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Fragment rozdziału 13 "Complex of operations" w książce Analysis and Decision Making in Uncertain Systems

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13.3 Special Cases and Examples

In many cases an expert gives the value x_i^* and the interval of the approximate values of $\bar{x}_i$: $x_i^* - d_i \le x_i \le x_i^* + d_i$. Then we assume that $h_{xi}(x_i)$ has a triangular form presented in Fig. 13.3 where $d_i \le x_i^*$. Let us consider the relation (13.11) in the form $T_i \le x_i u_i$ where $x_i > 0$ and u_i denotes the size of a task. In this case, using (13.17) and (13.19) it is easy to obtain the following formulas for the functions $v_i(u_i)$ and $\hat{v}_i(u_i)$:

$$v_i(u_i) = \begin{cases} 1 & \text{for} \quad u_i \le \dfrac{\alpha}{x_i^*} \\ \dfrac{1}{d_i}(\dfrac{\alpha}{u_i} - x_i^*) + 1 & \text{for} \quad \dfrac{\alpha}{x_i^*} \le u_i \le \dfrac{\alpha}{x_i^* - d_i} \\ 0 & \text{for} \quad u_i \ge \dfrac{\alpha}{x_i^* - d_i}, \end{cases} \tag{13.23}$$

$$\hat{v}_i(u_i) = \begin{cases} 0 & \text{for} \quad u_i \le \dfrac{\alpha}{x_i^* + d_i} \\ -\dfrac{1}{d_i}(\dfrac{\alpha}{u_i} - x_i^*) + 1 & \text{for} \quad \dfrac{\alpha}{x_i^* + d_i} \le u_i \le \dfrac{\alpha}{x_i^*} \\ 1 & \text{for} \quad u_i \ge \dfrac{\alpha}{x_i^*}. \end{cases} \tag{13.24}$$

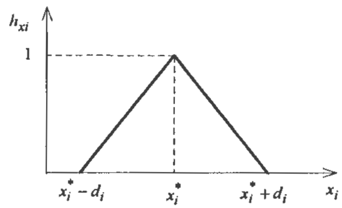

Figure 13.3. Example of the certainty distribution

For the relations $T_i \le x_i u_i^{-1}$ where u_i denotes the size of a resource, the functions $v_i(u_i)$ and $\hat{v}_i(u_i)$ have an analogous form with u_i^{-1} in place of u_i:

$$v_i(u_i)=\begin{cases}0 & \text{for} \quad u_i\le\dfrac{x_i^*-d_i}{\alpha}\\ \dfrac{1}{d_i}(\alpha u_i-x_i^*)+1 & \text{for} \quad \dfrac{x_i^*-d_i}{\alpha}\le u_i\le\dfrac{x_i^*}{\alpha}\\ 1 & \text{for} \quad u_i\ge\dfrac{x_i^*}{\alpha},\end{cases} \tag{13.25}$$

$$\hat{v}_i(u_i)=\begin{cases}1 & \text{for} \quad u_i\le\dfrac{x_i^*}{\alpha}\\ -\dfrac{1}{d_i}(\alpha u_i-x_i^*)+1 & \text{for} \quad \dfrac{x_i^*}{\alpha}\le u_i\le\dfrac{x_i^*+d_i}{\alpha}\\ 0 & \text{for} \quad u_i\ge\dfrac{x_i^*+d_i}{\alpha}.\end{cases} \tag{13.26}$$

Example 13.1.
Let us consider the resource allocation for two operations ($k=2$). Now in the maximization problem (13.13) the decision u_1^* may be found by solving the equation $v_1(u_1)=v_2(U-u_1)$ and $u_2^*=U-u_1^*$. Using (13.25), we obtain the following result:
1. For

$$\alpha\le\frac{x_1^*-d_1+x_2^*-d_2}{U} \tag{13.27}$$

$v(u)=0$ for any u_1.
2. For

$$\frac{x_1^*-d_1+x_2^*-d_2}{U}\le\alpha\le\frac{x_1^*+x_2^*}{U} \tag{13.28}$$

we obtain

$$u_1^*=\frac{\alpha d_1U+x_1^*d_2-x_2^*d_1}{\alpha(d_1+d_2)}, \tag{13.29}$$

$$v(u^*)=\frac{1}{d_1}[\alpha u_1^*-x_1^*]+1. \tag{13.30}$$

3. For

$$\alpha\ge\frac{x_1^*+x_2^*}{U} \tag{13.31}$$

we obtain $v(u^*)=1$ for any u_1 satisfying the condition

$$\frac{x_1^*}{\alpha}\le u_1\le U-\frac{x_2^*}{\alpha}.$$

In the case (13.27) α is too small (the requirement is too strong) and it is not possible to find the allocation for which $v(u)$ is greater than 0. In the case (13.28) we obtain one solution maximizing $v(u)$. For the numerical data $U=9$, $\alpha=0.5$, $x_1^*=2$, $x_2^*=3$, $d_1=d_2=1$, using (13.29) and (13.30) we obtain $u_1^*=3.5$, $u_2^*=5.5$ and $v=0.75$, which means that the requirement $T\le\alpha$ will be approximately satisfied with the certainty index 0.75. The solution of the optimization problem (13.18) based on (13.26) may be obtained in an analogous way:
1. For

$$\alpha\le\frac{x_1^*+x_2^*}{U} \tag{13.32}$$

$v_n(u) = 0$ for any u_1.
2. For

$$\frac{x_1^* + x_2^*}{U} \le \alpha \le \frac{x_1^* + d_1 + x_2^* + d_2}{U} \tag{13.33}$$

$u_{N1}^* = u_1^*$ in the formula (13.29) and

$$v_n(u^*) = \frac{1}{d_1}(\alpha u_{N1}^* - x_1^*). \tag{13.34}$$

3. For

$$\alpha \ge \frac{x_1^* + d_1 + x_2^* + d_2}{U}$$

we obtain $v_n(u^*) = 1$ for any u_1 satisfying the condition

$$\frac{x_1^* + d_1}{\alpha} \le u_1 \le U - \frac{x_2^* + d_2}{\alpha}.$$

For the numerical data we have the case (13.32) and $v_n(u) = 0$.

The optimization problem (13.22) for C-uncertain variables is much more complicated and should be considered in the different intervals of α introduced for v and v_n. For example, if

$$\frac{x_1^* + x_2^*}{U} \le \alpha \le \frac{x_1^* + d_1 + x_2^* + d_2}{U} \tag{13.35}$$

which means the combination of the cases (13.31) and (13.33), then $u_{c1}^* = u_{N1}^*$ and

$$v_c(u^*) = \frac{1}{2}[v(u^*) + 1 - v_n(u^*)].$$

Substituting $v(u^*) = 1$ and (13.34) yields

$$v_c(u^*) = 1 - \frac{1}{2d_1}(\alpha u_{c1}^* - x_1^*). \tag{13.36}$$

For the numerical data $U = 9$, $\alpha = 0.6$, $x_1^* = 2$, $x_2^* = 3$, $d_1 = d_2 = 1$ the inequality (13.35) is satisfied. Then, by using (13.29) and (13.36) we obtain $u_{c1}^* = 3.67$ and $v_c(u^*) = 0.9$. The results for these data in the case v and v_n are as follows: $u_{N1}^* = u_{c1}^* = 3.67$ and $v_n(u^*) = 0.2$; $v(u^*) = 1$ for any u_1 from the interval $[3.33,\ 4]$. □

Example 13.2.

Let us consider the task allocation for two operations. In the maximization problem (13.13) the decision u_1^* may be found by solving the equation $v_1(u_1) = v_2(U - u_1)$ and $u_2^* = U - u_1^*$. Using (13.23), we obtain the following result:
1. For

$$\alpha \le \frac{U(x_1^* - d_1)(x_2^* - d_2)}{x_1^* - d_1 + x_2^* - d_2} \tag{13.37}$$

$v(u) = 0$ for any u_1.
2. For

$$\frac{U(x_1^* - d_1)(x_2^* - d_2)}{x_1^* - d_1 + x_2^* - d_2} \le \alpha \le \frac{U x_1^* x_2^*}{x_1^* + x_2^*} \tag{13.38}$$

u_1^* is a root of the equation

$$\frac{1}{d_1}(\frac{\alpha}{u_1}-x_1^*)=\frac{1}{d_2}(\frac{\alpha}{U-u_1}-x_2^*)$$

satisfying the condition

$$\frac{\alpha}{x_1^*}\le u_1^*\le\frac{\alpha}{x_1^*-d_1},$$

and $v(u^*)=v_1(u_1^*)$.

3. For

$$\alpha\ge\frac{Ux_1^*x_2^*}{x_1^*+x_2^*} \qquad (3.39)$$

$v(u^*)=1$ for any u_1 satisfying the condition

$$U-\frac{\alpha}{x_2^*}\le u_1\le\frac{\alpha}{x_1^*}.$$

For example, if $U=2$, $\alpha=2$, $x_1^*=2$, $x_2^*=3$, $d_1=d_2=1$ then using (13.38) yields $u_1^*=1.25$, $u_2^*=0.75$, $v(u^*)=0.6$.

The result is simpler under the assumption

$$\frac{x_1^*}{d_1}=\frac{x_2^*}{d_2}\triangleq\gamma. \qquad (13.40)$$

Then in the case (13.40)

$$u_1^*=\frac{Ux_2^*}{x_1^*+x_2^*}, \qquad u_2^*=\frac{Ux_1^*}{x_1^*+x_2^*},$$

$$v(u^*)=v_1(u_1^*)=\frac{1}{d_1}(\frac{\alpha}{u_1^*}-x_1^*)+1=\gamma[\frac{\alpha(x_1^*+x_2^*)}{U\,x_1^*x_2^*}-1]+1. \qquad (13.41)$$

The formula (13.41) shows that $v(u^*)$ is a linear function of the parameter γ characterizing the expert's uncertainty. □

The result in point 3 of Example 13.2 may be easily generalized for *k* operations described by the inequalities $T_i\le x_iu_i$ and for any form of $h_{xi}(x_i)$. Let us denote by x_i^* the value maximizing $h_{xi}(x_i)$, i.e. $h_{xi}(x_i^*)=1$.

Theorem 13.2. If

$$\alpha\ge\frac{U}{\sum_{i=1}^{k}(x_i^*)^{-1}} \qquad (13.42)$$

then

$$D_u=\{u:(\bigwedge_{i\in\overline{1,k}}0\le u_i\le\frac{\alpha}{x_i^*})\wedge\sum_{i=1}^{k}u_i=U\} \qquad (13.43)$$

is the set of all allocations $u^*=(u_1^*,u_2^*,...,u_k^*)$ such that $v(u^*)=1$.

Proof: From (13.14) it follows that if

$$u_i\le(x_i^*)^{-1} \qquad (13.44)$$

then $x_i^* \in D_{xi}(u_i)$ and consequently $v_i(u_i) = 1$. It is easy to see that under the assumption (13.42) there exists an allocation $u = (u_1, u_2, ..., u_k)$ such that (13.44) is satisfied for each $i \in \overline{1,k}$ and $u_1 + u_2 + ... + u_k = U$. All allocations satisfying these conditions form the set D_u described by (13.43), and if $u \in D_u$ then, according to (13.13), $v(u^*) = 1$. □

13.4 Decomposition and Two-level Control

The determination of the control decision u^* may be difficult for $k > 2$ because of the great computational difficulties. To decrease these difficulties we can apply the decomposition of the complex into two subcomplexes and consequently to obtain a two-level control system (Fig. 13.4). This approach is based on the idea of decomposition and two-level control presented for the deterministic case [13]. At the upper level the value U is divided into U_1 and U_2 assigned to the first and the second subcomplex, respectively, and at the lower level the allocation $u^{(1)}$, $u^{(2)}$ for the subcomplexes is determined. Let us introduce the following notation:

n, m – the number of operations in the first and the second complex, respectively, $n + m = k$, $T^{(1)}$, $T^{(2)}$ – the execution times in the subcomplexes, i.e.

$$T^{(1)} = \max(T_1, T_2, ..., T_n), \quad T^{(2)} = \max(T_{n+1}, T_{n+2}, ..., T_{n+m}),$$

$u^{(1)}$, $u^{(2)}$ – the allocations in the subcomplexes, i.e.

$$u^{(1)} = (u_1, ..., u_n), \quad u^{(2)} = (u_{n+1}, ..., u_{n+m}).$$

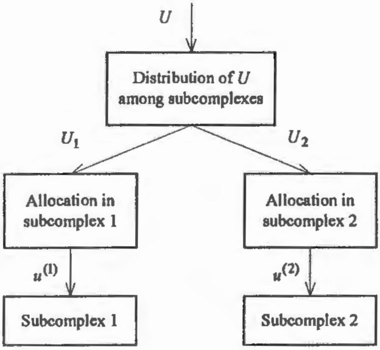

Figure 13.4. Two-level control system

The procedure of the determination of u^* is then the following:

1. To determine the allocation $u^{(1)*}(U_1)$, $u^{(2)*}(U_2)$ and the certainty indexes $v^{(1)*}(U_1)$, $v^{(2)*}(U_2)$ in the same way as u^*, v^* in Sect. 13.2, with U_1 and U_2 in place of U.

2. To determine U_1^*, U_2^* via the maximization of

$$v(T \,\tilde{\le}\, \alpha) = v[(T^{(1)} \,\tilde{\le}\, \alpha) \wedge (T^{(2)} \,\tilde{\le}\, \alpha)] \triangleq v(U_1, U_2).$$

Then

$$(U_1^*, U_2^*) = \arg \max_{U_1, U_2} \min\{v^{(1)*}(U_1), v^{(2)*}(U_2)\}$$

with the constraints: $U_{1,2} \geq 0$, $U_1 + U_2 = U$.

3. To find the values of $u^{(1)*}$, $u^{(2)*}$ and v^* putting U_1^* and U_2^* into the results $u^{(1)*}(U_1)$, $u^{(2)*}(U_2)$ obtained in point 1 and into $v(U_1, U_2)$ in point 2.

It may be shown that the result obtained via the decomposition is the same as the result of the direct approach presented in Sect. 13.2.

Example 13.3.

Let us consider the resource allocation problem the same as in Example 13.1 for $k = 4$ and introduce the decomposition into two subcomplexes with $n = m = 2$. Using the result obtained in Example 13.1 with $U^{(1)}$, $v^{(1)}$ in place of U, v, we have the following result for the first subcomplex:

1. For

$$U \leq \frac{x_1^* - d_1 + x_2^* - d_2}{\alpha}$$

$v^{(1)*}(U_1) = 0$.

2. For

$$\frac{x_1^* - d_1 + x_2^* - d_2}{\alpha} \leq U \leq \frac{x_1^* + x_2^*}{\alpha}$$

we obtain

$$v^{(1)*}(U_1) = A_1 U_1 + B_1$$

where

$$A_1 = \frac{\alpha}{d_1 + d_2}, \quad B_1 = \frac{x_1^* d_2 - x_2^* d_1}{d_1(d_1 + d_2)} - \frac{x_1^*}{d_1} + 1.$$

3. For

$$U \geq \frac{x_1^* + x_2^*}{\alpha}$$

$v^{(1)*}(U_1) = 1$.

The relationship $v^{(2)*}(U_2)$ is the same with x_3, x_4, d_3, d_4, A_2, B_2 in place of x_1, x_2, d_1, d_2, A_1, B_1. The value U_1^* may be determined by solving the equation $v^{(1)*}(U_1) = v^{(2)*}(U - U_1)$ and $U_2^* = U - U_1^*$.

The result is as follows:

1. For

$$\alpha \leq \frac{x_1^* - d_1 + x_2^* - d_2 + x_3^* - d_3 + x_4^* - d_4}{U}$$

$v(U_1, U_2) = 0$.

2. For

$$\frac{x_1^* - d_1 + x_2^* - d_2 + x_3^* - d_3 + x_4^* - d_4}{U} \leq \alpha \leq \frac{x_1^* + x_2^* + x_3^* + x_4^*}{U}$$

we obtain

$$U_1^* = \frac{A_2 U + B_2 - B_1}{A_1 + A_2}, \quad U_2^* = \frac{A_1 U + B_1 - B_2}{A_1 + A_2},$$

$$v(U_1^*, U_2^*) = \frac{A_1 A_2 U + A_1 B_2 + A_2 B_1}{A_1 + A_2}.$$

3. For

$$\alpha \geq \frac{x_1^* + x_2^* + x_3^* + x_4^*}{U}$$

we obtain $v(U_1^*, U_2^*) = 1$ for any U_1 satisfying the condition

$$\frac{x_1^* + x_2^*}{\alpha} \leq U_1 \leq U - \frac{x_3^* + x_4^*}{\alpha}.$$

For the numerical data $U = 20$, $\alpha = 0.5$, $x_1^* = 2$, $x_2^* = 3$, $x_3^* = 3$, $x_4^* = 4$, $d_1 = d_2 = 1$, $d_3 = d_4 = 2$ we obtain: $U_1^* = 8\frac{2}{3}$, $U_2^* = 11\frac{1}{3}$, $u_1^* = 3\frac{1}{3}$, $u_2^* = 5\frac{1}{3}$, $u_3^* = 4\frac{1}{3}$, $u_4^* = 7$ and $v^* = \frac{2}{3}$, which means that the requirement $T \leq \alpha$ will be approximately satisfied with the certainty index $\frac{2}{3}$. □