section 2, the equation (66) can be transformed into a more convenient form. As previously, we denote

$$x = \frac{N}{P_*}, \quad i = \frac{C}{N}, \tag{67}$$

$$R_1 = r_f, \quad R_2 = (1+r_f)^2 - 1, \quad R_3 = (1+r_f)^3 - 1, \; ... \; , R_n = (1+r_f)^n - 1, \tag{68}$$

– the risk-free interest rates over the periods of $1,2,3,...,n$ years; in particular,

$$R_f = R_n = (1+r_f)^n - 1 \text{ - the risk-free interest rate over the whole investment period of } n\text{-years.} \tag{69}$$

Let us also denote:

$$\begin{aligned}
&\Re_2 = (1+R_2)i, \\
&\Re_3 = (1+R_1)i + (1+R_2)i, \\
&\quad \vdots \\
&\Re_n = (1+R_1)i + (1+R_2)i + \ldots + (1+R_{n-1})i, \\
&\Re_{n+1} = (1+R_1)i + (1+R_2)i + \ldots + (1+R_{n-1})i + (1+i)
\end{aligned} \tag{70}$$

Considering the above notation, after some transformations, the formula (66) can be expressed as

$$R^* = \begin{cases} R_1^* = -1 & ; \quad \text{w.p.} \quad p_1 = \alpha \\ R_2^* = \Re_2 x - 1 & ; \quad \text{w.p.} \quad p_2 = (1-\alpha)\alpha \\ R_3^* = \Re_3 x - 1 & ; \quad \text{w.p.} \quad p_3 = (1-\alpha)^2 \alpha \\ \quad \vdots & \qquad\qquad \vdots \\ R_n^* = \Re_n x - 1 & ; \quad \text{w.p.} \quad p_n = (1-\alpha)^{n-1} \alpha \\ R_{n+1}^* = \Re_{n+1} x - 1 & ; \quad \text{w.p.} \quad p_{n+1} = (1-\alpha)^n \end{cases} \tag{71}$$

The formulae for the expected value R_e and the standard deviation σ of the random rate of return R^* are as follows.

Let us denote

$$a = p_2 \Re_2 + p_3 \Re_3 + \ldots + p_n \Re_n + p_{n+1} \Re_{n+1}, \tag{72}$$

$$b = [p_1 a^2 + p_2 (\Re_2 - a)^2 + \ldots + p_n (\Re_n - a)^2 + p_{n+1} (\Re_{n+1} - a)^2]^{1/2}. \tag{73}$$

Thus, from (71)-(73), after some transformations, we obtain

$$R_e \overset{\Delta}{=} \sum_{i=1}^{n+1} p_i R_i^* = ax - 1, \tag{74}$$

$$\sigma \overset{\Delta}{=} \left[\sum_{i=1}^{n+1} p_i (R_i^* - R_e)^2 \right]^{1/2} = bx; \quad \text{where} \quad x = \frac{N}{P_*}. \tag{75}$$

Let us observe that the expressions (74), (75) for the expected return R_e and σ are linear functions of the variable x as it was in the case of 3-year bond; see eqs. (6) and (7). The only difference is that the coefficients a and b defined by (72), (73) have some more general form as compared with the analogous equations (4), (5), determined in Section 2. Therefore, as it was already mentioned, all other parts of the valuation model considered in Sections 3-7 for the 3-year catastrophe bond are - in the case of n-year bond - identical.

Concluding, it should be pointed out that for the practical purposes, it would be more advantageous for the catastrophe bond issuer to consider a rather long-term investment horizon (i.e. much more longer than 3 years). This follows from the fact that the issue of the long-term insurance-linked securities (e.g. 10-15 year bonds) would allow for spreading the catastrophic risk consequences over a longer time period.

10. Conclusions

In the paper, we have considered the problem of valuation of catastrophe bonds. In this case, one main question arises. It is how much of the *default premium* (i.e. a difference between the expected and the risk-free rates of return) should be paid to an investor in order to compensate for his risk, attached to the consequences of catastrophe, which can occur with some estimated probability.

One possible approach refers in such a case to a rating system for bonds established by *Moody's*, *Standard & Poor's* or *Fitch IBCA*, for large bond-issuing organizations mainly. In many cases, e.g. for bonds issued by small municipalities or other institutions, these ratings are not available. Moreover, even if such a rating is provided for a particular bond, this is an evaluation of this bond's risk expressed in a qualitative scale only (e.g. *Moody's* ratings Aa, Baa, Ca, etc.). And it is still not clear how to transform these qualitative ratings into the quantitative measure of the risk premium that should be assigned to the bond under consideration. In other words, the methodology of determining the price discount of a risky bond as compared to the price of the (default) risk-free bond with the same financial cash flows - is still poorly developed. Only some empirical studies have been undertaken for this case; Fabozzi (2000), Elton, Gruber (1995).

In all such situations, an investor can evaluate a risky bond using the methodology proposed by the author. In the valuation model worked out, the notion of the bond *safety level* is used to solve the problem of acceptance/rejection of a risky bond by the investor, whose strategy is characterized by the two-factor utility function. Although, the application of the methodology considered is still related to some subjective assessments of the parameters of the investors' attitude to risk, the proposed model of decision support is, in the author's option, a meaningful tool in comparison with the present solutions in this area.

The need for a systematically developed and theoretically well justified methodology for valuation of the catastrophe bonds follows also from the fact that the insurance-linked securities (ILS) market is still in the early stages of its development. The market prices of the catastrophe bonds have not yet reached equilibrium levels. The major ILS deals indicate a spread over LIBOR between 367 basis points (*Trinity Re* bond) and 576 basis points (*Res Re* bond); see Cholnoky, Zief, et al. (1998). Thus, there are no well developed market benchmarks for pricing the catastrophe bonds via comparative means. This additionally justifies the importance of theoretical investigations on the valuation of risky instruments, such as the approach presented in this paper.

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The list of Greek Symbols:

α - alpha

β - beta

γ - gamma

σ - sigma

κ - kappa

The list of Gothic symbols:

$\mathfrak{R}_2, \mathfrak{R}_3, \mathfrak{R}_4, \ldots, \mathfrak{R}_n, \mathfrak{R}_{n+1}$ – Gothic $R_2, R_3, R_4, \ldots, R_n, R_{n+1}$.

The list of figures:

Notation: P_* - the price of catastrophe bond; C - the coupon; N - the bond principal value; R_i^* - the anticipated rate of return conditioned upon the flood occurrence or non-occurrence; $i = 1,\ldots,4$.

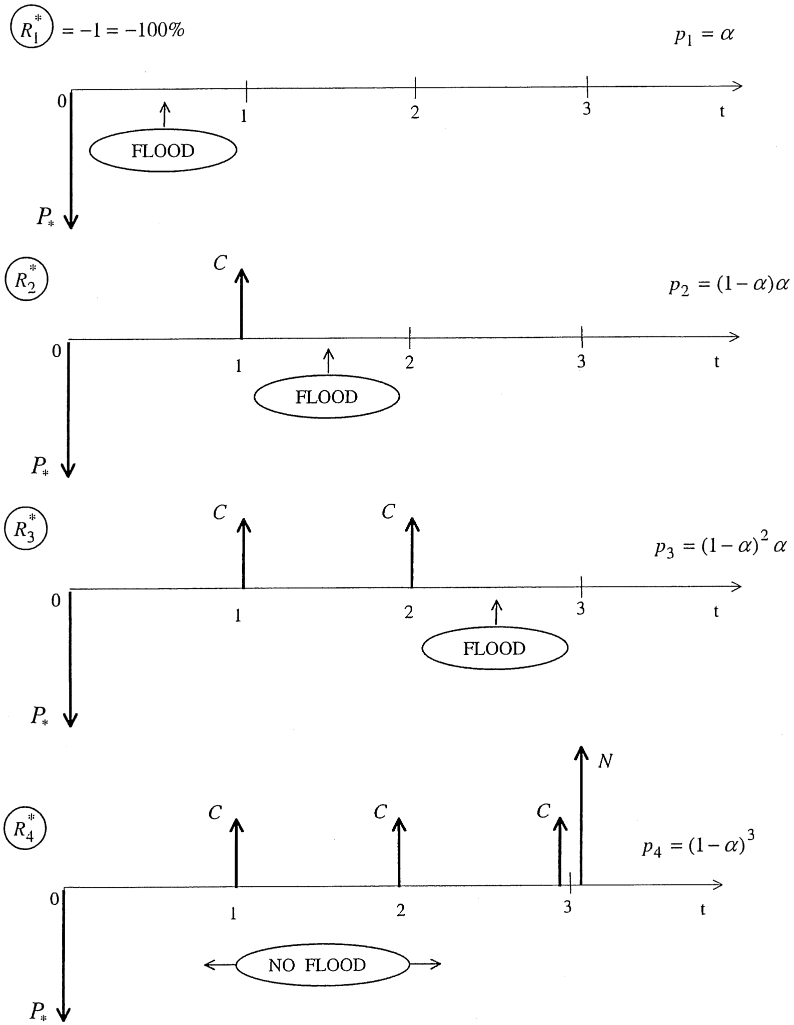

Fig. 1. The cash flows of the catastrophe bond conditioned upon the flood occurrence or non-occurrence in the successive years; α - the probability of flood per 1 year;
p_1, p_2, p_3 - the probabilities of flood in years $t = 1,2,3$;
p_4 - the probability of no flood occurrence in the whole period of 3 years.

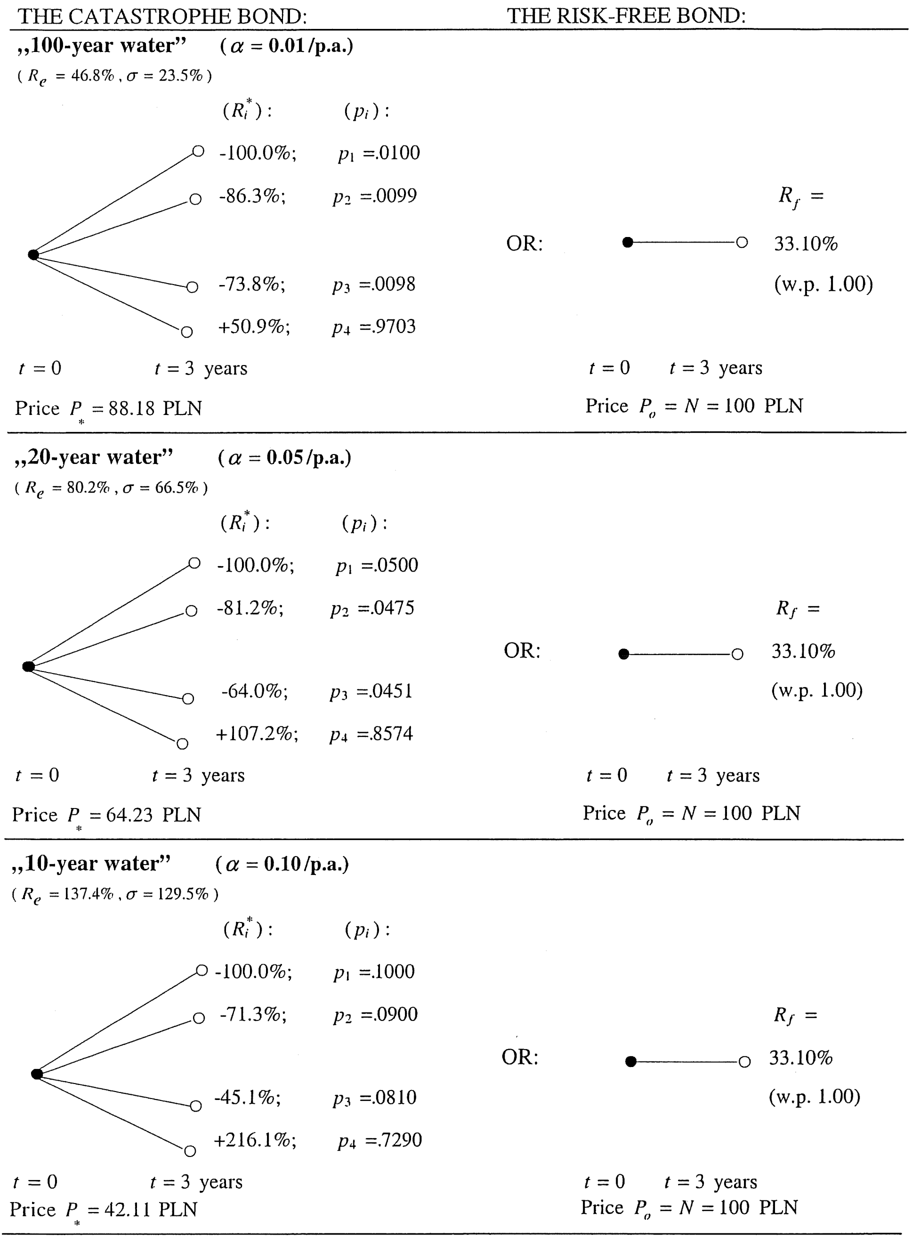

Fig. 2. The scenarios of possible outcomes resulting from investments in catastrophe bonds compared with the equivalent risk-free investments. The case of risk-tolerant investor: $\beta = 0.5$, $\kappa = 1.0$.