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Methods review for solving inverse optimization problems

M. Gola

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 8373578

fax: (+48) (22) 8372772

Kierownik Pracowni zgłaszający pracę:
Prof. dr hab. inż. Krzysztof C. Kiwiel

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Methods Review for Solving Inverse Optimization Problems

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Magdalena Gola

System Research Institute, Polish Academy of Science, Poland

Abstract: For a given optimization problem and some fixed feasible solution, the inverse problem is to find new cost function for which fixed solution is optimal. The new cost function should be different as little as possible from the initial cost function.

In this study we present review of methods for solving different inverse combinatorial optimization problems, for example: inverse shortest path problem, inverse problem of minimum spanning tree, inverse problem of minimal cuts etc. and also some general methods for solving inverse problems.

Key words: inverse problems, minimum spanning tree, shortest path tree, assignment problem, maximum-weight circulation problem, shortest path prob-

lem, maximum flow, minimum cut, graph theory, submodular functions on digraphs, polymatroidal flow problem.

1 Introduction

In the past few years we have observed the increase of interest in inverse optimization problem, which was probably first introduced in discrete optimization by Burton and Toint [3]. They discussed inverse shortest paths problem and formulated it as a quadratic programming problem. Ever since number of researchers have studied problems of this category and looked for some general methods for solving them.

It is worth to mention that such problems are strongly motivated by real problems. For example shortest path problem which was studied by Burton and Toint was motivated by seismic tomography. Next two examples show us real life examples of inverse optimization problems.

Model of earthquakes.

This model has network representation where cost arcs represents the transmission time of seismic waves between vertices which describe discrete points of the geologic zone. Earthquakes are observed and on the surface in chosen discrete points seismic perturbation is recorded. The problem is to reconstruct the transmission times between particular points, if we have some information about geological structure of zone and we assume that shortest time waves are known.

Model of transportation network.

Here we also can describe network representation and it is more natural than in previous example. Vertices represent particular junctions and arcs describe cost of routes between them. The question is to recover travel costs as perceived by network users from the knowledge of their actual route choices.

There are others applications of inverse optimization problems which motivate researchers to study this problems. (see e.g. [2])

The paper is organized as follows. In Section 2 we formulate the inverse optimization problem and some special cases of this problem. In Section 3, we consider the inverse shortest path problem. In Section 4 we present methods for solving inverse minimum spanning tree problem. In Section 5 we consider inverse problem of minimum cuts. In Section 6 we show two general methods for solving inverse optimization problems. In Section 7 we consider inverse combinatorial optimization problems. In Section 8 we present general model of some inverse combinatorial problems. In Section 9 we show some possible extensions of inverse optimization problems.

2 Inverse optimization problem

To describe the problem generally, we consider an optimization problem:

$$\min\{c^T x : x \in X\} \tag{1}$$

where $X \subset \mathbf{R}^n$ is a set of feasible solutions and $c \in \mathbf{R}^n$ is a vector of coefficients of the objective function. Let x^* be any solution of problem (1). The inverse problem for x^* is formulated as follows:
we are looking for such vector $\bar{c}$ for which $||c - \bar{c}||$ is minimal and

$$x^* = \arg\min\{\bar{c}^T x : x \in X\}.$$

As norm $||\cdot||$ is usually taken:

- l_1-norm, i.e.

$$\sum_{i=1}^{n} |\bar{c}_i - c_i|, \tag{2}$$

- l_2-norm, i.e.

$$\sqrt{\sum_{i=1}^{n} (\bar{c}_i - c_i)^2} \tag{3}$$

- l_∞-norm, i.e.

$$\max_{i=1,\cdots,n} |\bar{c}_i - c_i|. \tag{4}$$

Now we formulate special cases of the inverse problem.

The inverse shortest path problem

Let $G(V, E, C)$ be a oriented graph, where $V = \{v_1, \ldots, v_n\}$ is a set of vertices, $E = \{e_1, \ldots, e_m\}$ is a set of arcs, and $C = \{c(e_1), \ldots, c(e_m)\}$ is a set of nonnegative costs associated with the arcs.

Let s and t denote two specific nodes in graph G. The shortest $s - t$ path problem is to determinate a directed path (P) from node s to t in graph G, whose cost: $\sum_{e_j \in P} c(e_j)$ is minimum among all $s - t$ paths in G.

The inverse shortest path problem can be formulated as follows. For a given acyclic $s-t$ path (P^*) in G we are looking for such minimal (under fixed norm) perturbation $\bar{C}$ the set of arc costs C which makes path (P^*) the shortest $s-t$ path in graph G. We assume that all members of a new set of arc cost $\bar{C}$ are also nonnegative. In general, there can be not only one given path but a set of paths which should be shortest in graph G.

The inverse minimum spanning tree

Let $G = (V, E, C)$ be a weighted connected graph in which V is the set of vertices, E is a set of edges and C is the weight vector associated with the edges of G. Let $\mathcal{T}$ be the set of all spanning trees in graph G.

The minimum spanning tree problem is to determinate such tree (T^*) which contains all the vertices V and $\sum_{e_i \in T^*} c(e_i)$ is minimal among all spanning tries in $\mathcal{T}$.

The inverse minimum spanning tree can be formulated as follows. For a given spanning tree T^* we are looking for minimal perturbation $\bar{C}$ the set of costs of arcs C which makes fixed tree T^* the minimum spanning tree in graph G. The new arc costs should be nonnegative.

The inverse problem of minimum cuts

Let $N = (V, E, C)$ be a network with vertices set V, arcs set E and capacity vector C associated with the edges, where $s \in V$ is a source and $t \in V$ is a sink. Let $(A, V \setminus A)$ be partition of set V, where $s \in A$ and $t \in V \setminus A$. The $s-t$ cut is a a set $S_A \subset E$ which contains only these edges $e = (v', v'')$ for which $v' \in A$ and $v'' \in V \setminus A$.

The problem of minimum cuts is to find such $s - t$ cut S for which $\sum_{e_i \in S} c(e_i)$ is minimal.

The inverse problem of minimum cuts we can formulate as follows. For a given $s - t$ cut S^* we are looking for minimal perturbation $\bar{C}$ of capacity vector C which makes fixed $s - t$ cut S^* the minimum $s - t$ cut in network N.

In our problem we also assume that there are costs functions w^+, w^- on E for capacity increment and decrement so the inverse problem of minimum cuts is to find an adjusted capacity $\bar{C}$ which makes given cut S^* minimum cut in network and for which

$$\sum_{e_i \in E} [w^+(e_i) \max\{\bar{c}(e_i) - c(e_i), 0\} + w^-(e_i) \max\{c(e_i) - \bar{c}(e_i), 0\}]$$

is minimum.

The inverse combinatorial optimization problem

The generic combinatorial optimization problem we can formulate as follows:

$$\min(\max) f(x) \tag{5}$$

subject to $x \in X$, where feasible region X is combinatorial; e.g. a subset of $\{0,1\}^n$, edge set E of graph $G = (V, E)$ etc.

The inverse combinatorial optimization problem is to find such minimal perturbation coefficients of function f which makes fixed solution x^* optimal solution in set X.

3 Inverse shortest path problem

Burton and Toint [3] consider the inverse shortest path problem and use l_2-norm and obtain quadratic programming problem. They use the modified Goldfarb-Indnani (GI) method [9] for solving obtained quadratic programming problem. The idea of GI method is to solve sequence of quadratic programming problems involving only some constraints which contains the original problem and in that way to obtain a sequence of dual feasible points. At every iteration of the procedure the objective function value monotonically increases to reach the desired optimum.

Zhang, Ma and Yang [19] formulate an inverse shortest path problem as a special linear programming problem and consider it under l_1-norm. They formulate dual problem for this linear problem and use column generation scheme to solve obtained problem. This algorithm obtains optimal solution in finitely many steps.

Zhang and Liu [14] consider inverse linear programming problems under l_1-norm and observe a connection between this problem and an instance of the assignment problem. They obtain a method which reduces inverse shortest path problem (and other LP problem also) to assignment problem which can be solved by strongly polynomial algorithm.

Ahuja and Orlin [2] consider the inverse shortest path problem under l_1-norm and obtain more efficient method compared to method obtained by Zhang and Liu [14]. They show that the unit-weight inverse shortest path problem can be solved by solving a shortest path problem. If we can assume that all arc cost are nonnegative then there exists algorithm which can solve inverse shortest path problem in $O(m + n \log n)$ time. In case of a weighted version of the inverse shortest path problem we can use any efficient minimum cost flow algorithm to solve this problem.

As a generalization of the inverse shortest path problem, Burton and Toint [4] consider an inverse shortest path problem where arc costs are subject to correlation constraints.

This problem is also motivated by traffic modelling and seismic tomography but with additional constraints. In traffic case we consider not only delays on the links but also delays at signalizated junctions; in seismic tomography we consider costs associated with arcs and also the geological densities. This problem is solved by modified algorithm which was used in [3].

4 An Inverse Problem of Minimum Spanning Tree

J. Zhang, S. Xu and Z. Ma [20] discuss the inverse minimum spanning tree problem under l_1-norm.

For graph $G = (V, E, C)$ and for given spanning tree T^* we denote set of edges on T^* by $\{e_1, \dots, e_{n-1}\}$ and the other by $\{f_n, \dots, f_m\}$, where $|V| = n$, and $|E| = m$.

By Ω_T we denote the family of such sets of arc costs C which satisfy condition:

$$c(f_j) \geq \max\{c(e_i) : e_i \in P(f_j)\},\ j = n, \ldots, m,$$

where $P(f_j)$ is a unique cycle in $T^* \cup \{f_j\}$.

Now the inverse problem can be formulated as follows:

We are looking for such costs set $\bar{C}$ which fulfil the following conditions:

- $\bar{C} \in \Omega_T$
- For any edge weight vector $\widehat{C} \in \Omega_T$ satisfying

$$\widehat{c}(e_i) \leq c(e_i),\ \forall e_i \quad (6)$$

$$\widehat{c}(f_j) \geq c(f_j),\ \forall f_j \quad (7)$$

  inequality holds :

$$\|\widehat{C} - C\| \geq \|\bar{C} - C\|. \quad (8)$$

Authors propose algorithm which bases on theorem, that exist optimal solution $\bar{C}$ of the minimum spanning tree problem which satisfying condition:

$$\bar{C} \subseteq C. \quad (9)$$

This algorithm solve the inverse minimum spanning tree problem in a polynomial time.

Sokkalingam, Ahuja and Orlin [11] show that the inverse spanning tree problem, consider under l_1-norm, can be formulated as the dual of an unbalanced assignment problem on bipartite graph (see [11]). The algorithm solves obtained assignment problem, and also unit-weight inverse spanning tree problem in $O(n^3)$ time, where n is a number of vertices in graph.

They also consider weighted inverse spanning tree problem and formulate it as a unbalanced transportation problem, for which there exists an algorithm solving this problem in $O(n^2 m \log(n\bar{c}))$ time, where n is a number of vertices, m is a number of arcs and $\widehat{c}$ is maximum cost among all costs associated with arcs in the graph.

Sokkalingam, Ahuja and Orlin [11] considered the inverse spanning tree under l_∞-norm where the objective is to minimize the maximum arc deviation instead of minimizing the sum of the arc deviations. Let

$$\delta = \max\{c_i - c_j : \text{ for each } c_i \in P(f_j),\ j = k, \ldots, n\}. \tag{10}$$

If $\delta \leq 0$ then C is an optimal cost vector for T^*. If $\delta \geq 0$ then $\delta/2$ is a lower bound on the objective function value of the problem.

Autos bases on theorem that if we assume that $\bar{c}_i - c_i = \alpha_i$, and if $\alpha_i = \min\{0, -\delta/2\}$ for each i, $i = 1, \ldots, k-1$, and $\alpha_j = \max\{0, \delta/2\}$ for each j, $j = k, \ldots, n$, then it is an optimal solution of minimax inverse spanning tree problem.
Authors also propose the algorithm which solves inverse spanning tree problem in $O(n^2)$ time.

J. Zhang, Z. Liu, Z. Ma [16] consider the inverse problem when we have spanning tree T^* with partition constraints of graph G under l_1-norm.

Formally, let $G = (V, E, C)$ be a weighted connected graph and

$\mathbb{B} = \{B_1, B_2, \ldots, B_m\}$ be a partition of E, i.e.

$$B_i \cap B_j = \emptyset, \qquad i \neq j$$
$$\bigcup_{j=1}^{m} B_j = E,$$

Let a_i, d_i be nonnegative integers and

$$\mathcal{K} = \{T : \; T \in \mathcal{T} \wedge \; a_i \leq |T \cap B_i| \leq d_i, \; 1 \leq i \leq m\}$$

set of all partition constrained spanning trees of G. For given T^*, we define:

$$S(T^*) = \{\bar{C} \in R^n \; : \bar{C}(T^*) = \min_{T \in \mathcal{K}} \bar{C}(T)\}$$

where $\bar{C}(T) = \sum_{e_j \in T} \bar{c}(e_j)$. The inverse problem can be formulated as follows:

$$\min\{\|\bar{C} - C\| \; : \bar{C} \in S(T^*)\}$$

This problem is converted to linear programming problem for which dual problem can be formulated as a maximum cost flow problem (MCFP). There is a strongly polynomial algorithm for solving MCFP, so our inverse problem of minimum spanning tree with partition constraints can be solve by strongly polynomial algorithm.

5 Inverse Problem of Minimum Cuts

In [7] the authors consider methods for solving inverse problem of minimum cut. An important property of minimum cuts in their studies is theorem that

$s-t$ cut S is minimum with respect to the capacity vector C if and only if there exists a feasible flow f such that:

$$f(a)=\begin{cases} C(a) & \text{for all } a\in S,\\ 0 & \text{for all } a\in \bar{S}.\end{cases}$$

Obtained method transforms the inverse problem of minimum cut into a minimum cost circulation problem.Presented method allows to solve this problem by strongly polynomial algorithms. They formulate also necessary and sufficient condition for the feasibility of the inverse problem.

Ahuja and Orlin [2] give simpler proof of theorem obtained by Zhang and Cai [7].

6 Two General Methods

Yang and Zhang [13] formulate a group of inverse optimization problems as a linear programming model and present two general methods for solving this problem.

First method is the column generation method which consists in generating columns for simplex method by solving original optimization problem. The convergence of this method depends only on the convergence of algorithm which is used to solve the original problem.

Second method is the ellipsoid method. This method does not require listing all constraints. It is big advantage in case when we have a huge problem to solve. It also provides a polynomial order algorithm for many inverse problems.

Ahuja and Orlin [2] make an use of the ellipsoid algorithm and obtain a theorem, that if an inverse optimization problem is solvable for each linear cost function, then inverse versions of this problem under l_1-norm and l_∞-norm are polynomially solvable.

This theorem establishes the polynomial solvability of many inverse optimization problems, but the ellipsoid algorithm is not practical for large problems.

Ahuja and Orlin [2] also show that inverse linear programming problems under l_1-norm and l_∞- norm are also linear programming problems.

7 Inverse Combinatorial Optimization Problems

Some authors not only consider special cases of the inverse problem but also their generalization i.e. inverse combinatorial problem. Methods obtained by them are more universal and can be used to solve many problems such as: minimum spanning tree problem, assignment problem and shortest path tree problem.

7.1 Inverse problem as a minimum-weight circulation problem

Zhang and Ma in [20] propose an universal method to solve combinatorial optimization problems as a minimum - weight circulation problems. They convert considered inverse problems to linear programming problems (LPP) and then construct network models which correspond to dual problems of obtained LPP.

This method is general and has better time complexity than column generation method.

7.2 Solution sets of inverse combinatorial problems

In [18] Zhang and Ma consider some inverse combinatorial problems and forms of their solution sets. They use Fulkerson's theory of blocking and anti blocking polyhedra [8] and found that the solution set of each considered inverse combinatorial problems is characterized by solving another combinatorial optimization problem.

For example, after applying block theory to inverse directed spanning tree problem, they obtain minimal r-cuts problem of the same graph.

Theory of blocking and anti blocking polyhedra [8] is in a such case very useful and lets to derive results more easily than it was before, see Xu and Zhang [12].

7.3 Inverse problem as a case of submodular function

The combinatorial optimization problems such a shortest path problem, minimum spanning tree, weighted bipartite matching and weighted matroid intersection are special cases of submodular function on digraphs. Thus it is important to study inverse problem of submodular function on digraphs(IPSFD).

Authors in [5] present method to formulate IPSFD as a combinatorial linear program and also as a minimum cost circulation problem, which can be solved by a strongly polynomial algorithm. They also give a necessary and sufficient condition for the feasibility of IPSFD.

8 General Model

Zhang and Liu [15] present a general model of some inverse combinatorial problems and consider this problem under l_∞-norm.

For a given graph $G = (V, E, C)$, let $\mathfrak{F}$ be a family of subsets of E, $(P, \bar{P})$ be a given partition of E, and $q \in \mathbb{R}$.

For each $F \in \mathfrak{F}$ we define:

$$d_C(F) = \sum_{e_j \in F \cap \bar{P}} c(e_j) - \sum_{e_i \in F \cap P} c(e_i). \tag{11}$$

If $d_C(F) \leq q$ for all $F \in \mathfrak{F}$, then P is called a dominant set under C.

If the given set P is not a dominant then our problem can be stated as follows:

$$\min \|\bar{C} - C\| \tag{12}$$

$$\text{s.t.} \quad d_{\bar{C}}(F) \leq q, \ \forall F \in \mathfrak{F}.$$

If as $\|.\|$ we take l_∞-norm then our problem is equivalent to the following problem:

$$\min \lambda \tag{13}$$

$$\text{s.t.} \quad d_C(F) - \lambda|F| \le q, \ \forall f \in \mathcal{F},$$

$$\lambda \ge 0.$$

When we consider weighted case, the problem has form:

$$\min \lambda \tag{14}$$

$$\text{s.t.} \quad d_C(F) - \lambda \sum_{e \in F} \frac{1}{u(e)} \le q, \ \forall F \in \mathcal{F}$$

where $u(e)$ denote the cost of increasing (decreasing) one unit $c(e_i)$.

If P is not a dominant set under C, then λ^* is an optimal solution of our problem if and only if the optimal value of the following problem:

$$\max_{F \in \mathcal{F}} \{d_C(F) - \lambda^* \frac{1}{u(e)}\} \tag{15}$$

is q.

Authors prove that if there is an algorithm $\mathcal{A}$ which solves problem (15) with any fixed λ, then the Newton's method which they propose can find solution λ^* of

$$\max_{F \in \mathcal{F}} \{d_C(F) - \lambda \frac{1}{u(e)}\} = q \tag{16}$$

by using the algorithm $\mathcal{A}$ at most $O(n^2 \log^2 n)$ times.

Used Newton's method was previously obtained by Radzik [10] from a study of fractional combinatorial optimization problems.

9 Summation

Presenting methods do not explore all subject of inverse optimization problem. We can easily find possible extensions, for example consideration of other norms.

The scope of inverse optimization has been also expanded. New aspect of this issue is so called reverse optimization problem. When we have optimization problem and a target value which can not be reached by this problem, we looking for most economic adjustment of problem parameters for which optimal value of the problem is not worse than the given target value.

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