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ON ALLOCATION PROBLEMS FOR A COMPLEX OF PARALLEL OPERATIONS DESCRIBED BY UNCERTAIN VARIABLES

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Abstract. The paper is concerned with allocation problems for a class of parallel operations described by a relational knowledge representation. Unknown parameters in the relations are assumed to be values of uncertain variables described by certainty distributions given by an expert. Theorems concerning properties of the optimal allocation are presented. The equivalence of the solutions obtained by a direct approach to the allocation problem and by a decomposition is discussed. An example illustrates the presented method.

1. Introduction

The idea of uncertain variables has been introduced and developed as a tool for analysis and decision making in a class of uncertain systems described by traditional mathematical models and relational knowledge representations [2]–[4], [8] The uncertain variable is described by so called certainty distribution giving by an expert and characterizing his/her knowledge of approximate values of the variable. In [7], [8] it has been shown how to use the uncertain

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variables in allocation problems for complex of parallel operations. The purpose of this paper is to present new results in this area:

1. Theorems concerning properties of the optimal allocation (Sec. 2).

2. A theorem about the equivalence of the solutions obtained by a direct approach to the allocation problem and by a decomposition (Sec. 3).

We shall start with a short description of the uncertain variables and their applications for a decision problem. In the definition of the uncertain variable $\overline{x}$ we use a set of values $X \subset R^k$ (real number vector space) and two *soft properties*: "$\overline{x} \cong x$" which means that "$\overline{x}$ is approximately equal to *x*" or "*x* is the approximate value of $\overline{x}$", and "$\overline{x} \,\tilde{\in}\, D_x$" (where $D_x \subseteq X$) which means that "the approximate value of $\overline{x}$ belongs to D_x" or "$\overline{x}$ approximately belongs to D_x". The logic value of a soft property $v \in [0,1]$ is called a *certainty index* and a function $h(x) = v(\overline{x} \cong x)$ is called a *certainty distribution* ($\max h(x) = 1$). The uncertain variable $\overline{x}$ is defined by the set of values *X*, the certainty distribution $h(x)$ given by an expert, and the following definitions:

$$v(\overline{x} \,\tilde{\in}\, D_x) = \begin{cases} \max\limits_{x \in D_x} h(x) & \text{for } D_x \neq \varnothing \\ 0 & \text{for } D_x = \varnothing \end{cases},$$

$$v(\overline{x} \,\tilde{\notin}\, D_x) = 1 - v(\overline{x} \,\tilde{\in}\, D_x),$$

$$v(\overline{x} \,\tilde{\in}\, D_1 \vee \overline{x} \,\tilde{\in}\, D_2) = \max\{v(\overline{x} \,\tilde{\in}\, D_1), v(\overline{x} \,\tilde{\in}\, D_2)\},$$

$$v(\bar{x} \tilde{\in} D_1 \wedge \bar{x} \tilde{\in} D_2) = \begin{cases} \min \{v(\bar{x} \tilde{\in} D_1), v(\bar{x} \tilde{\in} D_2)\} & \text{for } D_1 \cap D_2 \neq \varnothing \\ 0 & \text{for } D_1 \cap D_2 = \varnothing . \end{cases}$$

So called *C-uncertain variable* $\bar{x}$ is defined by the set of values *X*, the function $h(x) = v(\bar{x} \cong x)$ given by an expert and the following definitions:

$$v_c(\bar{x} \tilde{\in} D_x) = \frac{1}{2}[v(\bar{x} \tilde{\in} D_x) + v(\bar{x} \tilde{\notin} X - D_x)], \qquad (1)$$

$$v_c(\bar{x} \tilde{\notin} D_x) = 1 - v_c(\bar{x} \tilde{\in} D_x),$$

$$v_c(\bar{x} \tilde{\in} D_1 \vee \bar{x} \tilde{\in} D_2) = v_c(\bar{x} \tilde{\in} D_1 \cup D_2) ,$$

$$v_c(\bar{x} \tilde{\in} D_1 \wedge \bar{x} \tilde{\in} D_2) = v_c(\bar{x} \tilde{\in} D_1 \cap D_2) .$$

In the case of *C*-uncertain variable the expert's knowledge is used in a better way but the calculations are more complicated.

Consider a static plant with the input vector $u \in U$ and the output vector $y \in Y$, described by a relation $R(u, y; x) \subset U \times Y$ where $x \in X$ is an unknown vector parameter which is assumed to be a value of an uncertain variable $\bar{x}$ with $h(x)$ given by an expert. If the relation R is not a function then the value u determines the set of possible outputs $D_y(u, x) = \{y \in Y : (u, y) \in R\}$. For the property $y \in D_y \subset Y$ required by a user, we can formulate the following **decision problem**: For the given R, $h(x)$ and D_y one should find the decision u^* maximizing the certainty index of the property: "the set of possible outputs approximately belongs to D_y". Then

$$u^* = \arg\max_{u\in U} v[D_y(u,\bar{x}) \tilde{\subseteq} D_y] = v[\bar{x} \tilde{\in} D_x(u)] = \arg\max_{u\in U} \max_{x\in D_x(u)} h(x)$$

where $D_x(u) = \{x \in X : D_y(u,x) \subseteq D_y\}$.

If $\bar{x}$ is considered as C-uncertain variable then one should determine u_c^* maximizing $v_c[\bar{x} \tilde{\in} D_x(u)]$.

2. Allocation problem

Let us consider a complex of k parallel operations described by a set of inequalities

$$T_i \le \varphi_i(u_i, x_i), \quad i = 1, 2, ..., k \tag{2}$$

where T_i is the execution time of the i-th operation, u_i is the size of a task in the problem of task allocation or the amount of a resource in the problem of resource allocation, an unknown parameter $x_i \in R^1$ is a value of an uncertain variable $\bar{x}_i$ described by a certainty distribution $h_i(x_i)$ given by an expert, $\bar{x}_1, ..., \bar{x}_k$ are independent variables, $\varphi_i(u_i, x_i)$ is a non-decreasing function of u_i in the case of task allocation and a non-increasing function of u_i in the case of resource allocation. The complex may be considered as a decision plant described in Sec. 1 where y is the execution time of the whole complex $T = \max\{T_1, ..., T_k\}$, $x = (x_1, ..., x_k)$, $u = (u_1, ..., u_k) \in \bar{U}$. The set $\bar{U} \subset R^k$ is determined by the constraints: $u_i \ge 0$ for each i and the equality $u_1 + ... + u_k = U$ where U is the total size of the task or the total amount of the resource to be distributed among the operations (Figure 1).

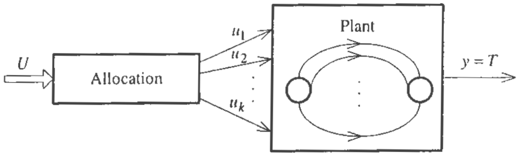

Figure 1. Complex of parallel operations as a decision plant

According to the general formulation of the decision problem presented in Sec. 1, the allocation problem may be formulated as an optimization problem consisting in finding the optimal allocation u^* that maximizes the certainty index of the soft property: "the set of possible values T approximately belongs to $[0, \alpha]$" (i.e. belongs to $[0, \alpha]$ for an approximate value of $\bar{x}$).

Optimal allocation problem: For the given φ_i, h_i $(i \in \overline{1,k})$, U and α find

$$u^* = \arg\max_{u \in \bar{U}} v(u)$$

where

$$v(u) = v\{D_T(u;\bar{x}) \tilde{\subseteq} [0, \alpha]\} = v(T(u, \bar{x}) \tilde{\leq} \alpha).$$

The soft property "$D_T(u;\bar{x}) \tilde{\subseteq} [0, \alpha]$" is denoted here by "$T(u, \bar{x}) \tilde{\leq} \alpha$", and $D_T(u;x)$ denotes the set of possible values T for the fixed u, determined by the inequality $T \leq \max_i \varphi_i(u_i, x_i)$. The property "$T(u, \bar{x}) \tilde{\leq} \alpha$" means that the maximum possible value of the executive time T is approximately (i.e. for the approximate value of $\bar{x}$) less or equal to α.

According to (2)

$$\nu(u) = \nu\{[T_1(u_1, \overline{x}_1) \tilde{\lesssim} \alpha)] \wedge [T_2(u_2, \overline{x}_2) \tilde{\lesssim} \alpha)] \wedge \ldots \wedge [T_k(u_k, \overline{x}_k) \tilde{\lesssim} \alpha)]\}.$$

Then

$$u^* = \arg\max_{u \in \overline{U}} \min_i \nu_i(u_i) \tag{3}$$

where

$$\nu_i(u_i) = \nu[T_i(u_i, \overline{x}_i) \tilde{\lesssim} \alpha)] = \nu[\varphi_i(u_i, \overline{x}_i) \tilde{\lesssim} \alpha)] = \nu[\overline{x}_i \tilde{\in} D_i(u_i)],$$

$$D_i(u_i) = \{x_i \in R^1 : \varphi_i(u_i, x_i) \le \alpha\}. \tag{4}$$

Finally

$$\nu_i(u_i) = \max_{x_i \in D_i(u_i)} h_i(x_i) \tag{5}$$

and

$$u^* = \arg\max_{u \in \overline{U}} \min_i \max_{x_i \in D_i(u_i)} h_i(x_i).$$

In many cases an expert gives the value x_i^* and the interval of the approximate values of $\overline{x}_i$: $x_i^* - d_i \le x_i \le x_i^* + d_i$. Then we assume that $h_i(x_i)$ has a triangular form presented in Figure 2 where $d_i \le x_i^*$. Let us consider the relation (2) in the form $T_i \le x_i u_i$ where $x_i > 0$ and u_i denotes the size of a task. In this case, using (5) it is easy to obtain the following formula for the function $\nu_i(u_i)$:

$$v_i(u_i)=\begin{cases} 1 & \text{for} \quad u_i \le \dfrac{\alpha}{x_i^*} \\ \dfrac{1}{d_i}(\dfrac{\alpha}{u_i}-x_i^*)+1 & \text{for} \quad \dfrac{\alpha}{x_i^*} \le u_i \le \dfrac{\alpha}{x_i^*-d_i} \\ 0 & \text{for} \quad u_i \ge \dfrac{\alpha}{x_i^*-d_i}. \end{cases}$$

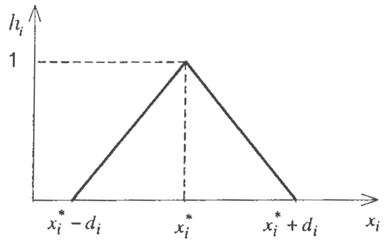

Figure 2. Example of the certainty distribution

For the relations $T_i \le x_i u_i^{-1}$ where u_i denotes the size of a resource, the function $v_i(u_i)$ has an analogous form with u_i^{-1} in place of u_i:

$$v_i(u_i)=\begin{cases} 0 & \text{for} \quad u_i \le \dfrac{x_i^*-d_i}{\alpha} \\ \dfrac{1}{d_i}(\alpha u_i - x_i^*)+1 & \text{for} \quad \dfrac{x_i^*-d_i}{\alpha} \le u_i \le \dfrac{x_i^*}{\alpha} \\ 1 & \text{for} \quad u_i \ge \dfrac{x_i^*}{\alpha}. \end{cases} \tag{6}$$

If $\overline{x}$ is considered as C-uncertain variable then, according to (1), the optimal decision u_c^* maximizing the certainty index $v_c[T(u,\overline{x}) \tilde{\le} \alpha]$ is as follows

$$u_c^* = \arg\max_{u\in\bar{U}}[\min_i v_i(u_i)+1-\max_i \hat{v}_i(u_i)] = \arg\max_{u\in\bar{U}}[\min_i v_i(u_i)-\max_i \hat{v}_i(u_i)] \quad (7)$$

where

$$\hat{v}_i(u_i) = v[T_i(u_i,\bar{x}_i)\tilde{\geq}\alpha] = \max_{x\in\bar{D}_i(u_i)} h_i(x_i)$$

and $\bar{D}_i$ is the complement of D_i, i.e. $\bar{D}_i = R^1 - D_i$ (see [7]).

Let us denote by x_i^* the value maximizing $h_i(x_i)$ (i.e. $h_i(x_i^*)=1$), and by D_{ui} the set of u_i for which $v_i(u_i)<1$. Let us assume for further considerations that $h_i(x_i)$ is a continuous function $(i\in\overline{1,k})$.

Lemma

If

– $\varphi_i(u_i,x_i)$ is an increasing (decreasing) function of u_i for every x_i,

– $\varphi_i(u_i,x_i)$ is a monotonic function of x_i for every u_i,

– $h_i(x_i)$ is an increasing function of x_i for $x_i \leq x_i^*$ and a decreasing function of x_i for $x_i \geq x_i^*$,

then $v_i(u_i)$ is a decreasing (increasing) function of u_i in D_{ui}.

Proof: Denote by $\tilde{x}_i(u_i)$ the solution of the equation $\varphi_i(u_i,x_i)=\alpha$ and by $\tilde{u}_i$ the solution of the equation $\tilde{x}_i(u_i)=x_i^*$, i.e. the solution of the equation $\varphi_i(u_i,x_i^*)=\alpha$. Assume that

$\varphi_i(u_i, x_i)$ is an increasing function of x_i. Then $D_i(u_i)$ in (4) is determined by the inequality $x_i \le \tilde{x}_i(u_i)$ and

$$v_i(u_i) = \max_{x_i \le \tilde{x}_i(u_i)} h_i(x_i). \tag{8}$$

If $\varphi_i(u_i, x_i)$ is an increasing function of u_i then $\tilde{x}_i(u_i)$ is a decreasing function of u_i and

$$v_i(u_i) = \begin{cases} 1 & \text{for } \tilde{x}_i(u_i) \ge x_i^*, \text{ i.e. } 0 \le u_i \le \tilde{u}_i, \\ h_i[\tilde{x}_i(u_i)] & \text{for } \tilde{x}_i(u_i) \le x_i^*, \text{ i.e. } u_i \ge \tilde{u}_i. \end{cases} \tag{9}$$

From (9) and the assumption that $h_i(x_i)$ is an increasing function for $x_i \le x_i^*$, it follows that for $u_i \ge \tilde{u}_i$ (i.e. for $u_i \in D_{ui}$) $v_i(u_i)$ is a decreasing function of u_i. If $\varphi_i(u_i, x_i)$ is a decreasing function of u_i then $\tilde{x}_i(u_i)$ is an increasing function of u_i and

$$v_i(u_i) = \begin{cases} h_i[\tilde{x}_i(u_i)] & \text{for } \tilde{x}_i(u_i) \le x_i^*, \text{ i.e. } 0 \le u_i \le \tilde{u}_i, \\ 1 & \text{for } \tilde{x}_i(u_i) \ge x_i^*, \text{ i.e. } u_i \ge \tilde{u}_i. \end{cases} \tag{10}$$

From (10) and the assumption that $h_i(x_i)$ is an increasing function for $x_i \le x_i^*$, it follows that for $0 \le u_i \le \tilde{u}_i$ (i.e. for $u_i \in D_{ui}$) $v_i(u_i)$ is an increasing function of u_i.

Assume now that $\varphi_i(u_i, x_i)$ is a decreasing function of x_i. Then $D_i(u_i)$ in (4) is determined by the inequality $x_i \ge \tilde{x}_i(u_i)$ and

$$v_i(u_i) = \max_{x_i \ge \tilde{x}_i(u_i)} h_i(x_i). \tag{11}$$

In a way analogous to that for the previous case, it is easy to see that if $\varphi_i(u_i, x_i)$ is an increasing (decreasing) function of u_i then $\tilde{x}_i(u_i)$ is an increasing (decreasing) function of u_i and for $u_i \in D_{ui}$ i.e. for $u_i \geq \tilde{u}_i$ (for $0 \leq u_i \leq \tilde{u}_i$) $v_i(u_i)$ is a decreasing (increasing) function of u_i.

Theorem 1

If the assumptions in Lemma are satisfied for each *i* and $u_1^*, u_2^*, ..., u_k^*$ is the optimal allocation then

$$v_1(u_1^*) = v_2(u_2^*) = ... = v_k(u_k^*). \tag{12}$$

Proof: The theorem follows directly from Lemma and (3). For $v(u^*) = 1$, (12) follows from the fact that $v_i(u_i^*) \leq 1$ for each *i*. Assume that $v(u^*) < 1$ and (12) is not satisfied.

Let

$$v_{p(s)}(u_{p(s)}^*) = \min_i v_i(u_i^*), \quad s \in \overline{1, r}, \tag{13}$$

i.e.

$$\arg\min_i v_i(u_i^*) \in \{p(1), ..., p(r)\},$$

and $v_j(u_j^*)$ be the smallest value $v_i(u_i^*)$ greater than $v_{p(s)}(u_{p(s)}^*)$. For $i = p(1), ..., p(r), j$ consider a new allocation $\overline{u}_{p(1)}, ..., \overline{u}_{p(r)}, \overline{u}_j$ such that

$$\bar{u}_{p(1)} + ... + \bar{u}_{p(r)} + \bar{u}_j = u^*_{p(1)} + ... + u^*_{p(r)} + u^*_j,$$

$$v_{p(1)}(\bar{u}_{p(1)}) = ... = v_{p(r)}(\bar{u}_{p(r)}) = v_j(\bar{u}_j).$$

Note that $v_{p(s)}(u^*_{p(s)}) < 1$, i.e. $u^*_{p(s)} \in D_{up(s)}$. For the task allocation (φ_i are increasing functions of u_i for each *i*), it follows from Lemma that $\bar{u}_{p(s)} < u^*_{p(s)}$ for each *s* and $\bar{u}_j > u^*_j$. For the decreasing functions φ_i (the resource allocation), it follows from Lemma that $\bar{u}_{p(s)} > u^*_{p(s)}$ and $\bar{u}_j < u^*_j$. Then in both cases $v_{p(s)}(\bar{u}_{p(s)}) > v_{p(s)}(u^*_{p(s)})$ and $v_{p(s)}(\bar{u}_{p(s)})$ is the smallest value $v_i\ (i \in \overline{1,k})$. Consequently, the new allocation gives the greater certainty index $v(u) = \min v_i(u_i)$ and if (12) is not satisfied then $u^*_1, u^*_2, ..., u^*_k$ is not the optimal allocation.

It is easy to note that the equality (12) is also a sufficient condition of the optimal allocation.

Theorem 2

If the assumptions in Lemma are satisfied for each *i* and (12) is satisfied then $u^*_1, u^*_2, ..., u^*_k$ is the optimal allocation.

Proof: Assume $v_1(u^*_1) = ... = v_k(u^*_k) = v(u^*) < 1$ and consider an allocation $\bar{u} \neq u^*$. It follows from Lemma (from the statement that v_i is a monotonic function of u_i) that there exists *j* such that $v_j(\overline{u}_j) < v_j(u_j^*)$. Then $\min v_i(\overline{u}_i) < v(u^*)$ which means that u^* is the optimal allocation.

Theorems 1 and 2 may be easily extended to the case of C-uncertain variables.

Theorem 3

If the assumptions in Lemma are satisfied then $u_{c1}^*, u_{c2}^*, ..., u_{ck}^*$ is the optimal allocation if

$$v_{c1}(u_{c1}^*) = v_{c2}(u_{c2}^*) = ... = v_{ck}(u_{ck}^*).$$

Proof: According to (7) $v_{ci}(u_i) = v_i(u_i) + 1 - \hat{v}_i(u_i)$. In the way analogous to that for $v_i(u_i)$ in Lemma, it may be proved that for the task allocation (φ_i is an increasing function of u_i) $\hat{v}_i(u_i)$ is an increasing function, and for the resource allocation (φ_i is a decreasing function of u_i) $\hat{v}_i(u_i)$ is a decreasing function. Then $v_{ci}(u_i)$ is a decreasing (increasing) function in the set $\{u_i : v_{ci}(u_i) < 1\}$. Consequently, the second part of the proof is the same as the proof of Theorems 1 and 2, with v_{ci} in place of v_i. □

The consideration should be completed with the case when there exist $\underline{x}_i$ and $\overline{x}_i$ such that $h_i(x_i) = 0$ for $x_i \notin [\underline{x}_i, \overline{x}_i]$. Now the assumption concerning $h_i(x_i)$ is as follows: $h_i(x_i)$ is an increasing function for $x_i \in [\underline{x}_i, x_i^*]$ and a decreasing function for $x_i \in [x_i^*, \overline{x}_i]$. In this case $v_i(u_i)$ is a monotonic function in the set $D_{ui} = \{u_i : 0 < v_i(u_i) < 1\}$.

Theorem 4

Assume that for each i the properties concerning φ_i in Lemma and the property concerning h_i presented above are satisfied. Assume that there exist $u_1^*, u_2^*, ..., u_k^*$ such that

$$v_1(u_1^*) = v_2(u_2^*) = ... = v_k(u_k^*) = v(u^*) = 0. \qquad (14)$$

Then $v(u) = 0$ for every $u \in \overline{U}$.

Proof: According to (8) and (11), in the case of task allocation there exists such a value of u_i that for u_i greater than this value $v_i(u_i) = 0$, and in the case of resource allocation there exists such a value of u_i that for u_i less than this value $v_i(u_i) = 0$. Consider an allocation $\overline{u} \neq u^*$. Then there exists j such that $\overline{u}_j > u_j^*$ and r such that $\overline{u}_r < u_j^*$. Consequently, it follows from (14) that in the case of task allocation $v_j(\overline{u}_j) = 0$ and in the case of resource allocation $v_r(\overline{u}_r) = 0$. Then, in both cases $v(u) = \min v_i(u_i) = 0$.

According to Theorems 1,2 and 4, to determine the optimal allocation one should solve the set of equations

$$v_1(u_1) = v_2(u_2) = ... = v_k(u_k),$$

$$u_1 + u_2 + ... + u_k = U.$$

Denote the set of the solutions by $D \subset \overline{U}$.

1. If $D = \{u^*\}$ then u^* is a unique optimal allocation and $0 < v(u^*) < 1$.

2. If there exists $u \in D$ such that $v_1(u_1) = ... = v_k(u_k) = 1$ then for every $u \in D_u$ $v(u) = 1$, i.e. D is a set of the optimal allocations. This means that α is sufficiently large for the given U, or U is sufficiently small in the case of a task (sufficiently large in the case of a resource) for the given α.

3. If there exists $u \in D$ such that $v_1(u_1) = ... = v_k(u_k) = 0$ then $v(u) = 0$ for any allocation $u \in \overline{U}$. This means that α is too small or U is too large in the case of a task (too small in the case of a resource).

3. Decomposition and two-level allocation

The determination of the allocation u^* may be difficult for $k > 2$ because of the great computational difficulties. To decrease these difficulties we can apply the decomposition of the complex into two subcomplexes and consequently to obtain a two-level allocation system (Figure 3). At the upper level the value U is divided into U_1 and U_2 assigned to the first and the second subcomplex, respectively, and at the lower level the allocation $u^{(1)}$, $u^{(2)}$ for the subcomplexes is determined. Let us introduce the following notation:

n, m – the number of operations in the first and the second complex, respectively, $n + m = k$,

$T^{(1)}$, $T^{(2)}$ – the execution times in the subcomplexes, i.e.

$$T^{(1)} = \max(T_1, T_2, ..., T_n), \qquad T^{(2)} = \max(T_{n+1}, T_{n+2}, ..., T_{n+m}),$$

$u^{(1)}$, $u^{(2)}$ – the allocations in the subcomplexes, i.e.

$$u^{(1)} = (u_1, \ldots, u_n), \qquad u^{(2)} = (u_{n+1}, \ldots, u_{n+m}).$$

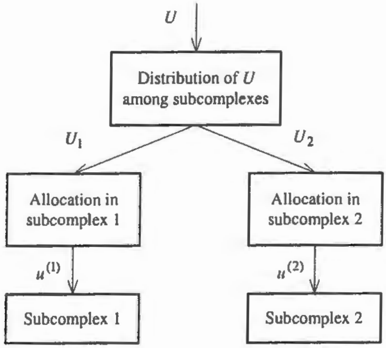

Figure 3. Two-level allocation system

The procedure of the determination of u^* is then the following:

1. To determine the allocation $u^{(1)*}(U_1)$, $u^{(2)*}(U_2)$ and the certainty indexes $v^{(1)*}(U_1)$, $v^{(2)*}(U_2)$ in the same way as u^*, v^* in Sec. 2, with U_1 and U_2 in place of U.

2. To determine U_1^*, U_2^* via the maximization of

$$v(T \mathrel{\tilde{\lesssim}} \alpha) = v[(T^{(1)} \mathrel{\tilde{\lesssim}} \alpha) \wedge (T^{(2)} \mathrel{\tilde{\lesssim}} \alpha)] \triangleq v(U_1, U_2).$$

Then

$$(U_1^*, U_2^*) = \arg \max_{U_1, U_2} \min \{v^{(1)*}(U_1), v^{(2)*}(U_2)\}$$

with the constraints: $U_{1,2} \geq 0$, $U_1 + U_2 = U$.

3. To find the values of $u^{(1)*}$, $u^{(2)*}$ and v^* putting U_1^* and U_2^* into the results $u^{(1)*}(U_1)$, $u^{(2)*}(U_2)$ obtained in point 1 and into $v(U_1,U_2)$ in point 2.

It may be shown that the result obtained via the decomposition is the same as the result of the direct approach presented in Sec. 2.

Theorem 5

$$\arg\max_{u\in\bar{U}} \min_{i\in\overline{1,k}} v_i(u_i) = \max_{U_1,U_2} \min\{ \max_{u^{(1)}\in\bar{U}_1} \min_{i\in\overline{1,n}} v_i(u_i), \max_{u^{(2)}\in\bar{U}_2} \min_{i\in\overline{n+1,m}} v_i(u_i)\}$$

where the sets $\bar{U}_1$, $\bar{U}_2$ are determined by the equalities $u_1+...+u_n=U_1$, $u_{n+1}+...+u_{n+m}=U_2$, respectively, i.e. the direct approach and the approach via the decomposition give the same results.

Proof: Denote by $v^*(U)$ the value v^* (i.e. the result of the optimal allocation considered in Sec. 2) as a function of U, and assume $0<v^*(U)<1$, i.e.

$$0<v_1(u_1^*)=...=v_k(u_k^*)=v^*(U)<1. \qquad (15)$$

It is easy to show that $v^*(U)$ is a monotonic function. Consider $\tilde{U}>U$ and the respective optimal allocation $\tilde{u}$, i.e.

$$v_1(\tilde{u}_1)=...=v_k(\tilde{u}_k)=v^*(\tilde{U}). \qquad (16)$$

From (15), (16) and the equality $\tilde{u}_1 + ... + \tilde{u}_k = \tilde{U}$, it is easy to see that $v_i(\tilde{u}_i) < v_i(u_i^*)$ for the increasing functions $v_i(u_i)$ and $v_i(\tilde{u}_i) > v_i(u_i^*)$ for the decreasing functions $v_i(u_i)$ $(i \in \overline{1,k})$. Then $v^*(\tilde{U}) < v^*(U)$ in the case of a task and $v^*(\tilde{U}) > v^*(U)$ in the case of a resource.

The optimal allocation u^* satisfies the equations

$$\left.\begin{array}{c} v_1(u_1) = ... = v_n(u_n) = v_{n+1}(u_{n+1}) = ... = v_{n+m}(u_{n+m}) \\ u_1 + ... + u_n + u_{n+1} + ... + u_{n+m} = U. \end{array}\right\} \quad (17)$$

Denote by $\bar{u} = (\bar{u}^{(1)}, \bar{u}^{(2)})$ the result of the decomposition. Then $\bar{u}^{(1)}$, $\bar{u}^{(2)}$ satisfy the equations

$$v_1(u_1) = ... = v_n(u_n) \stackrel{\Delta}{=} v^{(1)}(U_1), \quad u_1 + ... + u_n = U_1, \quad (18)$$

$$v_{n+1}(u_{n+1}) = ... = v_{n+m}(u_{n+m}) \stackrel{\Delta}{=} v^{(2)}(U_2), \quad u_{n+1} + ... + u_{n+m} = U_2, \quad (19)$$

respectively. From the monotonic properties of the functions $v^{(1)}(U_1)$, $v^{(2)}(U_2)$ (showed at the beginning of this proof, with U_1 and U_2 in place of U) and Theorems 1, 2 (with $k = 2$ and U_1, U_2 in place of u_1, u_2), it follows that for $u = \bar{u}$, U_1 and U_2 satisfy the equations

$$v^{(1)}(U_1) = v^{(2)}(U_2), \quad U_1 + U_2 = U.$$

Finally, it follows from (18), (19) that $\bar{u}$ satisfies the equations (17). Consequently, the result of the decomposition is the same as the result of the direct approach.

4. Example

Let us consider the resource allocation for $T_i \le x_i u_i^{-1}$, $k = 4$ and the certainty distributions presented in Figure 2, and introduce the decomposition into two subcomplexes with $n = m = 2$. For the first subcomplex the decision u_1^* may be found by solving the equation $v_1(u_1) = v_2(U_1 - u_1)$, and $u_2^* = U_1 - u_1^*$. Using (6) we obtain the following result for the first subcomplex:

1. For

$$U_1 \le \frac{x_1^* - d_1 + x_2^* - d_2}{\alpha}$$

$v^{(1)*}(U_1) = 0$.

2. For

$$\frac{x_1^* - d_1 + x_2^* - d_2}{\alpha} \le U_1 \le \frac{x_1^* + x_2^*}{\alpha}$$

we obtain

$$v^{(1)*}(U_1) = A_1 U_1 + B_1$$

where

$$A_1 = \frac{\alpha}{d_1 + d_2}, \quad B_1 = \frac{x_1^* d_2 - x_2^* d_1}{d_1(d_1 + d_2)} - \frac{x_1^*}{d_1} + 1.$$

3. For

$$U_1 \geq \frac{x_1^* + x_2^*}{\alpha}$$

$\nu^{(1)*}(U_1) = 1$.

The relationship $\nu^{(2)*}(U_2)$ is the same with x_3, x_4, d_3, d_4, A_2, B_2 in place of x_1, x_2, d_1, d_2, A_1, B_1.

The value U_1^* may be determined by solving the equation $\nu^{(1)*}(U_1) = \nu^{(2)*}(U - U_1)$, and $U_2^* = U - U_1^*$. The result is as follows:

1. For

$$\alpha \leq \frac{x_1^* - d_1 + x_2^* - d_2 + x_3^* - d_3 + x_4^* - d_4}{U} \tag{20}$$

$\nu(U_1, U_2) = 0$.

2. For

$$\frac{x_1^* - d_1 + x_2^* - d_2 + x_3^* - d_3 + x_4^* - d_4}{U} \leq \alpha \leq \frac{x_1^* + x_2^* + x_3^* + x_4^*}{U}$$

we obtain

$$U_1^* = \frac{A_2 U + B_2 - B_1}{A_1 + A_2}, \quad U_2^* = \frac{A_1 U + B_1 - B_2}{A_1 + A_2},$$

$$\nu(U_1^*, U_2^*) = \frac{A_1 A_2 U + A_1 B_2 + A_2 B_1}{A_1 + A_2}.$$

3. For

$$\alpha \geq \frac{x_1^* + x_2^* + x_3^* + x_4^*}{U}$$

we obtain $\nu(U_1^*, U_2^*) = 1$ for any U_1 satisfying the condition

$$\frac{x_1^* + x_2^*}{\alpha} \leq U_1 \leq U - \frac{x_3^* + x_4^*}{\alpha}.$$

In the case (20) α is too small (the requirement is too strong) and it is not possible to find the allocation for which $\nu(u)$ is greater than 0. For the numerical data $U = 20$, $\alpha = 0.5$, $x_1^* = 2$, $x_2^* = 3$, $x_3^* = 3$, $x_4^* = 4$, $d_1 = d_2 = 1$, $d_3 = d_4 = 2$ we obtain: $U_1^* = 8\frac{2}{3}$, $U_2^* = 11\frac{1}{3}$, $u_1^* = 3\frac{1}{3}$, $u_2^* = 5\frac{1}{3}$, $u_3^* = 4\frac{1}{3}$, $u_4^* = 7$ and $\nu^* = \frac{2}{3}$, which means that the requirement $T \leq \alpha$ will be approximately satisfied with the certainty index $\frac{2}{3}$.

5. Conclusions

According to Theorems 1-4, the difficult optimization problem (3) may be reduced to the solution of the set of equations presented at the end of Sec. 2. The decomposition described in Sec. 3 permits to decrease the computational difficulties and gives the exact result, the same as in the direct approach described in Sec. 2. It is worth nothing that at the upper level we have the deterministic optimization problem without uncertain parameters.

The results presented in the paper may be applied to a load distribution in a computer system with parallel processors, and may be extended to other decision problems in operation systems, to learning systems with the knowledge updating and to systems with a distributed knowledge [1], [5], [6], [9].

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