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Stability and accuracy functions in multicriteria linear combinatorial optimization problems*

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Abstract We consider a vector linear combinatorial optimization problem in which initial coefficients of objective functions are subject to perturbations. For Pareto and lexicographic principles of efficiency we introduce appropriate measures of the quality of a given feasible solution. These measures correspond to so-called stability and accuracy functions defined earlier for scalar optimization problems. Then we study properties of such functions and calculate the maximum norms of perturbations for which an efficient solution preserves the efficiency.

Key words multicriteria optimization, sensitivity analysis, stability and accuracy, Pareto and lexicographic optima.

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1 Introduction

The stability theory is an integral part of any traditional section of mathematics. J. Hadamar introduced the stability condition and treated it within the concept of a well-posed mathematical programming problem equally with the conditions of existence and uniqueness of the solution. In optimization a question of stability of a problem arises in the case where the set of feasible solutions and (or) the objective function depend on parameters. The presence of such parameters in optimization models is caused by inaccuracy of initial data, non-adequacy of models to real processes, errors of numerical methods, errors of rounding off and other factors. So it appears to be important to allocate classes of problems in which small changes of input data lead to small changes of the result. The problems with such properties are called stable. It is obvious that any optimization problem arising in practice cannot be correctly formulated and solved without use of results of the stability theory.

A vector (multicriteria) optimization problem is usually understood as the problem of finding a set of efficient solutions, i.e. choosing from the set of feasible solutions the alternatives which satisfy a given optimality principle. In the case where the partial criteria of the problem have an equal importance, the Pareto optimality principle (see e.g. [4], [5], [15], [18]) is more often used. If all the partial criteria are ordered by importance in such a manner that each of them is more important than all the subsequent, then the principle of lexicographic optimality is used. Investigating stability of a vector optimization problem means usually studying the behavior of the set of efficient solutions under perturbations of problem parameters.

In the literature a technique of studying the stability of optimization problems (both single criterion and multicriteria) is better developed and covered for problems with continuous set of feasible solutions and there are numerous results in sensitivity analysis for such problems. Unfortunately, these results are not very useful in discrete case although most of discrete optimization problems may be formally transformed, at least – in principle, to an equivalent continuous optimization problem. The reason is that such a transformation does not exploit the specific combinatorial structure.

There are also a lot of papers devoted to stability of combinatorial optimization problems. There is no chance to describe all variety of results in the frame of one article. However one can find excellent annotated bibliographies and surveys for sensitivity and post-optimal analysis in integer programming and combinatorial optimization problems in [2], [7], [16], [17].

In single objective case the most frequently considered object is so-called stability radius with respect to some given optimal solution (see e.g. [1]). It gives a subset of problem parameters for which this solution remains optimal. There are already similar investigations in multiobjective case. For example, in [3] the stability radius for multicriteria linear combinatorial optimization problem is calculated in the Pareto case. One can find also a large survey on sensitivity analysis of vector unconstrained integer linear programming in [2].

It is important to note that even in single objective case the stability radius does not provide us with any information about the quality of a given solution in the case when problem data are outside of the stability region. Some attempts to study a quality of the problem solution in this case are connected with concepts of stability and accuracy functions. These functions were firstly introduced in [12] for scalar combinatorial optimization problem. In this paper we give an extension of results obtained in [12] and [13] for the vector perturbed combinatorial optimization problem with Pareto and lexicographic optimality principles. To our knowledge this problem has not been approached earlier within the multicriteria framework.

The paper is organized as follows. In section 1 we consider vector linear combinatorial optimization problem which consists in finding the set of Pareto optimal solutions. For a given Pareto optimal solution we introduce an appropriate relative error as a function of the norm of data perturbations. This leads us to natural extension of stability and accuracy functions in multiobjective case. We give formulae to calculate values of both functions. Afterward, we define so called stability and accuracy radii as extreme norms of perturbations of problem parameters for which the stability and accuracy functions are equal to zero. In section 2 analogous results are stated for the case of lexicographic optimality. In this section both functions are defined in a different way which reflects lexicographic specific. At the end of paper we give small example which illustrates why it seems so important to calculate stability and accuracy functions which give us the most detailed information about efficient solution.

2 Stability and accuracy functions of Pareto optimal solution

Let $E = \{e_1, e_2, ..., e_n\}$, $n > 1$, be a given set, and let $T \subseteq 2^E \backslash \{0\}$, $|T| > 1$, be a family of non-empty subsets of E. For $e \in E$ we define a vector of positive weights

$$c(e) = (c_1(e), c_2(e), ..., c_m(e)),\ m \geq 1,$$

and a matrix $C = \{c_i(e_j)\} \in \mathbf{R}_+^{m\times n}$, where $\mathbf{R}_+ = \{u \in \mathbf{R} :\ u > 0\}$. Denote for $k \in \mathbf{N}$, $N_k = \{1, 2, ..., k\}$ and let for $t \in T$, $N(t) = \{j : e_j \in t\}$. We will consider a vector criterion

$$f(C,t) = (f_1(C,t), f_2(C,t), ..., f_m(C,t)),$$

where

$$f_i(C,t) = \sum_{j\in N(t)} c_i(e_j),\ i \in N_m.$$

For a matrix $C \in \mathbf{R}_+^{m\times n}$ and a feasible solution $t \in T$, let

$$\pi(C,t) = \{t' \in T :\ f(C,t') \leq f(C,t),\ f(C,t') \neq f(C,t)\}.$$

The Pareto set $P^m(C)$ is defined in a traditional way, namely:

$$P^m(C) = \{t \in T :\ \pi(C,t) = \emptyset\}.$$

In other words, a solution t is Pareto optimal if and only if there is no solution t' such that $f_i(C,t') \leq f_i(C,t)$ for all $i \in N_m$ and at least one strict inequality holds. If the sets E and T are fixed, then an instance of m-criteria combinatorial optimization problem is uniquely determined by the matrix $C \in \mathbf{R}_+^{m\times n}$. Therefore, we will denote it by $Z_P^m(C)$.

It is assumed that the set T is fixed, but the matrix of weights C may vary or is estimated with errors. Moreover, it is assumed that for some originally specified matrix $C^0 = \{c_i^0(e_j)\} \in \mathbf{R}_+^{m\times n}$ we know one Pareto optimal solution t^0.

When coefficients of objective functions change, then initially efficient solution may become no longer efficient. We will evaluate the quality of this solution from the point of view of its robustness on data perturbations. Namely, in case of Pareto optimality we introduce for $t^0 \in P^m(C^0)$ and a given matrix $C \in \mathbf{R}_+^{m\times n}$ so-called relative error of this solution:

$$\varepsilon_P(C,t^0) = \max_{t\in T} \min_{i\in N_m} \frac{f_i(C,t^0) - f_i(C,t)}{f_i(C,t)}.$$

In the scalar case, i.e. for m=1, the Pareto set transforms into the set of optimal solutions. Therefore the relative error $\varepsilon_P(C,t^0)$ converts into (see [13]):

$$\varepsilon_P(C,t^0) = \frac{f_1(C,t^0) - \min\limits_{t\in T} f_1(C,t)}{\min\limits_{t\in T} f_1(C,t)}.$$

In the scalar case the equality $\varepsilon_P(C,t^0) = 0$ gives necessary and sufficient optimality conditions of the optimality of the solution t^0 for problem $Z_P^1(C)$. But in the multicriteria case the situation is a bit different. Observe, that for arbitrary $C \in \mathbf{R}_+^{m\times n}$ we have $\varepsilon_P(C,t^0) \geq 0$. If $\varepsilon_P(C,t^0) > 0$, then $t^0 \notin P^m(C)$ and this positive value of the relative error may be treated as a measure of inefficiency of the solution t^0 for problem $Z_P^m(C)$. Obviously, if $t^0 \in P^m(C)$, then $\varepsilon_P(C,t^0) = 0$. But the inverse is not always through as the following example shows:

Consider $E = \{e_1, e_2, e_3\}$, $T = \{t_1, t_2, t_3\}$, where $t_1 = \{e_1\}$, $t_2 = \{e_2\}$, $t_3 = \{e_3\}$, and let

$$C^0 = \begin{bmatrix} 3 & 3 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix},\quad C = \begin{bmatrix} 3 & 3 & 3 \\ 2 & 1 & 10 \\ 1 & 2 & 10 \end{bmatrix}.$$

It is clear that $t^0 = t_3$ is Pareto optimal in the original problem $Z_P^3(C^0)$, but it is not Pareto optimal in the problem $Z_P^3(C)$ although $\varepsilon_P(C,t^0) = 0$.

Thus, in the multiobjective case the equality $\varepsilon_P(C,t^0) = 0$ formulates in general only necessary condition of the efficiency of the solution t^0 for problem $Z_P^m(C)$. But later we will show, that if the equality $\varepsilon_P(C,t^0) = 0$ is valid for any matrix in some open neighbourhood of C, i.e., there is $\epsilon > 0$ such that $\varepsilon_P(\tilde{C},t^0) = 0$ for any $\tilde{C}$,

$||\tilde{C}-C|| < \epsilon$, where $||\cdot||$ denotes a norm in $\mathbf{R}_+^{m\times n}$, then this equality provides also sufficient efficiency condition of the solution t^0 for problem $Z_P^m(C)$.

In the following we are interested, in fact, in the maximum value of the error $\varepsilon_P(C,t^0)$ when the matrix C belongs to some specified set. Two particular cases are considered:

In the first case we are interested in absolute perturbations of the weights of elements and the quality of a given solution is described by the so-called *stability function.* For a given $p \geq 0$ the value of the stability function is equal to the maximal relative error of a given solution under the assumption that no weights of elements are increased or decreased by more than p.

In the second case we deal with relative perturbations of weights. This leads to the concept of the *accuracy function.* The value of the accuracy function for a given $\delta \in [0,1)$ is equal to the maximum relative error of the solution t^0 under the assumption that the weights of the elements are perturbed by no more than $\delta \cdot 100\%$ of their original values.

Observe that if we compare two initially efficient solutions from the point of view of their robustness on data perturbations or inaccuracy, then smaller value of the stability or accuracy function for a given norm of data perturbation is more preferable. Thus, both defined functions may be used to evaluate the quality of solutions from this particular point of view.

Let X be the set of non-stable elements, i.e. elements for which weights may change, and let

$$C^0(X) = \{C \in \mathbf{R}_+^{m\times n}: \ c_i(e_j) = c_i^0(e_j), \ e_j \in E\backslash X, \ i \in N_m \ j \in N_n\}.$$

For a given $p \in [0, q(C^0,X))$, where $q(C^0,X) = \min\{c_i^0(e_j): \ e_j \in X, \ i \in N_m, \ j \in N_n\}$, we consider a set

$$\Omega_p(C^0,X) = \{C \in C^0(X): \ |c_i(e_j) - c_i^0(e_j)| \leq p, \ i \in N_m, \ j \in N_n\}.$$

For a Pareto optimal solution $t^0 \in P^m(C^0)$, an arbitrary set of non-stable elements X, and $p \in [0, q(C^0,X))$, the value of the stability function is defined as follows:

$$S_P(t^0,X,p) = \max_{C\in\Omega_p(C^0,X)} \varepsilon_P(C,t^0).$$

In a similar way, for a given $\delta \in [0,1)$, we consider a set

$$\Theta_\delta(C^0,X) = \{C \in C^0(X): \ |c_i(e_j) - c_i^0(e_j)| \leq \delta c_i^0(e_j), i \in N_m, \ j \in N_n\}.$$

For a Pareto optimal solution $t^0 \in P^m(C^0)$, an arbitrary set of non-stable elements X and $\delta \in [0,1)$ the value of the accuracy function is defined as follows:

$$A_P(t^0,X,\delta) = \max_{C\in\Theta_\delta(C^0,X)} \varepsilon_P(C,t^0).$$

It is easy to check that $S_P(t^0,X,p) \geq 0$ for any $p \in [0, q(C^0,X))$ as well as $A_P(t^0,X,\delta) \geq 0$ for each $\delta \in [0,1)$. Moreover, the following fact holds:

Proposition 1 *For $t^0 \in P^m(C^0)$ and $p \in (0, q(C^0,X))$,*

$$S_P(t^0,X,p) = 0 \quad \textit{if and only if} \quad t^0 \in P^m(C) \ \textit{for any} \ \ C \in \Omega_p(C^0,X).$$

Similarly, for $t^0 \in P^m(C^0)$ and $\delta \in (0,1)$,

$$A_P(t^0,X,p) = 0 \quad \textit{if and only if} \quad t^0 \in P^m(C) \ \textit{for any} \ \ C \in \Theta_\delta(C^0,X).$$

Proof We will prove only first statement, because the proof of the second part is analogous.
If for a given $p \in (0, q(C^0,X))$, $t^0 \in P^m(C^0)$ for any $C \in \Omega_p(C^0,X)$, then – directly from the definition of the stability function – we have $S_P(t^0,X,p) = 0$. Thus, it remains to prove the opposite implication.

Assume that this implication does not hold, i.e., suppose that $S_P(t^0,X,p) = 0$, but there exists a matrix $C^* \in \Omega_p(C^0,X)$, such that $t^0 \notin P^m(C^*)$. We will show that such assumption must lead to a contradiction. Indeed, $t^0 \notin P^m(C^*)$ means that there exists $t^* \in T$ such that for all $i \in N_m$, $f_i(C^*,t^*) \leq f_i(C^*,t^0)$ and $f(C^*,t^*) \neq f(C^*,t^0)$. Let $I \subseteq N_m$ denotes the set of indices, for which $f_i(C^*,t^*) = f_i(C^*,t^0)$, and consider for $0 < \alpha < p$ the matrix $\tilde{C} = \{\tilde{c}_i(e_j)\} \in \mathbf{R}_+^{m\times n}$, where

$$\tilde{c}_i(e_j) = \begin{cases} c_i^*(e_j) - \alpha & \text{if } i \in I, \ e_j \in (t^*\backslash t^0) \cap X, \\ c_i^*(e_j) + \alpha & \text{if } i \in I, \ e_j \in (t^0\backslash t^*) \cap X, \\ c_i^*(e_j) & \text{otherwise.} \end{cases}$$

Observe that $\tilde{C} \in \Omega_p(C^0,X)$ and $f_i(\tilde{C},t^*) < f_i(\tilde{C},t^0)$ for $i \in N_m$, which implies $S_P(t^0,X,p) > 0$. Thus we have a contradiction which completes the proof.

Proposition 1 suggests, that it is of special interest to know the largest values of p and δ, for which, respectively, $S_P(t^0, X, p) = 0$ and $A_P(t^0, X, \delta) = 0$, because these values give the maximum norms of perturbations which preserve the efficiency of a given solution. These values are close analogues of so-called stability and accuracy radii introduced earlier for single objective optimization problems. Formally, for any arbitrary set of non-stable elements X the stability radius $R_P^S(t^0, X)$ and the accuracy radius $R_P^A(t^0, X)$ are defined in the following way:

$$R_P^S(t^0, X) = \sup\{p \in [0, q(C^0, X)) : \ S_P(t^0, X, p) = 0\},$$
$$R_P^A(t^0, X) = \sup\{\delta \in [0, 1) : \ A_P(t^0, X, \delta) = 0\}.$$

Two following theorems give formulae for calculating values of the stability and accuracy functions as well as values of the corresponding radii. Let for $t, t' \in T$, $t \otimes t' = (t \backslash t') \cup (t' \backslash t)$. Thus $|t \otimes t'| = |(t \backslash t') \cup (t' \backslash t)| = |t| + |t'| - 2|t \cap t'|$.

Theorem 1 *For an optimal solution $t^0 \in P^m(C^0)$, an arbitrary set of non-stable elements X, and $p \in [0, q(C^0, X))$, the stability function can be expressed by the formula:*

$$S_P(t^0, X, p) = \max_{t \in T} \min_{i \in N_m} \frac{f_i(C^0, t^0) - f_i(C^0, t) + p|(t \otimes t^0) \cap X|}{f_i(C^0, t) - p|t \cap X|}. \tag{1}$$

For an optimal solution $t^0 \in P^m(C^0)$, an arbitrary set of non-stable elements X, and $\delta \in [0, 1)$,

$$A_P(t^0, X, \delta) = \max_{t \in T} \min_{i \in N_m} \frac{f_i(C^0, t^0) - f_i(C^0, t) + \delta f_i(C^0, (t \otimes t^0) \cap X)}{f_i(C^0, t) - \delta f_i(C^0, t \cap X)}. \tag{2}$$

Proof We will prove only (1). The proof of (2) is analogous.

$$S_P(t^0, X, p) = \max_{C \in \Omega_p(C^0, X)} \varepsilon_P(C, t^0) = \max_{C \in \Omega_p(C^0, X)} \max_{t \in T} \min_{i \in N_m} \frac{f_i(C, t^0) - f_i(C, t)}{f_i(C, t)} =$$
$$= \max_{t \in T} \max_{C \in \Omega_p(C^0, X)} \min_{i \in N_m} \frac{f_i(C, t^0) - f_i(C, t)}{f_i(C, t)} \leq$$
$$\leq \max_{t \in T} \min_{i \in N_m} \max_{C \in \Omega_p(C^0, X)} \frac{f_i(C, t^0) - f_i(C, t)}{f_i(C, t)}.$$

For any fixed $t \in T$ and $i \in N_m$ the maximum $\frac{f_i(C, t^0) - f_i(C, t)}{f_i(C, t)}$ over $C \in \Omega_p(C^0, X)$ is attained when

$$c_i(e_j) = \begin{cases} c_i^0(e_j) + p & \text{if } j \in N(t^0 \cap X), \\ c_i^0(e_j) - p & \text{if } j \in N(t \cap X). \end{cases}$$

Thus, we get

$$S_P(t^0, X, p) \leq \max_{t \in T} \min_{i \in N_m} \frac{f_i(C^0, t^0) - f_i(C^0, t) + p|(t \otimes t^0) \cap X|}{f_i(C^0, t) - p|t \cap X|}.$$

Now it remains to prove that

$$S_P(t^0, X, p) \geq \max_{t \in T} \min_{i \in N_m} \frac{f_i(C^0, t^0) - f_i(C^0, t) + p|(t \otimes t^0) \cap X|}{f_i(C^0, t) - p|t \cap X|}.$$

Consider a matrix $C^* = \{c_i^*(e_j)\} \in \mathbf{R}^{m \times n}$ with elements defined for any index $i \in N_m$ as follows:

$$c_i^*(e_j) = \begin{cases} c_i^0(e_j) + p & \text{if } j \in N(t^0 \cap X), \\ c_i^0(e_j) - p & \text{otherwise.} \end{cases}$$

Then

$$\max_{t \in T} \min_{i \in N_m} \frac{f_i(C^*, t^0) - f_i(C^*, t)}{f_i(C^*, t)} = \max_{t \in T} \min_{i \in N_m} \frac{f_i(C^0, t^0) - f_i(C^0, t) + p|(t \otimes t^0) \cap X|}{f_i(C^0, t) - p|t \cap X|}.$$

So, we have that

$$S_P(t^0, X, p) \geq \max_{t \in T} \min_{i \in N_m} \frac{f_i(C^0, t^0) - f_i(C^0, t) + p|(t \otimes t^0) \cap X|}{f_i(C^0, t) - p|t \cap X|}.$$

Theorem 2 *For an optimal solution $t^0 \in P^m(C^0)$ and an arbitrary set of non-stable elements X,*

$$R_P^S(t^0, X) = \min\{q(C^0, X), \min_{t \in T \setminus \{t^0\}} \max_{i \in N_m} \frac{f_i(C^0, t) - f_i(C^0, t^0)}{|(t \otimes t^0) \cap X|}\}, \quad (3)$$

and

$$R_P^A(t^0, X) = \min\{1, \min_{t \in T_\alpha} \max_{i \in N_m} \frac{f_i(C^0, t) - f_i(C^0, t^0)}{f_i(C^0, (t \otimes t^0) \cap X)}\}, \quad (4)$$

where $T_\alpha = \{t \in T : \ f_i(C^0, (t \otimes t^0) \cap X) \neq 0 \text{ for all } i \in N_m\}$.

Proof We will prove only (3). The proof of (4) is analogous. If $p = 0$, then $S_P(t^0, X, 0) = 0$. Let $S_P(t^0, X, p) > 0$. It holds if and only if

$$\max_{t \in T} \min_{i \in N_m} \frac{f_i(C^0, t^0) - f_i(C^0, t) + p|(t \otimes t^0) \cap X|}{f_i(C^0, t) - p|t \cap X|} > 0.$$

But last means that

$$p > \bar{p} = \min_{t \in T \setminus \{t^0\}} \max_{i \in N_m} \frac{f_i(C^0, t) - f_i(C^0, t^0)}{|(t \otimes t^0) \cap X|}.$$

Thus, if $\bar{p} \leq q(C^0, X)$, then we get that $S_P(t^0, X, p) = 0$ on interval $[0, \bar{p})$. Otherwise stability function is equal to zero on $[0, q(C^0, X))$.

3 Stability and accuracy functions of lexicographically optimal solution

The lexicographic optimality principle is widely spread in optimization (see e.g. [4], [5]). This principle is used, for example, for solving stochastic programming problems, to define structure of priorities in complex systems which consist of different sublevels, etc. Observe also that any scalar constrained optimization problem may be transformed to unconstrained bicriteria lexicographic problem by using as first criterion some exact penalty function for problem constrains, and an original objective function as second criterion.

In this section we will consider a variant of lexicographic optimization with respect to all permutations of partial criteria.

Let S_m be the set of all permutations of N_m. For $s = (s_1, s_2, ..., s_m) \in S_m$, the binary relation $\prec_s$ of a lexicographic order is defined as follows: $t \prec_s t'$ if and only if $f(C, t) = f(C, t')$ or there exists an index $j \in N_m$ such that for all $k \in N_{j-1}$ we have $f_{s_j}(C, t) < f_{s_j}(C, t')$ and $f_{s_k}(C, t) = f_{s_k}(C, t')$. Here $N_0 = \emptyset$ for $j = 1$.

Under the vector (m-criteria) combinatorial optimization problem $Z_L^m(C)$ we understand the problem of finding the lexicographic set $L^m(C)$ defined in the following way:

$$L^m(C) = \bigcup_{s \in S_m} L^m(C, s),$$

where

$$L^m(C, s) = \{t \in T : \ t \prec_s t' \ \ \forall \, t' \in T\}.$$

The elements of the set $L^m(C)$ are called lexicographic optima of the problem $Z_L^m(C)$. It is easy to see that any lexicographic optimum belongs to the Pareto set.

For a given matrix C, we will measure the quality of $t^0 \in L^m(C^0)$ by the value of the relative error $\varepsilon_L(C, t^0)$ which is introduced as follows:

$$\varepsilon_L(C, t^0) = \min_{i \in N_m} \max_{t \in T} \frac{f_i(C, t^0) - f_i(C, t)}{f_i(C, t)}.$$

While $t^0 \in L^m(C)$ for any instance of problem $Z_L^m(C)$, the equality $\varepsilon_L(C, t^0) = 0$ holds. The inverse statement is not true (see in the previous section). If t^0 looses lexicographic optimality in an $Z_L^m(C)$, then the relative error $\varepsilon_L(C, t^0)$ characterizes the quality of t^0.

For a lexicographically optimal solution $t^0 \in L^m(C^0)$, an arbitrary set of non-stable elements X and $p \in [0, q(C^0, X))$ the value of the stability function is defined as follows:

$$S_L(t^0, X, p) = \max_{C \in \Omega_p(C^0, X)} \varepsilon_L(C, t^0).$$

Similarly, for a lexicographically optimal solution $t^0 \in P^m(C^0)$, an arbitrary set of non-stable elements X and $\delta \in [0,1)$ the value of the accuracy function is defined as follows:

$$A_L(t^0, X, \delta) = \max_{C \in \Theta_\delta(C^0, X)} \varepsilon_L(C, t^0).$$

Next two theorems give formulae for calculating values of stability and accuracy functions and corresponding radii in lexicographic case. We will omit their proofs because they are similar to the Pareto case.

Theorem 3 *For a lexicographically optimal solution $t^0 \in L^m(C^0)$, an arbitrary set of non-stable elements X, and $p \in [0, q(C^0, X))$,*

$$S_L(t^0, X, p) = \min_{i \in N_m} \max_{t \in T} \frac{f_i(C^0, t^0) - f_i(C^0, t) + p|(t \otimes t^0) \cap X|}{f_i(C^0, t) - p|t \cap X|}.$$

For a lexicographically optimal solution $t^0 \in L^m(C^0)$, an arbitrary set of non-stable elements X, and $\delta \in [0,1)$,

$$A_L(t^0, X, \delta) = \min_{i \in N_m} \max_{t \in T} \frac{f_i(C^0, t^0) - f_i(C^0, t) + \delta f_i(C^0, (t \otimes t^0) \cap X)}{f_i(C^0, t) - \delta f_i(C^0, t \cap X)}.$$

By analogue, for an arbitrary set of non-stable elements X, we define stability radius and accuracy radius as follows:

$$R_L^S(t^0, X) = \sup\{p \in [0, q(C^0, X)) : \; S_L(t^0, X, p) = 0\},$$

$$R_L^A(t^0, X) = \sup\{\delta \in [0,1) : \; A_L(t^0, X, \delta) = 0\}.$$

Theorem 4 *For a lexicographically optimal solution $t^0 \in L^m(C^0)$ and an arbitrary set of non-stable elements X,*

$$R_L^S(t^0, X) = \min\{q(C^0, X), \max_{i \in N_m} \min_{t \in T \setminus \{t^0\}} \frac{f_i(C^0, t) - f_i(C^0, t^0)}{|(t \otimes t^0) \cap X|}\},$$

$$R_L^A(t^0, X) = \min\{1, \max_{i \in N_m} \min_{t \in T_a} \frac{f_i(C^0, t) - f_i(C^0, t^0)}{f_i(C^0, (t \otimes t^0) \cap X)}\}.$$

4 Example

Consider the vector traveling salesman problem defined on graph $G = K_4$. Let the ground set E be equal to the set of all edges of G, i.e., $E = \{e_1, e_2, ..., e_6\}$ and let the set of feasible solutions T represents a family of all subsets of edges which form Hamiltonian cycles in the graph G. For this particular graph there are only three such subsets (see Fig. 1), thus $T = \{t_1, t_2, t_3\}$, where $t_1 = \{e_1, e_2, e_5, e_6\}$, $t_2 = \{e_1, e_3, e_4, e_6\}$, $t_3 = \{e_2, e_3, e_4, e_5\}$.

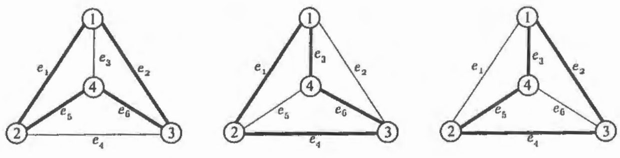

Fig. 1 All Hamiltonian cycles in graph K_4

We will consider 2-criteria optimization problem with the initial matrix of weights

$$C^0 = \begin{bmatrix} 2 & 2 & 3 & 2 & 1 & 1 \\ 3 & 1 & 1 & 4 & 3 & 3 \end{bmatrix}.$$

Then $f(C^0, t_1) = (9, 9)$, $f(C^0, t_2) = (7, 10)$, $f(C^0, t_3) = (6, 11)$, $P^2(C^0) = \{t_1, t_2, t_3\}$. Let all elements of E be non-stable, i.e. $X = E$. By Theorem 1, we calculate that

$$S_P(t_1, E, p) = \max\{0, \frac{4p-1}{10-4p}\} = \begin{cases} 0 & \text{if } p \in [0, \frac{1}{4}], \\ \frac{4p-1}{10-4p} & \text{if } p \in (\frac{1}{4}, 1), \end{cases}$$

$$S_P(t_2, E, p) = \max\{0, \frac{4p-2}{9-4p}, \frac{4p-1}{11-4p}\} = \begin{cases} 0 & \text{if } p \in [0, \frac{1}{4}], \\ \frac{4p-1}{11-4p} & \text{if } p \in (\frac{1}{4}, 1), \end{cases}$$

$$S_P(t_3, E, p) = \max\{0, \frac{4p-3}{9-4p}, \min\{\frac{4p-1}{7-4p}, \frac{4p+1}{10-4p}\}\} = \begin{cases} 0 & \text{if } p \in [0, \frac{1}{4}], \\ \frac{4p-1}{7-4p} & \text{if } p \in (\frac{1}{4}, \frac{17}{20}], \\ \frac{4p+1}{10-4p} & \text{if } p \in (\frac{17}{20}, 1). \end{cases}$$

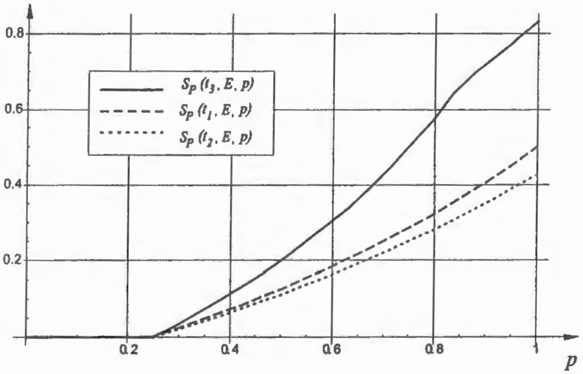

Fig. 2 Stability functions $S_P(t_1, E, p)$, $S_P(t_2, E, p)$, $S_P(t_3, E, p)$

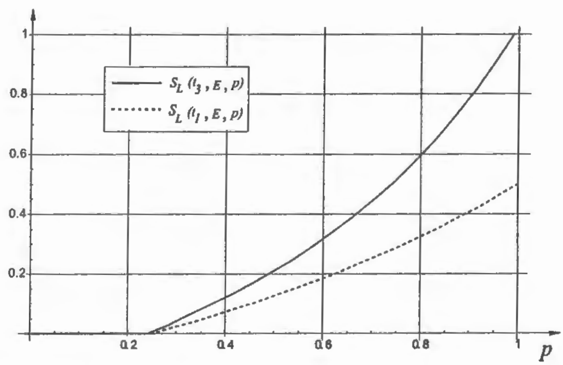

Fig. 3 Stability functions $S_L(t_1, E, p)$, $S_L(t_3, E, p)$

Observe that for any solution t_1, t_2, t_3 the stability radius is equal to $\frac{1}{4}$. But from the robustness point of view, t_2 may be regarded as 'better' than t_1 and t_3, since $S_P(t_2, E, p) \leq S_P(t_1, E, p)$ and $S_P(t_1, E, p) \leq S_P(t_3, E, p)$ for all $p \in [0, 1)$, with strict inequalities on some subinterval of $[0, 1)$ (see Fig. 2).

If we consider lexicographic optimality principle, then we get $L^2(C^0) = \{t_1, t_3\}$. By Theorem 3, we obtain that

$$S_L(t_1, E, p) = \max\{0, \frac{4p-1}{10-4p}\} = \begin{cases} 0 & \text{if } p \in [0, \frac{1}{4}], \\ \frac{4p-1}{10-4p} & \text{if } p \in (\frac{1}{4}, 1), \end{cases}$$

$$S_L(t_3, E, p) = \max\{0, \frac{4p-1}{7-4p}\} = \begin{cases} 0 & \text{if } p \in [0, \frac{1}{4}], \\ \frac{4p-1}{7-4p} & \text{if } p \in (\frac{1}{4}, 1). \end{cases}$$

We can see again that for both solutions t_1 and t_3 the stability radii are equal to $\frac{1}{4}$. But t_1 is 'better' than t_3, since $S_L(t_1, E, p) \leq S_L(t_3, E, p)$ for all $p \in [0, 1)$ with strict inequality on some subinterval of $[0, 1)$ (see Fig. 3).

5 Conclusions

The example in previous section suggests that small changes or inaccuracies in estimating objective function coefficients may influence significantly the set of efficient solutions of multicriteria combinatorial optimization problem. Moreover, some initially efficient solutions cannot be considered 'robust', because very small changes of data destroy their efficiency. Therefore, a possibility of ranking initially efficient solutions from the 'robustness' point of view is of special importance for a decision maker.

The simplest measure of the 'robustness' of the efficient solution is its stability radius or the accuracy radius. But frequently these radii are not sufficient to rank the efficient solutions and it is necessary to calculate complementary more general characteristics of solutions like stability and accuracy functions.

The accuracy and stability functions describe the quality of efficient solution in the situation when coefficients in criteria are subject to uncertainty. The definitions of these functions are directly connected to given optimality principle. The stability and accuracy radii give us the maximum values of independent perturbations which preserve the efficiency of a given solution.

The formulae provided in the paper do not lead directly to efficient methods of calculating the values of defined functions and radii. Moreover, one should not expect, that exact values of radii or defined functions may be computed easily for difficult combinatorial optimization problems. However, from the practical point of view it would be enough to have some approximate evaluations of these values. For single objective case such approximate methods, based on subsets of so-called k-best solutions [8], has been proposed in [12], [13] and [14]. It would be interesting to study whether this approach may be extended for multicriteria case.

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