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# APPLICATION OF UNCERTAIN VARIABLES TO A PROJECT MANAGEMENT UNDER UNCERTAINTY[†]

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## Abstract

The paper concerns task and resource allocation in a complex of operations which may be considered as a part of the knowledge-based project management system. The brief overview of concepts and results concerning the allocation problem under uncertainty described by uncertain variables is presented. The quality of the decision based on an expert's knowledge is considered. An application of two-level decomposition of the complex and the allocation taking into account uncertain and random parameters in the description of the operations are discussed. Three simple examples illustrate the presented approach.

## 1 Introduction

The paper deals with selected problems of task and resource allocation in a project management [see e.g. 14, 15], based on an uncertain knowledge. For decision making in a wide class of uncertain systems, a concept of uncertain variables has been developed [3, 4, 8, 13]. The uncertain variable is described by a certainty distribution given by an expert and characterizing his/her opinion of approximate values of the variable. The purpose of this paper is to show how the uncertain variables may be applied to task and resource allocation in the project considered as a complex of operations described by a relational knowledge representation with unknown parameters. The considerations are based on the general methods concerning the application of uncertain variables to the control of the complex of operations [5]. In Sect. 2 a brief description of the uncertain variables used in the next sections is presented. The details may be found in [4, 8], and the applications to related problems – in [6, 9, 11].

## 2 Uncertain variables and decision problem

In the definition of the uncertain variable $\bar{x}$ we consider two soft properties (i.e. such properties $\varphi(x)$ that for the fixed $x$ the logic value $v[\varphi(x)] \in [0,1]$ ): "$\bar{x} \cong x$" which means "$\bar{x}$ is approximately equal to $x$" or "$x$ is the approximate value of $\bar{x}$", and "$\bar{x} \,\tilde{\in}\, D_x$" which means "$\bar{x}$ approximately belongs to the set $D_x$" or "the approximate value of $\bar{x}$ belongs to $D_x$". The *uncertain variable* $\bar{x}$ is defined by a set of values $X$ (real number vector space), the function $h(x) = v(\bar{x} \cong x)$ (i.e. the certainty index that $\bar{x} \cong x$, given by an expert) and the following definitions for $D_x, D_1, D_2 \subseteq X$ :

$$v(\bar{x} \,\tilde{\in}\, D_x) = \begin{cases} \max\limits_{x \in D_x} h(x) & \text{for} \quad D_x \neq \varnothing \\ 0 & \text{for} \quad D_x = \varnothing, \text{ (empty set)} \end{cases}$$

$$v(\bar{x} \,\tilde{\notin}\, D_x) = 1 - v(\bar{x} \,\tilde{\in}\, D_x) \,,$$

[†] This paper is a modified and extended version of the work presented at 16[th] Int. Conf. on Systems Engineering, Coventry University, September 2003.

$$v(\bar{x} \tilde{\in} D_1 \vee \bar{x} \tilde{\in} D_2) = \max\{v(\bar{x} \tilde{\in} D_1), v(\bar{x} \tilde{\in} D_2)\},$$

$$v(\bar{x} \tilde{\in} D_1 \wedge \bar{x} \tilde{\in} D_2) = \begin{cases} \min\{v(\bar{x} \tilde{\in} D_1),\ v(\bar{x} \tilde{\in} D_2)\} & \text{for} \quad D_1 \cap D_2 \neq \varnothing \\ 0 & \text{for} \quad D_1 \cap D_2 = \varnothing. \end{cases}$$

The function $h(x)$ is called a *certainty distribution.*

*C-uncertain variable* $\bar{x}$ is defined by the set of values $X$, the function $h(x) = v(\bar{x} \cong x)$ given by an expert, and the following definitions:

$$v_c(\bar{x} \tilde{\in} D_x) = \frac{1}{2}[v(\bar{x} \tilde{\in} D_x) + 1 - v(\bar{x} \tilde{\in} \overline{D}_x)] \qquad (1)$$

where $\overline{D}_x = X - D_x$,

$$v_c(\bar{x} \tilde{\notin} D_x) = 1 - v_c(\bar{x} \tilde{\in} D_x),$$

$$v_c(\bar{x} \tilde{\in} D_1 \vee \bar{x} \tilde{\in} D_2) = v_c(\bar{x} \tilde{\in} D_1 \cup D_2),$$

$$v_c(\bar{x} \tilde{\in} D_1 \wedge \bar{x} \tilde{\in} D_2) = v_c(\bar{x} \tilde{\in} D_1 \cap D_2).$$

The application of C-uncertain variable means better using of the expert's knowledge, but may be more complicated from the computational point of view. Let us consider a static plant with the input vector $u \in U$ and the output vector $y \in Y$, described by a relation $R(u, y; x) \subset U \times Y$ where the vector of unknown parameters $x \in X$ is assumed to be a value of an uncertain variable described by the certainty distribution $h(x)$ given by an expert. If the relation $R$ is not a function then the value $u$ determines a set of possible outputs

$$D_y(u; x) = \{y \in Y : (u, y) \in R(u, y; x)\}. \qquad (2)$$

For the requirement $y \in D_y \subset Y$ given by a user, we can formulate the following **decision problem**: For the given $R(u, y; x)$, $h(x)$ and $D_y$ one should find the decision $u^*$ maximizing the certainty index that the set of possible outputs (2) approximately belongs to $D_y$ (i.e. belongs to $D_y$ for an approximate value of $\bar{x}$). Then

$$u^* = \arg\max_{u \in U} v[D_y(u; \bar{x}) \tilde{\subseteq} D_y] = \arg\max_{u \in U} \max_{x \in D_x(u)} h(x) \qquad (3)$$

where

$$D_x(u) = \{x \in X : D_y(u; x) \subseteq D_y\}.$$

This formulation of the decision problem will be used in the next sections in the formulation of allocation problems for the project under consideration.

## 3 Task and resource allocation in the complex of parallel operations

In the deterministic case the complex of parallel operations is described by a set of functions

$$T_i = \varphi_i(u_i), \quad i = 1, 2, ..., k, \qquad (4)$$

$$T = \max\{T_1, T_2, ..., T_k\} = \max_i \varphi_i(u_i) \triangleq \Phi(u) \qquad (5)$$

where $T_i$ is the execution time of the $i$-th operation, $u_i$ for each $i$ is the size of a task in the problem of task allocation or the amount of a resource in the problem of resource allocation, $T$ is the execution time of the whole complex and $u = (u_1, u_2, ..., u_k) \in \overline{U}$. The set $\overline{U} \subset R^k$ is determined by the constraints

$$\bigwedge_{i \in \overline{1,k}} u_i \geq 0, \qquad \sum_{i=1}^{k} u_i = U \qquad (6)$$

where $U$ is the total size of the task or the total amount of the resource to be distributed among the operations. The function $\varphi_i$ is an increasing function of $u_i$ (and $\varphi_i(0) = 0$) in the case of tasks and decreasing function of $u_i$ in the case of resources. The complex of the parallel operations may be considered as a specific decision (control) plant (Fig. 1) with the input vector $u$ and a single output $y = T$, described by a function $\Phi(u)$

determined according to (5).

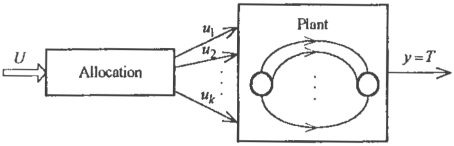


Figure 1. Complex of parallel operations as a decision plant

Let us consider a complex of parallel operations described by inequalities

$$T_i \le \varphi_i(u_i, x_i), \quad i = 1, 2, ..., k \qquad (7)$$

where $u_i$ is the size of a task assigned to the $i$-th operation, $x_i \in R^1$ is a parameter and $\varphi_i$ is a known increasing function of $u_i$. The parameter $x_i$ is unknown and is assumed to be a value of an uncertain variable $\bar{x}_i$ described by a certainty distribution $h_{xi}(x_i)$ given by an expert. Now the relational knowledge representation, consisting of (7) and the relationship $T = \max(T_1, T_2, ..., T_k)$, is completed by the functions $h_{xi}(x_i)$. We assume that $\bar{x}_1, \bar{x}_2, ..., \bar{x}_k$ are independent uncertain variables, i.e.

$$h_x(x) = \min_i h_{xi}(x_i) \quad \text{where } x = (x_1, x_2, ..., x_k).$$

The description of the complex is analogous for the resource allocation problem. When $u_i$ is the amount of a resource assigned to the $i$-th operation, $\varphi_i$ is a decreasing function of $u_i$ and $U$ denotes the total amount of the resource to be distributed. According to the general formulation of the decision problem presented in Sect. 2, the allocation problem may be formulated as an optimization problem consisting in finding the optimal allocation $u^*$ which maximizes the certainty index of the soft property: "the set of possible values $T$ approximately belongs to $[0, \alpha]$" (i.e. belongs to $[0, \alpha]$ for an approximate value of $\bar{x}$).

**Optimal allocation (decision) problem:** For the given $\varphi_i$, $h_{xi}$ $(i \in \overline{1,k})$, $U$ and $\alpha$ find

$$u^* = \arg\max_{u \in U} v(u)$$

where

$$v(u) = v\{D_T(u; \bar{x}) \tilde{\subseteq} [0, \alpha]\} = v[T(u, \bar{x}) \tilde{\le} \alpha]. \qquad (8)$$

The soft property "$D_T(u; \bar{x}) \tilde{\subseteq} [0, \alpha]$" is denoted here by "$T(u, \bar{x}) \tilde{\le} \alpha$", and $D_T(u; x)$ denotes the set of possible values $T$ for the fixed $u$, determined by the inequality $T \le \max_i \varphi_i(u_i, x_i)$. The property "$T(u, \bar{x}) \tilde{\le} \alpha$" means that the maximum possible value of the execution time $T$ is approximately (i.e. for the approximate value of $\bar{x}$) less or equal to $\alpha$.

According to (8)

$$v(u) = v\{[T_1(u_1, \bar{x}_1) \tilde{\le} \alpha] \wedge [T_2(u_2, \bar{x}_2) \tilde{\le} \alpha] \wedge ... \wedge [T_k(u_k, \bar{x}_k) \tilde{\le} \alpha]\}. \qquad (8a)$$

Then

$$u^* = \arg\max_{u \in U} \min_i v_i(u_i) \qquad (9)$$

where

$$v_i(u_i) = v[T_i(u_i, \bar{x}_i) \tilde{\le} \alpha] = v[\varphi_i(u_i, \bar{x}_i) \tilde{\le} \alpha] = v[\bar{x}_i \tilde{\in} D_{xi}(u_i)], \qquad D_{xi}(u_i) = \{x_i \in R^1 : \varphi_i(u_i, x_i) \le \alpha\}.$$

Finally

$$v_i(u_i) = \max_{x_i \in D_{xi}(u_i)} h_{xi}(x_i) \qquad (10)$$

and

$$u^* = \arg\max_{u \in U} \min_i \max_{x_i \in D_{xi}(u_i)} h_{xi}(x_i) \qquad (11)$$

The value $v_i(u_i)$ denotes the certainty index that in the $i$-th operation an approximate value of the execution time is less than $\alpha$. The procedure of finding the optimal allocation $u^*$ is then the following:

1. To determine $v_i(u_i)$ using (10).
2. To determine $u^*$ according to (9), subject to constraints (6).

It may be proved [5] that for the optimal solution $v_1(u_1^*) = v_2(u_2^*) = ... = v_k(u_k^*)$.

In many cases an expert gives the value $x_i^*$ and the interval of the approximate values of $\bar{x}_i$: $x_i^* - d_i \le x_i \le x_i^* + d_i$. Then we assume that $h_{xi}(x_i)$ has a triangular form presented in Fig. 2 where $d_i \le x_i^*$. Let us consider the relation (7) in the form $T_i \le x_i u_i$ where $x_i > 0$ and $u_i$ denotes the size of a task. In this case, using (10) it is easy to obtain the following formulas for the functions $v_i(u_i)$:

$$v_i(u_i) = \begin{cases} 1 & \text{for} \quad u_i \le \dfrac{\alpha}{x_i^*} \\ \dfrac{1}{d_i}\left(\dfrac{\alpha}{u_i} - x_i^*\right) + 1 & \text{for} \quad \dfrac{\alpha}{x_i^*} \le u_i \le \dfrac{\alpha}{x_i^* - d_i} \\ 0 & \text{for} \quad u_i \ge \dfrac{\alpha}{x_i^* - d_i}. \end{cases} \tag{12}$$

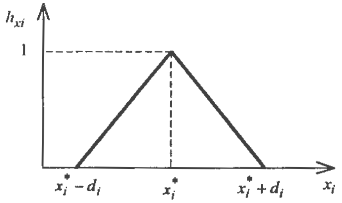

Figure 2. Example of the certainty distribution

For the relations $T_i \le x_i u_i^{-1}$ where $u_i$ denotes the size of a resource, the functions $v_i(u_i)$ have an analogous form with $u_i^{-1}$ in the place of $u_i$:

$$v_i(u_i) = \begin{cases} 0 & \text{for} \quad u_i \le \dfrac{x_i^* - d_i}{\alpha} \\ \dfrac{1}{d_i}(\alpha u_i - x_i^*) + 1 & \text{for} \quad \dfrac{x_i^* - d_i}{\alpha} \le u_i \le \dfrac{x_i^*}{\alpha} \\ 1 & \text{for} \quad u_i \ge \dfrac{x_i^*}{\alpha}, \end{cases} \tag{13}$$

**Example 1**

Let us consider the resource allocation for two operations ($k = 2$). Now in the maximization problem (9) the decision $u_1^*$ may be found by solving the equation $v_1(u_1) = v_2(U - u_1)$, and $u_2^* = U - u_1^*$. Using (13) we obtain the following result:

1. For

$$\alpha \le \frac{x_1^* - d_1 + x_2^* - d_2}{U} \tag{14}$$

$v(u) = 0$ for any $u_1$.

2. For

$$\frac{x_1^* - d_1 + x_2^* - d_2}{U} \le \alpha \le \frac{x_1^* + x_2^*}{U} \tag{15}$$

we obtain

$$u_1^* = \frac{\alpha d_1 U + x_1^* d_2 - x_2^* d_1}{\alpha(d_1 + d_2)}, \tag{16}$$

$$v(u^*) = \frac{1}{d_1}[\alpha u_1^* - x_1^*] + 1. \qquad (17)$$

3. For

$$\alpha \geq \frac{x_1^* + x_2^*}{U} \qquad (18)$$

we obtain $v(u^*) = 1$ for any $u_1$ satisfying the condition

$$\frac{x_1^*}{\alpha} \leq u_1 \leq U - \frac{x_2^*}{\alpha}.$$

In the case (14) $\alpha$ is too small (the requirement is too strong) and it is not possible to find the allocation for which $v(u)$ is greater than 0. In the case (15) we obtain one solution maximizing $v(u)$. For the numerical data $U = 9$, $\alpha = 0.5$, $x_1^* = 2$, $x_2^* = 3$, $d_1 = d_2 = 1$, using (16) and (17) we obtain $u_1^* = 3.5$, $u_2^* = 5.5$ and $v = 0.75$ what means that the requirement $T \leq \alpha$ will be approximately satisfied with the certainty index 0.75.

In the case of $C$-uncertain variable, according to (I) one should determine

$$u^* = \arg\max_u v_c(u)$$

where

$$v_c(u) = \frac{1}{2}\{v(u) + 1 - v[T(u,\overline{x}) \tilde{>} \alpha]\}.$$

The optimization problem is now much more complicated and should be considered in the different intervals of $\alpha$. For example, if

$$\frac{x_1^* + x_2^*}{U} \leq \alpha \leq \frac{x_1^* + d_1 + x_2^* + d_2}{U} \qquad (19)$$

then $u_{c1}^* = u_1^*$ in (16) and

$$v_c(u^*) = 1 - \frac{1}{2d_1}(\alpha u_{c1}^* - x_1^*). \qquad (20)$$

For the numerical data $U = 9$, $\alpha = 0.6$, $x_1^* = 2$, $x_2^* = 3$, $d_1 = d_2 = 1$ the inequality (19) is satisfied. Then, using (20) we obtain $u_{c1}^* = 3.67$, $u_{c2}^* = 5.33$ and $v_c(u^*) = 0.9$.

## 4 Quality of decisions based on the expert's knowledge

Let us assume that the exact descriptions of the operations have a form of the equations

$$T_i = f_i(u_i), \quad i = 1, 2, ..., k.$$

If the functions $f_i$ are known, it is possible (see e.g. [1,10]) to determine the optimal decisions $\overline{u}_i$ minimizing the execution time $T = \max\{T_1, T_2, ..., T_k\}$ and satisfying the constrains (6). If the functions $f_i$ are not known by a designer of the system, the nonoptimal decisions $u_i^*$ based on the expert's knowledge described in the previous section are determined and applied. Consequently, the execution time is the following

$$T^* = \max\{f_1(u_1^*), f_2(u_2^*), ..., f_k(u_k^*)\}.$$

To evaluate the result of the decisions based on the imperfect knowledge given by an expert, one may introduce a quality index

$$\frac{T^*}{\overline{T}} \triangleq Q$$

where $\overline{T}$ is the execution time obtained by applying the optimal decisions $\overline{u}_i$. The index $Q$ may be used to investigate how the quality depends on forms and parameters of the certainty distributions $h_{xi}$ (i.e. on the knowledge given by an expert), and to compare different experts.

**Example 2**

Consider the task allocation for two operations $(k = 2)$ described by

$$T_1 = a_1 u_1, \qquad T_2 = a_2 u_2.$$

It is easy to show that

$$\overline{T} = \frac{Ua_1a_2}{a_1 + a_2}.$$

Assume that

$$\frac{d_1}{x_1^*} = \frac{d_2}{x_2^*} \triangleq \gamma .$$

Using (12), it may be shown (see [13]) that

$$u_1^* = \frac{x_2^* U}{x_1^* + x_2^*}, \qquad u_2^* = \frac{x_1^* U}{x_1^* + x_2^*},$$

under the assumption

$$\frac{Ux_1^* x_2^*(1-\gamma)}{x_1^* + x_2^*} \le \alpha \le \frac{Ux_1^* x_2^*}{x_1^* + x_2^*}.$$

Using $\overline{T}$ and $T^* = \max\{a_1u_1^*, a_2u_2^*\}$, we obtain the following final result

$$Q = \frac{T^*}{\overline{T}} = \begin{cases} \dfrac{\varepsilon+1}{\varepsilon+\delta} & \text{for} \quad 0 < \delta \le 1 \\ \dfrac{\delta(\varepsilon+1)}{\varepsilon+\delta} & \text{for} \quad \delta \ge 1 \end{cases}$$

where

$$\delta = \frac{x_2^*}{a_2}\left(\frac{x_1^*}{a_1}\right)^{-1}, \qquad \varepsilon = \frac{a_1}{a_2}.$$

It is worth noting that the quality index $Q$ strongly depends on $\delta$ for small values $\delta$.

## 5 Decomposition and two-level management

The determination of the decision $u^*$ may be difficult for $k > 2$ because of the great computational difficulties. To decrease these difficulties we can apply the decomposition of the complex into two subcomplexes and consequently to obtain two-level management system (Fig. 3). This approach is based on the idea of decomposition and two-level control presented for the deterministic case [1]. On the upper level the value $U$ is divided into $U_1$ and $U_2$ assigned to the first and the second subcomplex, respectively, and on the lower level the allocation $u^{(1)}$, $u^{(2)}$ for the subcomplexes are determined. Let us introduce the following notation:

$n, m$ – the number of operations in the first and the second complex, respectively, $n + m = k$,

$T^{(1)}$, $T^{(2)}$ – the execution times in the subcomplexes, i.e.

$$T^{(1)} = \max(T_1, T_2, ..., T_n),\ T^{(2)} = \max(T_{n+1}, T_{n+2}, ..., T_{n+m}),$$

$u^{(1)}$, $u^{(2)}$ – the allocations in the subcomplexes, i.e.

$$u^{(1)} = (u_1, ..., u_n), \qquad u^{(2)} = (u_{n+1}, ..., u_{n+m}).$$

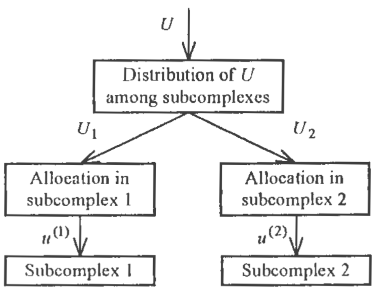

Figure 3. Two-level management system

The procedure of the determination of $u^*$ is then the following:

1. To determine the allocation $u^{(1)*}(U_1)$, $u^{(2)*}(U_2)$ and the certainty indexes $v^{(1)*}(U_1)$, $v^{(2)*}(U_2)$ in the same way as $u^*$, $v^*$ in Sect. 2, with $U_1$ and $U_2$ in the place of $U$.

2. To determine $U_1^*$, $U_2^*$ via the maximization of

$$v(T \tilde{\leq} \alpha) = v[(T^{(1)} \tilde{\leq} \alpha) \wedge (T^{(2)} \tilde{\leq} \alpha)] \triangleq v(U_1, U_2).$$

Then

$$(U_1^*, U_2^*) = \arg \max_{U_1, U_2} \min\{v^{(1)*}(U_1), v^{(2)*}(U_2)\}$$

with the constraints: $U_{1,2} \geq 0$, $U_1 + U_2 = U$.

3. To find the values of $u^{(1)*}$, $u^{(2)*}$ and $v^*$ putting $U_1^*$ and $U_2^*$ into the results $u^{(1)*}(U_1)$, $u^{(2)*}(U_2)$ obtained in the point 1 and into $v(U_1, U_2)$ in the point 2.

It may be shown that the result obtained via the decomposition is the same as the result of the direct approach presented in Sect. 3.

## 6 Resource allocation in a project with cascade structure

Let us consider the resource allocation in a complex of operations with cascade structure in which the result obtained from the $i$-th operation is put at the input of the $(i+1)$-th operation ($i = 1, 2, ..., k-1$). The operations are described by the inequalities (7) and the unknown parameters $x_i$ are described by certainty distributions $h_{xi}$, the same as in Sect. 3. The **optimal allocation problem** may now be formulated as follows:

For the given $h_{xi}$ and $U$ find the allocation $u^* = (u_1^*, u_2^*, ..., u_k^*)$ maximizing the certainty index

$$v(u) = v\{D_y(u; \bar{x}) \tilde{\subseteq} [0, \alpha]\} = v[y(u; \bar{x}) \tilde{\leq} \alpha] \qquad (21)$$

where

$$y(u; x) = \varphi_1(u_1, x_1) + \varphi_2(u_2, x_2) + ... + \varphi_k(u_k, x_k)$$

denotes the maximum possible value of the execution time $T = T_1 + T_2 + ... + T_k$. The determination and maximization of $v(u)$ may be very complicated. It is much easier to solve the following problem: Find the allocation $\hat{u} = (\hat{u}_1, \hat{u}_2, ..., \hat{u}_k)$ satisfying the constraint (6) and maximizing the certainty index

$$\hat{v}(u) = \max_{\alpha_1, ..., \alpha_k} v\{[\varphi_1(u_1, \bar{x}_1) \tilde{\leq} \alpha_1] \wedge ... \wedge [\varphi_k(u_k, \bar{x}_k) \tilde{\leq} \alpha_k]\}, \qquad (22)$$

subject to constraint $\alpha_1 + \alpha_2 + ... + \alpha_k = \alpha$. It is easy to note that $\hat{v}(u) \leq v(u)$. Then $\hat{u}$ is the allocation maximizing the lower bound of the certainty index that the maximum possible value of the execution time $T$ is approximately less than $\alpha$. According to (22)

$$\hat{v}(u) = \max_{\alpha_1, ..., \alpha_k} \min \hat{v}_i(u_i, \alpha_i) \qquad (23)$$

where $\hat{v}_i(u_i, \alpha_i)$ is obtained in the same way as $v_i$ in (10), with $\alpha_i$ in the place of $\alpha$, i.e.

$$\hat{v}_i(u_i, \alpha_i) = \max_{x_i \in D_{xi}(u_i, \alpha_i)} h_{xi}(x_i)$$

where

$$D_{xi}(u_i) = \{x_i \in R^1 : \varphi_i(u_i, x_i) \leq \alpha\}.$$

The presented approach may be extended to cascade-parallel structure (Fig. 4).

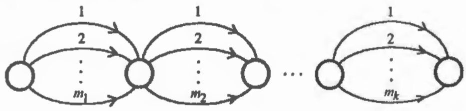


Figure 4. Cascade-parallel structure of the complex

The $i$-th subsystem ($i = 1, 2, ..., k$) is a complex of $m_i$ parallel operations denoted by 1, 2, ..., $m_i$. Let us denote by $U_i$ the amount of the resource assigned to the $i$-th subsystem by

$$\overline{u}_i^* = (u_{i1}^*, u_{i2}^*, ..., u_{im_i}^*) \triangleq f_i(U_i, \alpha_i) \qquad (24)$$

the allocation in the $i$-th subsystem, determined in the way described in the previous section with $(U_i, \alpha_i)$ in the place of $(U, \alpha)$, and by

$$\overline{v}_i^* = \overline{v}_i(\overline{u}_i^*) \triangleq g_i(U_i, \alpha_i) \qquad (25)$$

the maximum value of the certainty index (21) for the $i$-th subsystem. The determination of the functions (24) and (25) may be considered as a local optimization of the subsystems. The global optimization consists in maximization of the certainty index corresponding to (23), with respect to $\alpha = (\alpha_1, ..., \alpha_k)$ and $\overline{U} = (U_1, ..., U_k)$, i.e. maximization

$$\max_{\overline{U}} \max_{\alpha} \min_{i} \; g_i(U_i, \alpha_i) \qquad (26)$$

with constraints $U_1 + ... + U_k = U$, $\alpha_1 + ... + \alpha_k = \alpha$.

Putting the results of maximization (20) into $f_i$ in (24) we obtain the allocation $\overline{u}_i^*$ for $i = 1, 2, ..., k$.

## 7 Transferring of resources during the project execution

In the case of more complicated structures as those considered in Sects 2+5, it may be possible to apply a decomposition of the execution time into intervals in which parts of the operations may be executed. In each interval, the parts of the operations create a subcomplex with parallel operations for which the resource allocation can be determined. Such an approach may be applied if it is possible to use the different values $u_i$ in the different intervals, i.e. to transfer the resource from one operation to another during the execution of the project. Assume that the operations are described by the inequalities $\dot{z}_i(t) \geq \Phi_i(u_i; x_i)$, $i = 1, 2, ..., k$ where $z_i(t) \in R^1$ denotes a state of the operation; $z_i(0) = 0$. If $z_i^* = z_i(T_i)$ denotes a size of the operation and $u_i$ is constant, then $T_i \leq \varphi_i(u_i, z_i^*; x_i)$ where

$$\varphi_i(u_i, z_i^*; x_i) = \frac{z_i^*}{\Phi_i(u_i; x_i)}.$$

Denote by $z_{ij}$ the part of $z_i^*$ executed in the $j$-th interval, by $Q_j$ a set of operations which may be executed in this interval and by $\overline{z}_j$ a set of $z_{ij}$ for all $i \in Q_j$. The allocation problem for the whole complex may be decomposed into two subproblems:

1. The determination of the optimal allocations $\overline{u}_j(\overline{z}_j, \alpha)$ for the successive intervals (i.e. for the subcomplex of parallel operations) and the respective certainty indexes

$$v_j^*[\overline{u}_j(\overline{z}_j, \alpha_j)] \triangleq g_j(\overline{z}_j, \alpha_j).$$

These are the certainty indexes (10) for $j = 1, 2, ..., m$ ($m$ is a number of intervals) for which the requirement is determined by $\alpha_j$. The constraints (6) for the $j$-th subcomplex take the form

$$(\bigwedge_{i \in Q_j} u_{ij} \geq 0) \wedge \sum_{i \in Q_j} u_{ij} = U \qquad (27)$$

where $u_{ij}$ denotes the amount of the resource assigned to the $i$-th operation in the $j$-th subcomplex.

2. The maximization of the certainty index $v[(\overline{T}_1 \tilde{\leq} \alpha_1) \wedge (\overline{T}_2 \tilde{\leq} \alpha_2) \wedge ... \wedge (\overline{T}_m \tilde{\leq} \alpha_m)]$, where $\overline{T}_j$ is the execution time of the $j$-th subcomplex – with respect to $\overline{z} = (\overline{z}_1, ..., \overline{z}_m)$ and $\overline{\alpha} = (\alpha_1, ..., \alpha_m)$, i.e.,

$$\max_{\overline{z}, \overline{\alpha}} \min\{g_1(\overline{z}_1, \alpha_1), g_2(\overline{z}_2, \alpha_2), ..., g_m(\overline{z}_m, \alpha_m)\} \quad (28)$$

subject to constraints

$$\sum_{j=1}^{m} \alpha_j = \alpha, \qquad \sum_{j \in P_i} z_{ij} = z_i^*$$

where $P_i$ denotes a set of the intervals in which the $i$-th operation may be executed, and $\alpha$ determines the requirement for the complex as a whole. Putting the results of the maximization (28) into $\bar{u}_j(\bar{z}_j, \alpha_j)$ we obtain the allocation $\bar{u}_j^*$ $(j \in \overline{1,m})$. The values $u_{ij}^*$ may be different for the different $j$ and the application of this allocation requires the transferring of the resources among the operations.

**Remark:** The approach presented above is based on the decomposition of the complex similar to that in Sect. 5. In the case presented in Sect. 5 the resource used in $j$-th subcomplex could not be used in the next complexes. In the case considered in this section $u_i$ means the size of a stream of the resource and the second part of (27) concern $j$-th interval. It is worth noting that the result of the decomposition $u_{ij}^*$ $(j \in \overline{1,m})$ does not maximize the certainty index that $T = \overline{T}_1 + ... + \overline{T}_m$ is approximately not greater than $\alpha$, but maximize the lower band of this index. It follows from the fact that $(\overline{T}_1 \le \alpha_1) \wedge ... \wedge (\overline{T}_m \le \alpha_m) \rightarrow T \le \alpha$ but not on the contrary.

# 8 Knowledge representation with uncertain and random parameters

The relation knowledge representation may contain two kinds of unknown parameters: uncertain parameters characterized by certainty distributions given by an expert and random parameters described by probability distributions [12]. Let us consider a complex of parallel operations described by the inequalities $T_i \le \varphi_i(u_i; x_i, w_i)$, $i \in \overline{1,k}$ where $x_i$ are values of uncertain variables as in Sect. 3, and $w_i$ are values of random variables $\widetilde{w}_i$ with the known probability densities. Consequently, the certainty index $v(u; w)$ defined by (8a) and the optimal allocation $u^*(w)$ are functions of $w = (w_1, w_2, ..., w_k) \in W$. Then two versions of the allocation problem may be formulated.

1. The determination of $u_{\mathrm{I}}^*$:

$$u_{\mathrm{I}}^* = \arg\max_{u \in \overline{U}} \mathrm{E}[v(u; \widetilde{w})]$$

$$= \arg\max_{u \in \overline{U}} \int_W v(u; w) f(w) dw \qquad (29)$$

where $f(w)$ denotes the probability density of $\widetilde{w}$.

2. The determination of $u_{\mathrm{II}}^*$:

$$u_{\mathrm{II}}^* = \mathrm{E}[u^*(\widetilde{w})] = \int_W u^*(w) f(w) dw. \qquad (30)$$

The results have the different interpretations: $u_{\mathrm{I}}^*$ denotes the allocation maximizing the expected value of the certainty index that the requirement is approximately satisfied, and $u_{\mathrm{II}}^*$ denotes the expected value of the optimal allocation for the fixed $w$. The similar approach may be applied in the case with the inequalities $T_i \le \varphi_i(u_i; x_i)$ considered in Sect. 3 where the uncertain variable $x_i$ is characterized by $h_{xi}(x_i; w_i)$ where the unknown parameter $w_i$ is a value of $\widetilde{w}_i$ with the probability density $f_i(w_i)$. It means that the operations are characterized by a set of experts giving the different values $w_i$ "chosen randomly" from the set of values described by the underlying statistics. Now $v(u)$ and $u^*$ determined in Sect. 3 depend on $w$ (because $h_{xi}$ depend on $w_i$) and we can apply two versions according to (29), (30).

**Example 3**

Let us assume that in the resource allocation problem considered in Example 1 $\delta_1 \le d_1 \le \varepsilon_1$, $\delta_2 \le d_2 \le \varepsilon_2$ and $d_2 d_1^{-1} \triangleq w$ is a value of a random variable $\widetilde{w}$ with a triangular probability density $f(w) = \gamma - \beta w$ for $w_1 \le w \le w_2$ and $f(w) = 0$ otherwise, where

$$\gamma = \frac{2w_2}{(w_2 - w_1)^2}, \qquad \beta = \frac{2}{(w_2 - w_1)^2},$$

$w_1 = \delta_2 \varepsilon_1^{-1}$, $w_2 = \varepsilon_2 \delta_1^{-1}$. If

$$\frac{x_1^* - \delta_1 + x_2^* - \delta_2}{U} \le \alpha \le \frac{x_1^* + x_2^*}{U}$$

then, according to (16),

$$u_1^*(w) = \frac{(\alpha U - x_2^*) + x_1^* w}{\alpha(1+w)}.$$

Applying the second version one obtains

$$u_{\mathrm{II}1}^* = \mathrm{E}[u_1^*(\widetilde{w})] = \beta(w_1^2 - w_2^2) + [(\gamma + \beta)B - \beta A](w_2 + w_1)$$

$$+ (A - B)(\gamma + \beta)\ln\frac{w_2 + 1}{w_1 + 1}$$

where $A = U - x_2^* \gamma^{-1}$, $B = x_1^* \gamma^{-1}$,

and $u_{\mathrm{II}2}^* = U - u_{\mathrm{II}1}^*$.

## 9 Conclusions

An approach to task and resource allocation in the complex of operations considered as a part of the knowledge-based project management system has been presented. The formalism based on the uncertain variables may be useful as a non-probabilistic description of the uncertain operations characterized by an expert. The presented approach may be extended to more complicated structures of the complex of operations with distributed knowledge [6] and may be applied to related problems concerning the control of manufacturing operations [2, 7].

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