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Global existence to a three-dimensional nonlinear thermoelasticity system arising in shape memory materials

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Abstract. The paper is concerned with the unique global solvability of a three-dimensional $(3-D)$ nonlinear thermoelasticity system arising from the study of shape memory materials. The system consists of the coupled evolutionary problems of viscoelasticity with nonconvex elastic energy and nonlinear heat conduction with mechanical dissipation.
The present paper extends the previous $2-D$ existence result of the authors [7] to $3-D$ case. This goal is achieved by means of the Leray-Schauder fixed point theorem using technique based on energy arguments and DeGiorgi method.
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1. Introduction

In the present paper we study the issue of existence of solutions to a three-dimensional $(3-D)$ nonlinear thermoelasticity system arising as a model of structural phase transitions in shape memory materials. The system consists of the coupled evolutionary problems of viscoelasticity with nonconvex elastic energy and nonlinear heat conduction with mechanical dissipation.

In the previous paper [7] we have investigated such system in $2-D$ case applying the Leray-Schauder fixed point theorem and the technique based on energy methods. The main difficulty we have been faced with comes from the nonlinear coupling of mechanical and thermal effects. The key issue to solve the problem has been concerned with deriving L_∞-norm and then Hölder-norm estimates for temperature.
In $2-D$ case this has been accomplished with the help of technical energy estimates and theorems of Sobolev's imbeddings. In $3-D$ case such procedure is not sufficient and additional methods are required.

The goal of the present paper is to extend the existence result of [7] to $3-D$ case. This is achieved by combining the procedure of recursive improvement of energy estimates with DeGiorgi method employed to derive Hölder-norm estimate on temperature. The existence result applies to $2-D$ and $3-D$ problems.
The place of our study in the present theory of thermoviscoelasticity systems with nonconvex energy is discussed in the previous paper [7].

The problem under consideration has the following form:

$$(1.1)\qquad \begin{aligned} &\boldsymbol{u}_{tt} - \nu Q\boldsymbol{u}_t + \frac{\varkappa_0}{4}Q^2\boldsymbol{u} = \nabla\cdot F_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\theta) + \boldsymbol{b} && \text{in } \Omega^T = \Omega\times(0,T),\\ &\boldsymbol{u}\big|_{t=0} = \boldsymbol{u}_0,\quad \boldsymbol{u}_t\big|_{t=0} = \boldsymbol{u}_1 && \text{in } \Omega,\\ &\boldsymbol{u} = 0,\quad Q\boldsymbol{u} = 0 && \text{on } S^T = S\times(0,T), \end{aligned}$$

$$
(1.2) \qquad \begin{aligned} & c_0(\varepsilon,\theta)\theta_t - k_0\Delta\theta = \theta F_{,\theta\varepsilon}(\varepsilon,\theta) : \varepsilon_t + \nu(A\varepsilon_t) : \varepsilon_t + g && \text{in } \Omega^T, \\ & \theta\big|_{t=0} = \theta_0 && \text{in } \Omega, \\ & n \cdot \nabla\theta = 0 && \text{on } S^T, \end{aligned}
$$

where

$$
(1.3) \qquad c_0(\varepsilon,\theta) = c_v - \theta F_{,\theta\theta}(\varepsilon,\theta).
$$

Here $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, is a bounded domain with a smooth boundary S, representing the material points of a solid body with constant mass density $\rho = 1$; $T > 0$ is an arbitrary fixed time.
The vector field $u : \Omega^T \to \mathbb{R}^n$ is the displacement and $\theta : \Omega^T \to \mathbb{R}_+$ is the absolute temperature. The second order tensors

$$
\varepsilon = \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \quad \text{and} \quad \varepsilon_t = \varepsilon(u_t) = \frac{1}{2}(\nabla u_t + (\nabla u_t)^T)
$$

denote respectively the linearized strain and the strain rate.
The elastic energy $F(\varepsilon,\theta)$ is a multiwell function of $\varepsilon = (\varepsilon_{ij})$ with the shape strongly depending on θ; a physical example is given below.
The fourth order tensor $A = (A_{ijkl})$ represents the elasticity tensor given by

$$
\varepsilon(u) \mapsto A\varepsilon(u) = \lambda tr\varepsilon(u)I + 2\mu\varepsilon(u),
$$

where $I = (\delta_{ij})$ and λ, μ are Lamé constants with values within elasticity range (see assumption (A2)). Furhermore, Q stands for the second order linear elasticity operator defined by

$$
u \mapsto Qu = \nabla \cdot (A\varepsilon(u)) = \mu\Delta u + (\lambda + \mu)\nabla(\nabla \cdot u).
$$

Correspondingly, the operator $Q^2 = QQ$ is given by

$$
u \mapsto Q^2u = \nabla \cdot (A\varepsilon(Qu)) = \mu\Delta(Qu) + (\lambda + \mu)\nabla(\nabla \cdot (Qu)).
$$

The remaining quantities in (1.1), (1.2) have the following meaning: $c_0(\varepsilon,\theta)$ is the specific heat coefficient, c_v, k_0, ν and $\varkappa_0$ are positive numbers representing respectively thermal specific heat, heat conductivity, viscosity and strain-gradient energy coefficient. The fields $b : \Omega^T \to \mathbb{R}^n$ and $g : \Omega^T \to \mathbb{R}$ are external body forces and heat sources.
System (1.1), (1.2) represents balance laws of linear momentum and energy. The associated free energy density has the Landau-Ginzburg form

$$
(1.4) \qquad f(\varepsilon(u), \nabla\varepsilon(u), \theta) = -c_v\theta\log\theta + F(\varepsilon(u),\theta) + \frac{\varkappa_0}{8}|Qu|^2,
$$

where the three terms represent respectively thermal, elastic and strain-gradient energy.
For physical background of the model we refer to [7], [8].

To motivate the structural assumptions of our existence result we recall here the model of the elastic energy for $3-D$ CuAlNi shape memory alloy, proposed by Falk-Konopka [4]:

$$F(\varepsilon,\theta)=\sum_{i=1}^{3}a_i^2(\theta)J_i^2(\varepsilon)+\sum_{i=1}^{5}a_i^4(\theta)J_i^4(\varepsilon)+\sum_{i=1}^{2}a_i^6(\theta)J_i^6(\varepsilon), \tag{1.5}$$

where $J_i^k(\varepsilon)$ are crystal invariants given by k-th order polynomials in ε_{ij}, and $a_i^k(\theta)$ are experimentally determined coefficients which in the vicinity of phase transition temperature $\theta_c>0$ have the form

$$a_i^k(\theta)=\alpha_i^k+\tilde{\alpha}_i^k(\theta-\theta_c),\quad k=2,4\quad a_i^6(\theta)=\alpha_i^6, \tag{1.6}$$

where $\alpha_i^k,\tilde{\alpha}_i^k$ are constants, with $\alpha_i^6>0$.
We note that (1.5) generalizes the well-known Falk model for $1-D$ shape memory alloy

$$F(\varepsilon,\theta)=\alpha_1(\theta-\theta_c)\varepsilon^2-\alpha_2\varepsilon^4+\alpha_3\varepsilon^6,$$

where $\alpha_i>0,\ \theta_c>0$ are constant parameters.

Similarly as in $2-D$ case studied in [7] our main structure hypotheses concern the function $F(\varepsilon,\theta)$. We require the non-negativity of $F(\varepsilon,\theta)$ and its concavity with respect to θ. Such requirements are obviously satisfied by the model (1.5), (1.6).
We point on two important from mathematical point of view consequences of the concavity assumption. Firstly, in view of definition (1.3), it implies that the specific heat coefficient remains bounded from below by positive constant c_v (see (2.4)).
Secondly, it implies that the elastic part of the internal energy is nonnegative (see (2.7)) what is of importance in derivation of the first energy estimate.
We do not require any convexity assumptions on F with respect to ε, only growth conditions which are imposed by Sobolev's imbeddings.
The most restrictive is the condition on the growth of F with respect to θ. From the technique used to get temperature estimates (see Lemma 4.3) it follows that θ-growth exponent s has to satisfy condition $s<3/4$ if $n=2$ and $s<2/3$ if $n=3$.
At this point we stress that in $2-D$ case a technically different method applied in [7] has admitted less restrictive condition $s<7/8$.
Clearly, in view of growth conditions our existence result has physical relevance only in a finite range of strains and temperatures.
We add also that to remove the growth condition $s<1$ and to admit the linear growth of F in θ seems to be a serious mathematical obstacle.
The content of the paper is as follows.

In Section 2 we present the assumptions and state the main results for problem (1.1), (1.2) on global in time existence and uniqueness of solutions in $2-D$ and $3-D$ cases.
The proof of the existence theorem is presented in Sections $3\div 6$.
In Section 3 we prepare the setting for the application of the Leray-Schauder fixed point theorem. In particular, following [8] and [7], we introduce the parabolic decomposition of elasticity system (1.1) and define the solution map. Furthermore, following closely the

arguments of [7], we check the complete continuity and the uniform equicontinuity of the solution map.
A central new part of the proof constitute a priori bounds for a fixed point. Their derivation is for clarity partitioned into the distinct pieces which appear in Sections 4 ÷ 6. Section 4 contains lemmas on positivity of temperature, energy estimates and the procedure of recursive improvement of energy estimates. In Section 5 we prove the crucial $L_\infty(\Omega^T)$ - norm estimate on θ. The idea of the proof consists in deriving a bound in $L_r(\Omega^T)$ - norm and passing to the limit with $r \to \infty$. In Section 6, applying DeGiorgi method in a way presented in [5], we prove that $\theta \in \mathcal{B}_2(\Omega^T, M, \gamma, r, \delta, \varkappa)$ and consequently that it is Hölder continuous. Furthermore, in Section 6 we establish the final a priori estimates which complete the existence proof.

We use the following notations:

$$f_{,i} = \frac{\partial f}{\partial x_i}, \quad i = 1, \dots, n, \quad f_t = \frac{df}{dt}, \quad \boldsymbol{\varepsilon} = (\varepsilon_{ij})_{i,j=1,\dots,n},$$

$$F_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \theta) = \left(\frac{\partial F(\boldsymbol{\varepsilon}, \theta)}{\partial \varepsilon_{ij}}\right)_{i,j=1,\dots,n}, \quad F_{,\theta}(\boldsymbol{\varepsilon}, \theta) = \frac{\partial F(\boldsymbol{\varepsilon}, \theta)}{\partial \theta},$$

where space and time derivatives are material. For simplicity, whenever there is no danger of confusion, we omit the arguments $(\boldsymbol{\varepsilon}, \theta)$. The specification of tensor indices is omitted as well. Vector - and tensor-valued mappings are denoted by bold letters.
The summation convention over repeated indices is used. Moreover, for vectors $\boldsymbol{a} = (a_i)$, $\tilde{\boldsymbol{a}} = (\tilde{a}_i)$ and tensors $\boldsymbol{B} = (B_{ij})$, $\tilde{\boldsymbol{B}} = (\tilde{B}_{ij})$, $\boldsymbol{A} = (A_{ijkl})$, we write

$$\begin{aligned} &\boldsymbol{a} \cdot \tilde{\boldsymbol{a}} = a_i \tilde{a}_i, \qquad \boldsymbol{B} : \tilde{\boldsymbol{B}} = B_{ij} \tilde{B}_{ij}, \\ &\boldsymbol{AB} = (A_{ijkl} B_{kl}), \ \ \boldsymbol{BA} = (B_{ij} A_{ijkl}), \ \ |\boldsymbol{a}| = (a_i a_i)^{1/2}, \ \ |\boldsymbol{B}| = (B_{ij} B_{ij})^{1/2}, \end{aligned}$$

and denote by

$$\|\boldsymbol{a}\|_{\boldsymbol{L}_p(\Omega)} = \left(\int_\Omega |\boldsymbol{a}|^p dx\right)^{1/p}, \quad etc.,$$

the corresponding $\boldsymbol{L}_p(\Omega)$ - norms of tensor-valued functions.
The symbols ∇ and $\nabla\cdot$ denote the gradient and the divergence operators. For the divergence we use the convention of the contraction over the last index, that is,

$$\nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{x}) = (\varepsilon_{ij,j}(\boldsymbol{x}))_{i=1,\dots,n}.$$

We use the Sobolev spaces notation of [5].
Throughout the paper c and $c(T)$ denote generic constants, different in various instances, depending on the data of the problem and domain Ω. The argument T indicates time horizon dependence which is always of the form $T^a, a \in \mathbb{R}_+$.

2. Assumptions and main results

Throughout the paper we impose the following assumptions.
(A1) Domain $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, with the boundary of class C^4. The C^4 - regularity is needed to apply the classical regularity result for parabolic systems (see estimate (4.38)).
(A2) The coefficients of the operator $\boldsymbol{Q}$ satisfy

$$\mu > 0, \qquad n\lambda + 2\mu > 0.$$

These conditions assure the following properties:
(i) Coercivity and boundedness of the operator $\boldsymbol{A}$

$$\underline{c}|\varepsilon|^2 \leq (\boldsymbol{A}\varepsilon) : \varepsilon \leq \bar{c}|\varepsilon|^2, \tag{2.1}$$

where $\underline{c} = \min\{n\lambda + 2\mu, 2\mu\}, \bar{c} = \max\{n\lambda + 2\mu, 2\mu\}$;
(ii) Strong ellipticity of the operator $\boldsymbol{Q}$ (see [8], Sec.7). Thanks to this property the following estimate due to Nečas [6] holds true

$$c\|\boldsymbol{u}\|_{W_2^2(\Omega)} \leq \|\boldsymbol{Q}\boldsymbol{u}\|_{L_2(\Omega)} \quad \text{for } \left\{\boldsymbol{u} \in \boldsymbol{W}_2^2(\Omega) | \ \boldsymbol{u}|_s = 0\right\}; \tag{2.2}$$

(iii) Parabolicity in general (Solonnikov) sense of systems in the form (3.1), (3.2) (see [8], Sec. 7).
The next assumption concerns the structure of elastic energy.
(A3) Function $F(\varepsilon, \theta) : \mathcal{S}^2 \times [0, \infty) \to \mathbb{R}$ is of class C^3, where $\mathcal{S}^2$ denotes the set of symmetric second order tensors in $\mathbb{R}^n$. We assume the splitting

$$F(\varepsilon, \theta) = F_1(\varepsilon, \theta) + F_2(\varepsilon),$$

where F_1 and F_2 are subject to the following conditions:
(A3-1) Conditions on $F_1(\varepsilon, \theta)$
(i) concavity with respect to θ

$$-F_{1,\theta\theta}(\varepsilon, \theta) \geq 0 \quad \text{for} \ \ (\varepsilon, \theta) \in S^2 \times [0, \infty). \tag{2.3}$$

(ii) Nonnegativity

$$F_1(\varepsilon, \theta) \geq 0 \quad \text{for} \ \ (\varepsilon, \theta) \in S^2 \times [0, \infty).$$

(iii) Boundedness of the norm

$$\|F_1\|_{C^3(S^2 \times [0,\infty))} < \infty.$$

(iv) Growth conditions. There exist a positive constant c and numbers $s, K_1 \in (0, \infty)$ such that

$$\begin{aligned}
&|F_1(\varepsilon, \theta)| \leq c(1 + \theta^s |\varepsilon|^{K_1}), \\
&|F_{1,\varepsilon}(\varepsilon, \theta)| \leq c(1 + \theta^s |\varepsilon|^{K_1 - 1}), \\
&|F_{1,\varepsilon\varepsilon}(\varepsilon, \theta)| \leq c(1 + \theta^s |\varepsilon|^{K_1 - 2}), \\
&|F_{1,\theta\varepsilon}(\varepsilon, \theta)| \leq c(1 + \theta^{s-1} |\varepsilon|^{K_1 - 1}), \\
&|F_{1,\theta\theta}(\varepsilon, \theta)| \leq c(1 + \theta^{s-2} |\varepsilon|^{K_1}), \\
&|F_{1,\theta\theta\varepsilon}(\varepsilon, \theta)| \leq c(1 + \theta^{s-2} |\varepsilon|^{K_1 - 1})
\end{aligned}$$

for large values of θ and ε_{ij}, where admissible ranges of s and K_1 are given by

$$0 < s < \frac{n+1}{2n} = \begin{cases} 3/4 & \text{if } n = 2 \\ 2/3 & \text{if } n = 3, \end{cases}$$

$$0 < K_1 < 1 + \frac{q_n}{2}\left[\frac{n+2}{2n} + \frac{1}{n(n+1)}\right] = \begin{cases} \text{any finite number if} & n = 2 \\ 15/4 \qquad\qquad\qquad \text{if} & n = 3. \end{cases}$$

Moreover, in case $K_1 > 1$ the numbers s and K_1 are linked by the equality

$$\frac{2sn}{n+1} + \frac{4n(K_1-1)}{q_n(n+2)} = 1 + \frac{2}{(n+1)(n+2)}.$$

Here q_n is the Sobolev exponent for which the imbedding of $W_2^1(\Omega)$ into $L_{q_n}(\Omega)$ is continuous, i.e., $q_n = 2n/(n-2)$ for $n \geq 3$ and q_n is any finite number for $n = 2$.

Concernig the part $F_2(\varepsilon)$ we impose:

(A3-2) Conditions on $F_2(\varepsilon)$

(i) Nonnegativity

$$F_2(\varepsilon) \geq 0 \quad \text{for} \quad \varepsilon \in S^2.$$

(ii) Boundedness of the norm

$$\|F_2\|_{C^2(S^2)} < \infty.$$

(iii) Growth conditions

$$|F_2(\varepsilon)| \leq c(1 + |\varepsilon|^{K_2}),$$
$$|F_{2,\varepsilon}(\varepsilon)| \leq c(1 + |\varepsilon|^{K_2-1}),$$
$$|F_{2,\varepsilon\varepsilon}(\varepsilon)| \leq c(1 + |\varepsilon|^{K_2-2})$$

for large values of ε_{ij}, where

$$0 < K_2 \leq 1 + \frac{q_n(n+4)}{4n} = \begin{cases} \text{any finite number if} & n = 2 \\ 9/2 \qquad\qquad\qquad \text{if} & n = 3. \end{cases}$$

Before formulating regularity requirements on the data we note some consequences of assumption (A3-1) which are of importance in further considerations.

In view of (A3-1) (i), by definition of $c_0(\varepsilon, \theta)$,

$$0 < c_v \leq c_0(\varepsilon, \theta) \quad \text{for} \quad (\varepsilon, \theta) \in S^2 \times [0, \infty). \tag{2.4}$$

Moreover, (A3-1) (iii) and (iv) imply the bounds

$$\begin{aligned} &|c_0(\varepsilon, \theta)|, \quad |c_{0,\theta}(\varepsilon, \theta)| \leq c(1 + |\varepsilon|^{K_1}), \\ &|c_{0,\varepsilon}(\varepsilon, \theta)| \leq c(1 + |\varepsilon|^{\max\{0, K_1 - 1\}}) \quad \text{for} \quad (\varepsilon, \theta) \in S^2 \times [0, \infty). \end{aligned} \tag{2.5}$$

From (A3-1)(i) and (ii) it follows that

$$F_1(\varepsilon, \theta) - \theta F_{1,\theta}(\varepsilon, \theta) \geq 0 \quad \text{for} \quad (\varepsilon, \theta) \in S^2 \times [0, \infty), \tag{2.6}$$

and owing to (A3-2) (i),

(2.7) $$(F_1(\varepsilon,\theta) - \theta F_{1,\theta}(\varepsilon,\theta)) + F_2(\varepsilon) \geq 0 \quad \text{for} \quad (\varepsilon,\theta) \in S^2 \times [0,\infty),$$

what means that the elastic part of the internal energy is nonnegative.
The later bound is of importance in derivation of energy estimate (see Lemma 4.2).

Concerning the data we assume:

(A4) Source terms satisfy

$$b \in L_p(\Omega^T), \quad n+2 < p < \infty,$$
$$g \in L_q(\Omega^T), \quad n+2 < q < \infty, \quad \text{and} \quad g \geq 0 \ \text{ a.e. in } \Omega^T.$$

Initial data satisfy

$$u_0 \in W_p^{4-2/p}(\Omega) \quad u_1 \in W_p^{2-2/p}(\Omega), \quad n+2 < p < \infty,$$
$$\theta_0 \in W_q^{2-2/q}(\Omega), \quad n+2 < q < \infty, \quad \text{and} \quad \theta_* = \min_{\Omega} \theta_0 > 0.$$

Moreover, the initial data are supposed to satisfy compatibility conditions for the classical solvability of parabolic problems.
For further use we note that, by imbeddings,

$$\theta_0 \in C^{1,\alpha_0}(\Omega) \quad 0 < \alpha_0 < 1 - \frac{n+2}{q},$$
$$\varepsilon_0 \in C^{2,\alpha_0'}(\Omega) \quad 0 < \alpha_0' < 1 - \frac{n+2}{p}.$$

We shall give now an example of the function $F_1(\varepsilon,\theta)$ which satisfies the structure assumptions (A3-1) (i)–(iv). This example is motivated by the Falk-Konopka energy model (1.5), (1.6).

Example. Let

$$F_1(\varepsilon,\theta) = \sum_{i=1}^{N} \tilde{F}_{1i}(\theta)\tilde{F}_{2i}(\varepsilon),$$

with functions $\tilde{F}_{1i} \in C^3([0,\infty))$ given by

$$\tilde{F}_{1i}(\theta) = \begin{cases} \theta & \text{for } 0 \leq \theta \leq \theta_1 \\ \varphi_i(\theta) & \text{for } \theta_1 < \theta < \theta_2 \\ \theta^{s_i} & \text{for } \theta_2 \leq \theta < \infty. \end{cases}$$

Here $N \in \mathcal{N}$, $0 < s_i < s < 1$, θ_1, θ_2 are numbers satisfying $0 < \theta_1 < \theta_2$, $s_i\theta_2^{s_i-1} < 1$, and functions φ_i are nondecreasing, concave such that $\tilde{F}_{1i} \in C^3([0,\infty))$. Moreover, functions $\tilde{F}_{2i} \in C^3(S^2)$ are supposed to satisfy

$$\tilde{F}_{2i}(\varepsilon) \geq 0,$$
$$|\tilde{F}_{2i}(\varepsilon)| \leq c(1+|\varepsilon|^{K_1}),$$
$$|\tilde{F}_{2i,\varepsilon}(\varepsilon)| \leq c(1+|\varepsilon|^{\max\{0,K_1-1\}}),$$
$$|\tilde{F}_{2i,\varepsilon\varepsilon}(\varepsilon)| \leq c(1+|\varepsilon|^{\max\{0,K_1-2\}})$$

for all $\varepsilon \in S^2$, where numbers s and K_1 are subject to conditions specified in (A3-1) (iv).

For the presentation convenience the above example is used in the proof of pointwise estimates on temperature (see Lemmas 5.1, 5.2).

The main result of the paper is the following existence theorem.

Theorem 2.1. *Let assumptions (A1)-(A4) be satisfied and the coefficients $\varkappa_0, \nu$ fulfil the condition*

$$0 < \sqrt{\varkappa_0} \le \nu. \tag{2.8}$$

Then for any $T > 0$ there exists a solution $(\boldsymbol{u}, \theta)$ to problem (1.1), (1.2) in the space

$$V(p,q) \equiv \{(\boldsymbol{u},\theta)|\ \boldsymbol{u} \in W_p^{4,2}(\Omega^T),\ \theta \in W_q^{2,1}(\Omega^T),\ n+2 < p \le q < \infty\} \tag{2.9}$$

such that

$$\|\boldsymbol{u}\|_{W_p^{4,2}(\Omega^T)} \le c(T), \quad \|\theta\|_{W_q^{2,1}(\Omega^T)} \le c(T), \tag{2.10}$$

with a positive constant $c(T)$ depending on the data of the problem and T^a, $a \in \mathbb{R}_+$. Moreover, there exists a positive finite number ω, satisfying

$$[g + \nu(A\boldsymbol{\varepsilon}_t) : \boldsymbol{\varepsilon}_t]\exp(\omega t) + [\omega c_0(\boldsymbol{\varepsilon},\theta) + F_{,\theta\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\theta) : \boldsymbol{\varepsilon}_t]\theta_* \ge 0 \quad \text{in} \quad \Omega^T,$$

such that

$$\theta \ge \theta_* \exp(-\omega t) \quad \text{in} \quad \Omega^T. \tag{2.11}$$

The proof of this theorem is presented in Sections $3 \div 6$.
We add here some comments concerning the solution space.
The condition $n+2 < p, q$ is implied by the required regularity of solutions. More precisely, from Sobolev's imbeddings it follows that solutions in the space $V(p,q)$ enjoy the following properties:

$$\boldsymbol{u}, \boldsymbol{u}_t, \boldsymbol{\varepsilon}, \nabla\boldsymbol{\varepsilon}, \nabla^2\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_t, \theta, \nabla\theta \tag{2.12}$$

are Hölder continuous in Ω^T and satisfy the corresponding a priori bounds with constant $c(T)$. The continuity of $\boldsymbol{\varepsilon}, \theta$ and $\boldsymbol{\varepsilon}_t$ is used in the proof of Lemma 4.1.
The relation $p \le q$ is needed in the proof of the complete continuity of the solution map (see Section 3).

For completeness we recall also the uniqueness result which follows by repeating the arguments of the $2-D$ proof presented in [7]. In this proof the continuity of $\boldsymbol{\varepsilon}_t$ and $\nabla\theta$ is required.

Theorem 2.2. *Let the assumptions of Theorem 2.1 be satisfied and in addition suppose that*

$$F(\boldsymbol{\varepsilon},\theta) : S^2 \times [0,\infty) \text{ is of class } C^4, \quad \text{and} \quad g \in L_\infty(\Omega^T). \tag{A5}$$

Then the solution $(\boldsymbol{u},\theta) \in V(p,q)$ to problem (1.1), (1.2) is unique.

3. The proof of Theorem 2.1. Application of the Leray-Schauder fixed point theorem

First we recall (see [8]) that system $(1.1)_1$ admits the following decomposition into two parabolic systems, for vector field $\boldsymbol{w}$:

$$
\begin{aligned}
&\boldsymbol{w}_t - \beta Q\boldsymbol{w} = \nabla \cdot F_{,\varepsilon}(\varepsilon, \theta) + \boldsymbol{b} && \text{in } \Omega^T,\\
&\boldsymbol{w}\big|_{t=0} = \boldsymbol{w}_0 \equiv \boldsymbol{u}_1 - \alpha Q\boldsymbol{u}_0 && \text{in } \Omega,\\
&\boldsymbol{w} = 0 && \text{on } S^T,
\end{aligned} \tag{3.1}
$$

and vector field $\boldsymbol{u}$:

$$
\begin{aligned}
&\boldsymbol{u}_t - \alpha Q\boldsymbol{u} = \boldsymbol{w} && \text{in } \Omega^T,\\
&\boldsymbol{u}\big|_{t=0} = \boldsymbol{u}_0 && \text{in } \Omega,\\
&\boldsymbol{u} = 0 && \text{on } S^T,
\end{aligned} \tag{3.2}
$$

where α, β are numbers satisfying

$$
\alpha + \beta = \nu, \qquad \alpha\beta = \frac{\varkappa_0}{4}. \tag{3.3}
$$

We assume that the viscosity and capillarity coefficients, ν and $\varkappa_0$, satisfy relation (2.8). Under such condition $\alpha, \beta, \in \mathbb{R}_+$. Systems (3.1) and (3.2) are coupled with problem (1.2) for θ.

We apply the following formulation of the Leray-Schauder fixed point theorem (see [3]):

Theorem 3.1. *Let $\mathcal{B}$ be a Banach space. Assume that $T : [0,1] \times \mathcal{B} \to \mathcal{B}$ is a map with the following properties:*

(i) For any fixed $\tau \in [0,1]$ the map $T(\tau, \cdot) : \mathcal{B} \to \mathcal{B}$ is completely continuous.

(ii) For every bounded subset $\mathcal{C}$ of $\mathcal{B}$, the family of maps $T(\cdot, \chi) : [0,1] \to \mathcal{B}, \chi \in \mathcal{C}$, is uniformly equicontinuous.

(iii) There is a bounded subset $\mathcal{C}$ of $\mathcal{B}$ such that any fixed point in $\mathcal{B}$ of $T(\tau, \cdot), 0 \le \tau \le 1$, is contained in $\mathcal{C}$.

(iv) $T(0, \cdot)$ has precisely one fixed point in $\mathcal{B}$.
Then $T(1, \cdot)$ has at least one fixed point in $\mathcal{B}$.

In order to define the corresponding solution map we extend the definition of $F_1(\varepsilon, \theta)$ to all $\theta \in \mathbb{R}$ in such a way that it is of class C^3, and that

$$
F_{1,\theta\theta}(\varepsilon, \theta) \ge 0 \quad \text{for all} \quad (\varepsilon, \theta) \in \mathcal{S}^2 \times (-\infty, 0).
$$

With such extension the lower bound (2.4) on $c_0(\varepsilon, \theta)$ remains valid for all $(\varepsilon, \theta) \in \mathcal{S}^2 \times \mathbb{R}$. The solution space is $\boldsymbol{V}(p,q)$ defined by (2.9). The solution map

$$
T(\tau, \cdot) : (\bar{\boldsymbol{u}}, \bar{\theta}) \in \boldsymbol{V}(p,q) \to (\boldsymbol{u}, \theta) \in \boldsymbol{V}(p,q), \quad \tau \in [0,1], \tag{3.4}
$$

is defined by means of the following initial-boundary value problems:

$$
(3.5) \qquad \begin{aligned}
&w_t - \beta Q w = \tau[\nabla \cdot F_{,\varepsilon}(\bar{\varepsilon}, \bar{\theta}) + b] && \text{in } \Omega^T, \\
&w\big|_{t=0} = \tau w_0 && \text{in } \Omega, \\
&w = 0 && \text{on } S^T,
\end{aligned}
$$

$$
(3.6) \qquad \begin{aligned}
&u_t - \alpha Q u = w && \text{in } \Omega^T, \\
&u\big|_{t=0} = \tau u_0 && \text{in } \Omega, \\
&u = 0 && \text{on } S^T,
\end{aligned}
$$

$$
(3.7) \qquad \begin{aligned}
&c_0(\varepsilon, \bar{\theta}, \tau)\theta_t - k_0 \Delta\theta = \tau[\bar{\theta} F_{,\theta\varepsilon}(\varepsilon, \bar{\theta}) : \varepsilon_t + \nu(A\varepsilon_t) : \varepsilon_t + g] && \text{in } \Omega^T, \\
&\theta\big|_{t=0} = \tau\theta_0 && \text{in } \Omega, \\
&n \cdot \nabla\theta = 0 && \text{on } S^T,
\end{aligned}
$$

where

$$
c_0(\varepsilon, \bar{\theta}, \tau) = c_v - \tau\bar{\theta} F_{,\theta\theta}(\varepsilon, \bar{\theta}), \quad \bar{\varepsilon} = \varepsilon(\bar{u}).
$$

Clearly, a fixed point of $T(1, \cdot)$ in $V(p,q)$ is equivalent to a solution (u, θ) of the decomposed system (3.1), (3.2), (1.2), and thereby constitutes a solution to problem (1.1), (1.2) in $V(p,q)$. Therefore, the proof of Theorem 2.1 reduces to checking that the solution map $T(\tau, \cdot)$ satisfies properties (i)-(iv) of the Leray-Schauder fixed point theorem.
Here we check properties (i), (ii) and (iv). The property (iii) will be proved in Sections 4÷6.

The property (i) follows by showing that for any fixed $\tau \in [0,1], T(\tau, \cdot)$ maps the bounded subsets into precompact subsets in $V(p,q)$. Let $(\bar{u}^n, \bar{\theta}^n)$ be a bounded sequence in $V(p,q)$ such that for $n \to \infty$

$$
(3.8) \qquad \begin{aligned}
&\bar{u}^n \rightharpoonup \bar{u} \ \text{ weakly in } \ W_p^{4,2}(\Omega^T), \quad n+2 < p < \infty, \\
&\bar{\theta}^n \rightharpoonup \bar{\theta} \ \text{ weakly in } \ W_q^{2,1}(\Omega^T), \quad n+2 < q < \infty.
\end{aligned}
$$

Our aim is to show that for the values of $T(\tau, \cdot)$ given by

$$
(3.9) \qquad (u^n, \theta^n) = T(\tau, \bar{u}^n, \bar{\theta}^n)
$$

the following convergences hold for $n \to \infty$

$$
(3.10) \qquad u^n \to u \ \text{ strongly in } \ W_p^{4,2}(\Omega^T), \quad n+2 < p < \infty,
$$

$$
(3.11) \qquad \theta^n \to \theta \ \text{ strongly in } \ W_q^{2,1}(\Omega^T), \quad n+2 < q < \infty,
$$

where

$$(u,\theta) = T(\tau, \bar{u}, \bar{\theta}). \tag{3.12}$$

With the help of compact imbeddings theorems [1] it follows from (3.8) that for $n \to \infty$

$$\begin{aligned} &\bar{u}^n \to \bar{u} \quad \text{strongly in } \ W_p^{3,3/2}(\Omega^T), \ \ n+2 < p < \infty, \\ &\bar{\theta}^n \to \bar{\theta} \quad \text{strongly in } \ W_q^{1,1/2}(\Omega^T), \ \ n+2 < q < \infty. \end{aligned} \tag{3.13}$$

This, by virtue of continuous imbeddings, implies that

$$\bar{\varepsilon}^n \to \bar{\varepsilon}, \ \ \nabla\bar{\varepsilon}^n \to \nabla\bar{\varepsilon}, \ \ \bar{\theta}^n \to \bar{\theta} \tag{3.14}$$

strongly in spaces of Hölder continuous functions in Ω^T, where

$$\bar{\varepsilon}^n = \varepsilon(\bar{u}^n), \quad \bar{\varepsilon} = \varepsilon(\bar{u}).$$

Thanks to the above convergences, it follows that

$$\begin{aligned} &\nabla \cdot F_{,\varepsilon}(\bar{\varepsilon}^n, \bar{\theta}^n) = F_{,\varepsilon\varepsilon}(\bar{\varepsilon}^n, \bar{\theta}^n)\nabla\bar{\varepsilon}^n + F_{,\varepsilon\theta}(\bar{\varepsilon}^n, \bar{\theta}^n)\nabla\bar{\theta}^n \\ &\to F_{,\varepsilon\varepsilon}(\bar{\varepsilon}, \bar{\theta})\nabla\bar{\varepsilon} + F_{,\varepsilon\theta}(\bar{\varepsilon}, \bar{\theta})\nabla\bar{\theta} = \nabla \cdot F_{,\varepsilon}(\bar{\varepsilon}, \bar{\theta}) \\ &\text{strongly in } \ L_p(\Omega^T) \ \text{ for } \ n+2 < p \le q < \infty. \end{aligned} \tag{3.15}$$

Consequently, by the theory of parabolic systems [9],

$$w^n \to w \ \text{ strongly in } \ W_p^{2,1}(\Omega^T),$$

where w^n and w are the corresponding solutions to problem (3.1). In turn, owing to the latter convergence, for solutions of problem (3.2) it holds (3.10). Furthermore, we note that, by (3.10),

$$\varepsilon^n \to \varepsilon, \ \ \varepsilon_t^n \to \varepsilon_t \tag{3.16}$$

strongly in spaces of Hölder continuous functions in Ω^T, where

$$\varepsilon^n = \varepsilon(u^n), \ \ \varepsilon_t^n = \varepsilon(u_t^n), \ \ \varepsilon = \varepsilon(u), \ \ \varepsilon_t = \varepsilon(u_t).$$

In order to prove convergence (3.11) we consider the difference

$$\eta = \theta^n - \theta.$$

By definition, η satisfies the following problem

$$\begin{aligned} &c_0(\varepsilon, \bar{\theta}, \tau)\eta_t^n - k_0\Delta\eta^n = P^n(\varepsilon^n, \bar{\theta}^n, \tau) - P(\varepsilon, \bar{\theta}, \tau) \\ &\qquad - (c_0(\varepsilon^n, \bar{\theta}^n, \tau) - c_0(\varepsilon, \bar{\theta}, \tau))\theta_t^n && \text{in } \Omega^T, \\ &\eta^n\big|_{t=0} = 0 && \text{in } \Omega, \\ &n \cdot \nabla\eta^n = 0 && \text{on } S^T, \end{aligned} \tag{3.17}$$

where

$$P^n(\varepsilon^n, \bar\theta^n, \tau) = \tau[\bar\theta^n F_{,\theta\varepsilon}(\bar\varepsilon^n, \bar\theta^n) : \varepsilon_t^n + \nu(A\varepsilon_t^n) : \varepsilon_t^n + g],$$
$$P(\varepsilon, \bar\theta, \tau) = \tau[\bar\theta F_{,\theta\varepsilon}(\varepsilon, \bar\theta) : \varepsilon_t + \nu(A\varepsilon_t) : \varepsilon_t + g].$$

In view of Hölder continuity of the coefficient $c_0(\varepsilon, \bar\theta, \tau)$, in order to prove that for $n \to \infty$

$$\eta^n \to 0 \text{ strongly in } W_q^{2,1}(\Omega^T),$$

it is sufficient, by virtue of the classical parabolic theory, to show that the right-hand side of $(3.17)_1$ converges to 0 in $L_q(\Omega^T)$-norm. Indeed, we have

$$\begin{aligned}
&\|P^n(\varepsilon^n, \bar\theta^n, \tau) - P(\varepsilon, \bar\theta, \tau)\|_{L_q(\Omega^T)} \\
&\le c\Big(\| |\bar\theta^n - \bar\theta| |F_{,\theta\varepsilon}(\varepsilon^n, \bar\theta^n)| |\varepsilon_t^n| \|_{L_q(\Omega^T)} + \| \bar\theta |\varepsilon_t^n| (|\varepsilon^n - \varepsilon| + |\bar\theta^n - \bar\theta|) \|_{L_q(\Omega^T)} \\
&\qquad + \| \bar\theta |F_{,\theta\varepsilon}(\varepsilon, \bar\theta)| |\varepsilon_t^n - \varepsilon_t| \|_{L_q(\Omega^T)} + \| |\varepsilon_t^n - \varepsilon_t| (|\varepsilon_t^n| + |\varepsilon_t|) \|_{L_q(\Omega^T)}\Big) \\
&\qquad\qquad \to 0 \quad \text{as} \quad n \to 0,
\end{aligned}$$

where we have used uniform with respect to n Hölder bounds on ε^n, ε_t^n, $\bar\theta^n$ and the convergences (3.14), (3.16). Furthermore

$$\begin{aligned}
&\|(c_0(\varepsilon^n, \bar\theta^n, \tau) - c_0(\varepsilon, \bar\theta, \tau))\theta_t^n\|_{L_q(\Omega^T)} \\
&\le \sup_{\Omega^T} |c_0(\varepsilon^n, \bar\theta^n, \tau) - c_0(\varepsilon, \bar\theta, \tau)| \, \|\theta_t^n\|_{L_q(\Omega^T)} \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

This shows (3.11) and thereby the complete continuity of $T(\tau, \cdot)$.

The uniform equicontinuity property (ii) follows by direct comparison of two solutions $(\boldsymbol{w}, \boldsymbol{u}, \theta)$ and $(\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{u}}, \tilde\theta)$ to problem (3.5)-(3.7) corresponding to parameters τ and $\tilde\tau$, respectively, and applying the classical regularity theory (see [7] for details).

The property (iv) is obvious, by definition of $T(\tau, \cdot)$.

4. Energy estimates and recursively improved estimates

In this section we begin the derivation of a priori bounds for a fixed point of the solution map $T(\tau, \cdot)$. Without loss of generality we may set $\tau = 1$. Let then $(\boldsymbol{u}, \theta) \in \boldsymbol{V}(p, q)$ be a fixed point of $T(1, \cdot)$, i.e., a solution to problem (1.1), (1.2). Our goal is to obtain estimates (2.10). To this end we follow the procedure described in [7]. First of all, before establishing the energy estimates, we prove that for solutions $(\boldsymbol{u}, \theta) \in \boldsymbol{V}(p, q)$ temperature θ stays positive what is in accordance with thermodynamics. This is proved under sufficient regularity of solutions. The regularity requirements are satisfied for solutions in the space $\boldsymbol{V}(p, q)$, where ε, θ and ε_t are Hölder continuous in Ω^T. By repeating the proof of Lemma 3.1 [7], we have the following

Lemma 4.1. *Let*

$$\theta_* \equiv \min_{\Omega} \theta_0 > 0, \qquad g \geq 0 \ \text{ in } \ \Omega^T,$$

and $(\boldsymbol{u}, \theta)$ be a solution to (1.1), (1.2) such that $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_t \in \boldsymbol{L}_\infty(\Omega^T)$, $\theta \in L_\infty(\Omega^T)$, $\theta_t \in L_1(0,T; L_q(\Omega))$, $1 < q < \infty$.
Then there exists a positive finite number ω satisfying

$$[g + \nu(A\boldsymbol{\varepsilon}_t) : \boldsymbol{\varepsilon}_t] \exp(\omega t) + [\omega c_0(\boldsymbol{\varepsilon}, \theta) + F_{,\theta\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \theta) : \boldsymbol{\varepsilon}_t]\theta_* \geq 0 \quad \text{in } \ \Omega^T,$$

such that

$$\theta \geq \theta_* \exp(-\omega t) \quad \text{in } \ \Omega^T. \tag{4.1}$$

In the next step, due to (4.1) and the bound (2.7), we can establish physical integral estimates. Repeating the proof of Lemma 3.2 in [7] we have the following:

Lemma 4.2. *Let*

$$\begin{aligned}
&\boldsymbol{u}_0 \in \boldsymbol{W}_2^2(\Omega), \quad \boldsymbol{u}_1 \in \boldsymbol{L}_2(\Omega), \quad \theta_0 \in L_1(\Omega),\\
&(F_1(\boldsymbol{\varepsilon}_0, \theta_0) - \theta_0 F_{1,\theta}(\boldsymbol{\varepsilon}_0, \theta_0)) + F_2(\boldsymbol{\varepsilon}_0) \in L_1(\Omega),\\
&\boldsymbol{b} \in L_1(0,T; \boldsymbol{L}_2(\Omega^T)), \quad g \in L_1(\Omega^T).
\end{aligned}$$

Assume that $\theta \geq 0$ in Ω^T and the bound (2.7) holds. Then a solution $(\boldsymbol{u}, \theta)$ to (1.1), (1.2) satisfies estimate

$$\begin{aligned}
&c_v \|\theta\|_{L_\infty(0,T;L_1(\Omega))} + \frac{1}{4}\|\boldsymbol{u}_t\|^2_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \frac{\varkappa_0}{8}\|\boldsymbol{Q}\boldsymbol{u}\|^2_{L\infty(0,T;\boldsymbol{L}_2(\Omega))}\\
&+ \|(F_1(\boldsymbol{\varepsilon}, \theta) - \theta F_{1,\theta}(\boldsymbol{\varepsilon}, \theta)) + F_2(\boldsymbol{\varepsilon})\|_{L_\infty(0,T;L_1(\Omega))} \leq c
\end{aligned} \tag{4.2}$$

with the constant c given by

$$\begin{aligned}
c = {} & c_v\|\theta_0\|_{L_1(\Omega)} + \frac{1}{2}\|\boldsymbol{u}_1\|^2_{\boldsymbol{L}_2(\Omega)} + \|(F_1(\boldsymbol{\varepsilon}_0, \theta_0) - \theta_0 F_{1,\theta}(\boldsymbol{\varepsilon}_0, \theta_0)) + F_2(\boldsymbol{\varepsilon}_0)\|_{L_1(\Omega)}\\
&+ \frac{\varkappa_0}{8}\|\boldsymbol{Q}\boldsymbol{u}_0\|^2_{\boldsymbol{L}_2(\Omega)} + \|\boldsymbol{b}\|^2_{L_1(0,T;\boldsymbol{L}_2(\Omega))} + \|g\|_{L_1(\Omega^T)}.
\end{aligned}$$

We note the following implications of energy estimate (4.2). In view of property (2.2) of the operator $\boldsymbol{Q}$, it follows that

$$\|\boldsymbol{u}\|_{L_\infty(0,T;\boldsymbol{W}_2^2(\Omega))} \leq c. \tag{4.3}$$

Consequently,

$$\|\boldsymbol{\varepsilon}\|_{L_\infty(0,T;\boldsymbol{W}_2^1(\Omega))} \leq c,$$

so, by Sobolev's imbedding,

$$\|\boldsymbol{\varepsilon}\|_{L_\infty(0,T;\boldsymbol{L}_\sigma(\Omega))} \leq c, \quad \text{for} \quad 1 < \sigma \leq q_n = \begin{cases} \text{any finite number} & \text{if } n = 2,\\ 6 & \text{if } n = 3.\end{cases} \tag{4.4}$$

Moreover, (4.2) implies the bound

$$\|u\|_{W^{2,1}_{2,\infty}(\Omega^T)} \le c. \tag{4.5}$$

Hence,

$$\|\varepsilon\|_{W^{1,1/2}_{2,\infty}(\Omega^T)} \le c,$$

so, in view of imbedding,

$$\|\varepsilon\|_{L_\sigma(\Omega^T)} \le c \ \text{ for } \ 1 < \sigma \le \frac{q_n(n+2)}{n} = \begin{cases} \text{any finite number} & \text{if } n = 2, \\ 10 & \text{if } n = 3. \end{cases} \tag{4.6}$$

Our further procedure will consist in iterative improvement of energy estimates. Similarly as in [7], the main tool is the regularity theory of parabolic systems.
Applying Lemmas A2, A3 [7] to problems (3.1), (3.2) we obtain at first estimates on $\boldsymbol{w}$:

$$\|\boldsymbol{w}\|_{W^{1,1/2}_p(\Omega^T)} \le c\left(\|F_{,\varepsilon}(\varepsilon,\theta)\|_{L_p(\Omega^T)} + \|\boldsymbol{b}\|_{L_p(\Omega^T)} + \|Q\boldsymbol{w}_0\|_{L_p(\Omega)}\right), \tag{4.7}$$

and next on $\boldsymbol{u}$:

$$\|\varepsilon\|_{W^{2,1}_p(\Omega^T)} \le c\|\boldsymbol{u}\|_{W^{3,3/2}_p(\Omega^T)} \le c(\|\boldsymbol{w}\|_{W^{1,1/2}_p(\Omega^T)} + \|\boldsymbol{u}_0\|_{W^{3-2/p}_p(\Omega)}), \quad 1 < p < \infty.$$

With the help of these estimates we prove now additional bounds on temperature. To achieve this we have to impose appropriate growth restrictions on $F(\varepsilon,\theta)$.

Lemma 4.3. *Suppose that s, K_1 and K_2 satisfy conditions*

$$0 < s < \frac{n+1}{2n}, \quad 0 < K_1 < 1 + \frac{q_n}{2}\left[\frac{n+2}{2n} + \frac{1}{n(n+1)}\right],$$
$$0 < K_2 \le 1 + \frac{q_n(n+4)}{4n}.$$

In case $K_1 > 1$ the numbers s and K_1 are linked by

$$\frac{2sn}{n+1} + \frac{4n(K_1-1)}{q_n(n+2)} = 1 + \frac{2}{(n+1)(n+2)}.$$

Moreover,

$$\boldsymbol{u}_0 \in W^{3-(n+4)/2(n+2)}_{4(n+2)/(n+4)}(\Omega), \quad Q\boldsymbol{w}_0 \in L_{4(n+2)/(n+4)}(\Omega), \ \theta_0 \in L_2(\Omega),$$

$$G(\varepsilon_0,\theta_0) \equiv -\theta_0^2 F_{1,\theta}(\varepsilon_0,\theta_0) + 2\theta_0 F_1(\varepsilon_0,\theta_0) - 2\int_0^{\theta_0} F_1(\varepsilon_0,\xi)d\xi \in L_1(\Omega),$$

$$\boldsymbol{b} \in L_{4(n+2)/(n+4)}(\Omega^T), \quad g \in L_{2(n+2)/(n+4)}(\Omega^T).$$

Then there exists a constant $c(T)$ depending on the data and T^a, $a \in \mathbb{R}_+$ such that

$$\|\theta\|_{V_2(\Omega^T)} \equiv \|\theta\|_{L_\infty(0,T;L_2(\Omega))} + \|\nabla\theta\|_{L_2(\Omega^T)} \le c(T). \tag{4.8}$$

Proof. Multiplying $(1.2)_1$ by θ and integrating over Ω we get

$$\begin{aligned} &\frac{c_v}{2}\frac{d}{dt}\int_\Omega \theta^2 dx - \int_\Omega (\theta^2 F_{1,\theta\theta}\theta_t + \theta^2 F_{1,\theta\varepsilon} : \varepsilon_t) dx + k_0 \int_\Omega |\nabla\theta|^2 dx \\ &= \nu \int_\Omega \theta(A\varepsilon_t) : \varepsilon_t dx + \int_\Omega \theta g dx. \end{aligned} \tag{4.9}$$

Now we introduce the function

$$G(\varepsilon,\theta) = -\theta^2 F_{1,\theta}(\varepsilon,\theta) + 2\theta F_1(\varepsilon,\theta) - 2\hat{F}_1(\varepsilon,\theta), \tag{4.10}$$

where

$$\hat{F}_1(\varepsilon,\theta) = \int_0^\theta F_1(\varepsilon,\xi) d\xi.$$

It is seen that $G(\varepsilon,\theta)$ is the primitive of $-\theta^2 F_{1,\theta\theta}(\varepsilon,\theta)$ with respect to θ such that

$$G(\varepsilon,0) = 0 \quad \text{and} \quad G_{,\theta}(\varepsilon,\theta) = -\theta^2 F_{1,\theta\theta}(\varepsilon,\theta) \ge 0,$$

so

$$G(\varepsilon,\theta) \ge 0 \quad \text{for} \quad (\varepsilon,\theta) \in \mathcal{S}^2 \times [0,\infty). \tag{4.11}$$

Furthermore, according to (4.10),

$$G_{,\varepsilon}(\varepsilon,\theta) = -\theta^2 F_{1,\theta\varepsilon}(\varepsilon,\theta) + 2\theta F_{1,\varepsilon}(\varepsilon,\theta) - 2\hat{F}_{1,\varepsilon}(\varepsilon,\theta). \tag{4.12}$$

In view of (4.10) and (4.12), identity (4.9) takes on the form

$$\begin{aligned} &\frac{c_v}{2}\frac{d}{dt}\int_\Omega \theta^2 dx + \frac{d}{dt}\int_\Omega G(\varepsilon,\theta) dx + k_0 \int_\Omega |\nabla\theta|^2 dx \\ &= \nu \int_\Omega \theta(A\varepsilon_t) : \varepsilon_t dx + 2\int_\Omega (\theta F_{1,\varepsilon} - \hat{F}_{1,\varepsilon}) : \varepsilon_t dx + \int_\Omega \theta g dx. \end{aligned} \tag{4.13}$$

Integrating (4.13) with respect to time and using (4.11) we get

$$\begin{aligned} cX^2(t) &\le \frac{c_v}{2}\int_\Omega \theta^2 dx + k_0 \int_{\Omega^t} |\nabla\theta|^2 dx dt' \\ &\le \nu \int_{\Omega^t} \theta(A\varepsilon_{t'}) : \varepsilon_{t'} dx dt' + 2\int_{\Omega^t} (\theta F_{1,\varepsilon} - \hat{F}_{1,\varepsilon}) : \varepsilon_{t'} dx dt' \\ &\quad + \int_{\Omega^t} \theta g dx dt' + \frac{c_v}{2}\int_\Omega \theta_0^2 dx + \int_\Omega G(\varepsilon_0,\theta_0) dx, \quad 0 \le t \le T, \end{aligned} \tag{4.14}$$

where

$$X(t) \equiv (\|\theta\|^2_{L_\infty(0,t;L_2(\Omega))} + \|\nabla\theta\|^2_{L_2(\Omega^t)})^{1/2}.$$

We proceed now to estimate the terms on the right-hand side of (4.14). To this end first we note that by virtue of the imbedding of the space $V_2(\Omega^T)$ in $L_{2(n+2)/n}(\Omega^T)$, it holds

$$\begin{aligned}\|\theta\|_{L_{\frac{2(n+2)}{n}}(\Omega^T)}, &\le c\|\theta\|_{V_2(\Omega^T)}\\ &\le c\big(\|\theta\|^2_{L_\infty(0,T;L_2(\Omega))} + \|\nabla\theta\|^2_{L_2(\Omega^T)}\big)^{1/2} = cX(T).\end{aligned} \tag{4.15}$$

In view of (4.15) the first term on the right-hand side of (4.14) is estimated as follows

$$\nu\int\limits_{\Omega^t}\theta(A\varepsilon_{t'}) : \varepsilon_{t'}\,dx dt' \le c\|\theta\|_{L_{\frac{2(n+2)}{n}}(\Omega^T)}\|\varepsilon_t\|^2_{L_{\frac{4(n+2)}{n+4}}(\Omega^T)} \equiv Y_1. \tag{4.16}$$

To find bounds on $\|\varepsilon_t\|_{L_{4(n+2)/(n+4)}(\Omega^T)}$ we make use of estimate (4.7). From now on we set

$$p = \frac{4(n+2)}{n+4}.$$

In view of assumptions, by the Hölder inequality, it follows that

$$\begin{aligned}\|F_{,\varepsilon}(\varepsilon,\theta)\|_{L_p(\Omega^T)} &\le c\Bigg(\int\limits_{\Omega^T}\theta^{ps}|\varepsilon|^{p\max\{0,K_1-1\}}dxdt + \int\limits_{\Omega^T}|\varepsilon|^{p\max\{0,K_2-1\}}dxdt\Bigg)^{\frac{1}{p}} + c,\\ &\le c\Bigg(\int\limits_{\Omega^T}\theta^{ps\lambda_1}dxdt\Bigg)^{\frac{1}{p\lambda_1}}\Bigg(\int\limits_{\Omega^T}|\varepsilon|^{p\lambda_2\max\{0,K_1-1\}}dxdt\Bigg)^{\frac{1}{p\lambda_2}}\\ &\quad + c\Bigg(\int\limits_{\Omega^T}|\varepsilon|^{p\max\{0,K_2-1\}}dxdt\Bigg)^{\frac{1}{p}} + c,\end{aligned}$$

where $1/\lambda_1 + 1/\lambda_2 = 1$. Now we set

$$\lambda_2 = \begin{cases}\dfrac{q_n(n+4)}{4n(K_1-1)} & \text{if } K_1 > 1\\ \text{any number from the interval } (1,\infty) & \text{if } K_1 \le 1.\end{cases}$$

Then, by virtue of the bound (4.6),

$$\Bigg(\int\limits_{\Omega^T}|\varepsilon|^{p\lambda_2\max\{0,K_1-1\}}dxdt\Bigg)^{\frac{1}{p\lambda_2}} \le c.$$

Moreover, in view of (A3-2) (iii),

$$\left(\int_{\Omega^T} |\varepsilon|^{p\max\{0,K_2-1\}} dxdt\right)^{\frac{1}{p}} \leq c.$$

Consequently,

$$\|F_{,\varepsilon}(\varepsilon,\theta)\|_{L_p(\Omega^T)} \leq c(\|\theta\|^s_{L_{ps\lambda_1}(\Omega^T)} + 1).$$

We examine now the term

$$\|\theta\|^s_{L_{ps\lambda_1}(\Omega^T)} = \left(\int_0^T \|\theta\|^{ps\lambda_1}_{L_{ps\lambda_1}(\Omega)} dt\right)^{\frac{1}{p\lambda_1}} \equiv I_1.$$

To this end we make use of the interpolation inequality

$$\|\theta\|_{L_{ps\lambda_1}(\Omega)} \leq c\|\nabla\theta\|^{\vartheta_1}_{L_2(\Omega)}\|\theta\|^{1-\vartheta_1}_{L_1(\Omega)} + c\|\theta\|_{L_1(\Omega)},$$

where parameter ϑ_1 is determined by the relation

$$\frac{n}{ps\lambda_1} = (1-\vartheta_1)\frac{n}{1} + \vartheta_1(\frac{n}{2}-1), \quad \text{so} \quad \vartheta_1 = \frac{2n}{n+2}(1-\frac{1}{ps\lambda_1}).$$

Then

$$\begin{aligned} I_1 &\leq c\left[\int_0^T (\|\nabla\theta\|^{ps\lambda_1\vartheta_1}_{L_2(\Omega)}\|\theta\|^{ps\lambda_1(1-\vartheta_1)}_{L_1(\Omega)} + \|\theta\|^{ps\lambda_1}_{L_1(\Omega)})dt\right]^{\frac{1}{p\lambda_1}} \\ &\leq c\left(\int_0^T \|\nabla\theta\|^{\frac{2n}{n+2}(ps\lambda_1-1)}_{L_2(\Omega)} dt\right)^{\frac{1}{p\lambda_1}} + cT^{\frac{1}{p\lambda_1}} \equiv I_2, \end{aligned}$$

where in the last inequality we have used estimate (4.2). Imposing the condition

$$\frac{2n}{n+2}(ps\lambda_1 - 1) = 2,$$

equivalent to

$$\lambda_1 = \frac{(n+1)(n+4)}{2ns(n+2)},$$

it follows that

$$I_2 \leq c(T)\left(\|\nabla\theta\|^{\frac{2}{p\lambda_1}}_{L_2(\Omega^T)} + 1\right).$$

In the case $ps\lambda_1 \leq 1$, by virtue of energy estimate (4.2), $I_1 \leq c$.

In view of the above estimates, we obtain

$$\|F_{,\varepsilon}(\varepsilon,\theta)\|_{L_p(\Omega^T)} \le c(T)\Big(\|\nabla\theta\|_{L_2(\Omega^T)}^{\frac{2}{p\lambda_1}} + 1\Big),$$

so, by virtue of (4.7),

$$\|\varepsilon\|_{W_p^{2,1}(\Omega^T)} \le c(T)\Big(\|\nabla\theta\|_{L_2(\Omega^T)}^{\frac{2}{p\lambda_1}} + 1\Big). \tag{4.17}$$

Consequently, using the above estimate in (4.16) gives

$$Y_1 \le c(T)\|\theta\|_{L_{\frac{2(n+2)}{n}}(\Omega^T)}\left(\|\nabla\theta\|_{L_2(\Omega^T)}^{\frac{4}{p\lambda_1}} + 1\right) \le c(T)X(T)\left(X(T)^{\frac{4}{p\lambda_1}} + 1\right).$$

Now we assume the condition

$$1 + \frac{4}{p\lambda_1} < 2, \tag{4.18}$$

implying that

$$\lambda_1 > \frac{n+4}{n+2}.$$

Hence, in view of expression for λ_1,

$$s < \frac{n+1}{2n}. \tag{4.19}$$

Moeover, we note that the equality $1/\lambda_1 + 1/\lambda_2 = 1$ can be always satisfied in case $K_1 \le 1$, whereas in case $K_1 > 1$ it imposes an additional conditon between s and K_1, namely

$$\frac{2sn(n+2)}{(n+1)(n+4)} + \frac{4n(K_1-1)}{q_n(n+4)} = 1. \tag{4.19'}$$

In view of (4.18), by Young's inequality,

$$Y_1 \le \varepsilon_1 X(T)^2 + c(\varepsilon_1)c(T), \quad \varepsilon_1 = \text{const} > 0.$$

Therefore, for sufficiently small ε_1 the term $\varepsilon_1 X(T)^2$ can be absorbed by the left-hand side of (4.14).

The second integral on the right-hand side of (4.14) is handled as follows:

$$\begin{aligned}
&2\int\limits_{\Omega^t}(\theta F_{1,\varepsilon} - \hat{F}_{1,\varepsilon}) : \varepsilon_{t'}\,dx\,dt' \le c\int\limits_{\Omega^T}\theta(1+\theta^s)|\varepsilon|^{\max\{0,K_1-1\}}|\varepsilon_t|\,dx\,dt \\
&\le c\left[\left(\int\limits_{\Omega^T}\theta^{(1+s)\lambda_3}dx\,dt\right)^{\frac{1}{\lambda_3}} + c(T)\right]\left(\int\limits_{\Omega^T}|\varepsilon|^{\lambda_4\max\{0,K_1-1\}}dx\,dt\right)^{\frac{1}{\lambda_4}}\left(\int\limits_{\Omega^T}|\varepsilon_t|^{\lambda_5}dx\,dt\right)^{\frac{1}{\lambda_5}} \\
&\equiv Y_2,
\end{aligned}$$

where $1/\lambda_3 + 1/\lambda_4 + 1/\lambda_5 = 1$. In view of the previous considerations we set

$$\lambda_4 = \begin{cases} \dfrac{q_n(n+2)}{n(K_1-1)} & \text{if } K_1 > 1 \\ \text{any number from the interval } (1,\infty) \text{ if } K_1 \le 1, \end{cases}$$

$$\lambda_5 = \frac{4(n+2)}{n+4}.$$

In that case, utilizing estimates (4.6) and (4.17),

$$Y_2 \le c(T)\left(\|\theta\|^{1+s}_{L_{(1+s)\lambda_3}(\Omega^T)} + 1\right)\left(\|\nabla\theta\|^{\frac{2}{p\lambda_1}}_{L_2(\Omega^T)} + 1\right).$$

Similarly as I_2, the term

$$\|\theta\|^{1+s}_{L_{(1+s)\lambda_3}(\Omega^T)} = \left(\int_0^T \|\theta\|^{(1+s)\lambda_3}_{L_{(1+s)\lambda_3}(\Omega)}dt\right)^{\frac{1}{\lambda_3}} \equiv I_3$$

is examined by means of the interpolation inequality

$$\|\theta\|_{L_{(1+s)\lambda_3}(\Omega)} \le c\|\nabla\theta\|^{\vartheta_2}_{L_2(\Omega)}\|\theta\|^{1-\vartheta_2}_{L_1(\Omega)} + c\|\theta\|_{L_1(\Omega)},$$

where ϑ_2 is determined by the relation

$$\frac{n}{(1+s)\lambda_3} = (1-\vartheta_2)\frac{n}{1} + \vartheta_2(\frac{n}{2}-1), \quad \text{so} \quad \vartheta_2 = \frac{2n}{2+n}(1 - \frac{1}{(1+s)\lambda_3}).$$

Then

$$I_3 \le c\left[\int_0^T \left(\|\nabla\theta\|^{(1+s)\lambda_3\vartheta_2}_{L_2(\Omega)}\|\theta\|^{(1+s)\lambda_3(1-\vartheta_1)}_{L_1(\Omega)} + \|\theta\|^{(1+s)\lambda_3}_{L_1(\Omega)}\right)dt\right]^{\frac{1}{\lambda_3}}$$

$$\le c\left(\int_0^T \|\nabla\theta\|^{\frac{2n}{n+2}((1+s)\lambda_3-1)}_{L_2(\Omega)}dt\right)^{\frac{1}{\lambda_3}} + cT^{\frac{1}{\lambda_3}} \equiv I_4,$$

where in the last inequality we have used estimate (4.2). Here we set

$$\frac{2n}{n+2}((1+s)\lambda_3 - 1) = 2,$$

what implies that

$$\lambda_3 = \frac{2(n+1)}{n(1+s)}.$$

Then

$$I_4 \le c(T)\left(\|\nabla\theta\|_{L_2(\Omega^T)}^{\frac{2}{\lambda_3}} + 1\right).$$

Combining estimates on the terms I_3 and I_4, we arrive at

$$Y_2 \le c(T)\left(\|\nabla\theta\|_{L_2(\Omega^T)}^{\frac{2}{\lambda_3}} + 1\right)\left(\|\nabla\theta\|_{L_2(\Omega^T)}^{\frac{2}{p\lambda_1}} + 1\right) \le c(T)\left(X(T)^{\frac{2}{\lambda_3}} + 1\right)\left(X(T)^{\frac{2}{p\lambda_1}} + 1\right).$$

Now we assume the condition

$$\frac{2}{p\lambda_1} + \frac{2}{\lambda_3} < 2. \tag{4.20}$$

Consequently, by Young's inequality,

$$Y_2 \le \varepsilon_2 X(T)^2 + c(\varepsilon_2)c(T), \quad \varepsilon_2 = \text{const} > 0,$$

so for sufficiently small ε_2, the term $\varepsilon_2 X(T)^2$ can be absorbed by the left-hand side of (4.14). Using expressions for p, λ_1 and λ_3, condition (4.20) takes on the form

$$\frac{ns}{2(n+1)} + \frac{n(1+s)}{2(n+1)} < 1,$$

that is

$$s < \frac{n+2}{2n}.$$

Clearly, the latter condition is less restrictive than (4.19).
In case $K_1 > 1$ the equality $1/\lambda_3 + 1/\lambda_4 + 1/\lambda_5 = 1$ imposes the condition

$$\frac{(1+s)n}{2(n+1)} + \frac{n(K_1-1)}{q_n(n+2)} + \frac{n+4}{4(n+2)} = 1, \tag{4.21}$$

that is,

$$\frac{2sn(n+2)}{(n+1)(n+4)} + \frac{4n(K_1-1)}{q_n(n+4)} = \frac{(n+1)(n+2)+2}{(n+1)(n+4)}.$$

Obviously, since

$$\frac{(n+1)(n+2)+2}{(n+1)(n+4)} < 1,$$

condition (4.21) is more restrictive than (4.19′).
In case $K_1 \le 1$ it is seen, in view of

$$\frac{2sn(n+2)}{(n+1)(n+4)} < \frac{n+2}{n+4} < 1,$$

that the condition between λ_3, λ_4 and λ_5 can be always satisfied.

Summarizing restrictions, we see that s and K_1 have to satisfy (4.19) and (4.21).

Finally, the third integral on the right-hand side of (4.14) is bounded by

$$\int\limits_{\Omega^t} \theta g dx dt' \leq \|\theta\|_{L_{\frac{2(n+2)}{n}}(\Omega^T)} \|g\|_{L_{\frac{2(n+2)}{n+4}}(\Omega^T)} \leq \varepsilon_3 X(T)^2 + c(\varepsilon_3), \quad \varepsilon_3 = \text{const} > 0,$$

so again the term $\varepsilon_3 X(T)^2$ can be absorbed by the left-hand side of (4.14).
In this way, recalling assumptions on initial data, it follows from (4.14) that

$$X(T)^2 \leq c(T),$$

what completes the proof. □

We note some additional estimates resulting from the above lemma. From (4.15) it follows that

$$\|\theta\|_{L_{\frac{2(n+2)}{n}}(\Omega^T)} \leq c(T), \tag{4.22}$$

and from (4.17),

$$\|\varepsilon\|_{W^{2,1}_{\frac{4(n+2)}{n+4}}(\Omega^T)} \leq c(T). \tag{4.23}$$

Thus, by imbedding, ε is Hölder continuous in Ω^T, and

$$\|\varepsilon\|_{C^{\alpha_1,\alpha_1/2}(\Omega^T)} \leq c(T) \quad \text{with} \quad 0 < \alpha_1 < 1 - \frac{n}{4}. \tag{4.24}$$

Consequently,

$$\|F_{,\varepsilon}(\varepsilon,\theta)\|_{L_p(\Omega^T)} \leq c(T)\left(\|\theta\|^s_{L_{ps}(\Omega^T)} + 1\right),$$

and, with the help of (4.7), we conclude that

$$\|\varepsilon\|_{W^{2,1}_p(\Omega^T)} \leq c(T)\left(\|\theta\|^s_{L_{ps}(\Omega^T)} + 1\right) \leq c(T) \quad \text{for} \quad p = \frac{2(n+2)}{ns}. \tag{4.25}$$

We note also that, by virtue of (4.24), the bounds (2.5) imply

$$|c_0(\varepsilon,\theta)| + |c_{0,\varepsilon}(\varepsilon,\theta)| + |c_{0,\theta}(\varepsilon,\theta)| \leq c(T) \quad \text{in} \quad \Omega^T. \tag{4.26}$$

In the next lemma we obtain an estimate for θ_t.

Lemma 4.4. *Suppose that*

$$0 < s < \frac{n+1}{2n},$$
$$g \in L_2(\Omega^T), \qquad \nabla\theta_0 \in L_2(\Omega), \quad \text{and}$$
$$\|\theta\|_{L_{\frac{2(n+2)}{n}}(\Omega^T)} \le c(T), \quad \|\varepsilon\|_{C^{\alpha_1,\alpha_1/2}(\Omega^T)} \le c(T),$$
$$\|\varepsilon_t\|_{L_p(\Omega^T)} \le c(T) \quad \text{for} \quad p = \frac{2(n+2)}{ns} > 2.$$

Then there exists a constant $c(T) > 0$ *such that*

$$\|\theta_t\|_{L_2(\Omega^T)} + \|\theta\|_{L_\infty(0,T;W_2^1(\Omega))} \le c(T). \tag{4.27}$$

Proof. Multiplying $(1.2)_1$ by θ_t and integrating over Ω^t we obtain

$$c_v \int\limits_{\Omega^t} \theta_{t'}^2 dx dt' + \frac{k_0}{2} \int\limits_0^t \frac{d}{dt'} \int\limits_\Omega |\nabla\theta|^2 dx dt'$$
$$\le c \int\limits_{\Omega^t} (1 + \theta^s |\varepsilon|^{\max\{0,K_1-1\}}) |\varepsilon_{t'}| |\theta_{t'}| dx dt' + \bar{c} \int\limits_{\Omega^t} |\varepsilon_{t'}|^2 |\theta_{t'}| dx dt' + \int\limits_{\Omega^t} |g| |\theta_{t'}| dx dt'.$$

Hence, in view of assumptions, by Young's inequality it follows that

$$\begin{aligned} &\frac{c_v}{2} \|\theta_t\|_{L_2(\Omega^t)}^2 + \frac{k_0}{2} \|\nabla\theta\|_{L_\infty(0,t;L_2(\Omega))} \\ &\le c \int\limits_{\Omega^t} (1 + \theta^{2s}) |\varepsilon_{t'}|^2 dx dt' + c \int\limits_{\Omega^t} |\varepsilon_{t'}|^4 dx dt' + \int\limits_{\Omega^t} g^2 dx dt' + \frac{k_0}{2} \int\limits_\Omega |\nabla\theta_0|^2 dx. \end{aligned} \tag{4.28}$$

For the first term on the right-hand side of (4.28) we have

$$\begin{aligned} I \equiv \int\limits_{\Omega^t} (1 + \theta^{2s}) |\varepsilon_{t'}|^2 dx dt' &\le \|\varepsilon_t\|_{L_2(\Omega^t)}^2 + \|\theta\|_{L_{2s\lambda_1}(\Omega^t)}^{2s} \|\varepsilon_t\|_{L_{2\lambda_2}(\Omega^t)}^2 \\ &\le c(T) \|\theta\|_{L_{\frac{2(n+2)}{n}}(\Omega^T)}^{2s} \|\varepsilon_t\|_{L_{\frac{2(n+2)}{n+2-ns}}(\Omega^T)}^2 \\ &\le c(T)(1 + \|\varepsilon_t\|_{L_{\frac{2(n+2)}{n+2-ns}}(\Omega^T)}^2), \end{aligned}$$

where we have applied the Hölder inequality with

$$\lambda_1 = (n+2)/(ns), \quad \lambda_2 = (n+2)/(n+2-ns).$$

Since, $0 < s < 1 < \frac{n+2}{n}$, $1 < \lambda_1, \lambda_2 < \infty$.

Furthermore, in view of

$$s < \frac{n+1}{2n} < \frac{n+2}{2n}, \tag{4.29}$$

it follows that

$$\frac{2(n+2)}{n+2-ns} < \frac{2(n+2)}{ns}.$$

Hence, by virtue of the bound on ε_t

$$I \leq c(T).$$

Similarly, since

$$4 < \frac{2(n+2)}{ns},$$

the second integral on the right-hand side of (4.28) is bounded by

$$\int_{\Omega^t} |\varepsilon_{t'}|^4 dx dt' \leq \|\varepsilon_t\|^4_{L_4(\Omega^T)} \leq c(T).$$

Consequently, (4.28) implies

$$\|\theta_t\|_{L_2(\Omega^T)} + \|\nabla\theta\|_{L_\infty(0,T;L_2(\Omega))} \leq c(T),$$

what together with (4.8) shows the assertion. □

By virtue of imbedding, it follows from (4.27) that

$$\|\theta\|_{L_\infty(0,T;L_\sigma(\Omega))} \leq c(T) \quad \text{for} \quad 1 < \sigma \leq q_n. \tag{4.30}$$

We indicate more consequences of the estimates obtained so far.
Let us write equation $(1.2)_1$ in the form

$$-k_0\Delta\theta = -c_0(\varepsilon,\theta)\theta_t + \theta F_{,\theta\varepsilon}(\varepsilon,\theta) : \varepsilon_t + \nu(A\varepsilon_t) : \varepsilon_t + g. \tag{4.31}$$

Recalling the arguments used in the proof of the previous lemma we see that

$$\|\theta F_{,\theta\varepsilon}(\varepsilon,\theta) : \varepsilon_t + \nu(A\varepsilon_t) : \varepsilon_t + g\|_{L_2(\Omega^T)} \leq c(T).$$

Also, by (4.26), (4.27), we have $\|c_0(\varepsilon,\theta)\theta_t\|_{L_2(\Omega^T)} \leq c(T)$. Therefore, by virtue of the classical elliptic theory, it follows from (4.31) that

$$\|\theta\|_{L_2(0,T;W_2^2(\Omega))} \leq c(T), \quad \text{so} \quad \|\theta\|_{L_2(0,T;L_\infty(\Omega))} \leq c(T). \tag{4.32}$$

Furthermore, (4.32) and (4.27) imply that

$$\|\theta\|_{W_2^{2,1}(\Omega^T)} \leq c(T), \quad \|\nabla\theta\|_{W_2^{1,1/2}(\Omega^T)} \leq c(T), \tag{4.33}$$

so, by Sobolev's imbeddings,

$$\|\theta\|_{L_{\frac{q_n(n+2)}{n}}(\Omega^T)} \le c(T), \quad \|\nabla\theta\|_{L_{\frac{2(n+2)}{n}}(\Omega^T)} \le c(T). \tag{4.34}$$

Thus, repeating estimate (4.25), in view of (4.34), we arrive at

$$\|\varepsilon\|_{W_p^{2,1}(\Omega^T)} \le c(T)(\|\theta\|^s_{L_{ps}(\Omega^T)} + 1) \le c(T), \tag{4.35}$$

$$\|\nabla\varepsilon\|_{W_p^{1,1/2}(\Omega^T)} \le c(T)$$

for $p = q_n(n+2)/(ns) > q_n$. Hence, by imbedding,

$$\|\nabla\varepsilon\|_{C^{\alpha_2,\alpha_2/2}(\Omega^T)} \le c(T) \quad \text{with} \quad 0 < \alpha_2 < 1 - \frac{ns}{q_n}. \tag{4.36}$$

We close the sequence of estimates by the one resulting directly from the regularity theory of parabolic systems. Namely, in view of assumptions,

(4.37)
$$|\nabla\cdot F_{,\varepsilon}(\varepsilon,\theta)| \le c(1+\theta^s|\varepsilon|^{\max\{0,K_1-2\}}+|\varepsilon|^{\max\{0,K_2-2\}})|\nabla\varepsilon|+c(1+\theta^{s-1}|\varepsilon|^{\max\{0,K_1-1\}})|\nabla\theta|,$$

recalling bounds (4.24), (4.34) and (4.36), we get

$$\|\nabla\cdot F_{,\varepsilon}(\varepsilon,\theta)\|_{L_{\frac{2(n+2)}{n}}(\Omega^T)} \le c(T).$$

Therefore, owing to the classical regularity result for parabolic systems (see e.g. [7], Lemma A1), it follows that

$$\|u\|_{W^{4,2}_{\frac{2(n+2)}{n}}(\Omega^T)} \le c(T). \tag{4.38}$$

Consequently, by imbedding,

$$\|\varepsilon_t\|_{W^{1,1/2}_{\frac{2(n+2)}{n}}(\Omega^T)} \le c(T). \tag{4.39}$$

Hence, in particular, since $W^1_{\frac{2(n+2)}{n}}(\Omega) \subset L_\infty(\Omega)$, it follows that

$$\|\varepsilon_t\|_{L_2(0,T;\,L_\infty(\Omega))} \le c(T). \tag{4.40}$$

The latter estimate will be crucial for obtaining $L_\infty(\Omega^T)$ - norm bound for θ (see Lemma 5.2).

5. Pointwise estimate on temperature

In this section we prove the crucial $L_\infty(\Omega^T)$ - norm estimate on θ. Here we assume that the hypotheses (A1)–(A4) are satisfied and, for the presentation convenience, suppose that function $F_1(\varepsilon,\theta)$ has the form specified in Example in Section 2. The idea of the proof consists in deriving an uniform bound in $L_r(\Omega^T)$ - norm and then passing to the limit with $r\to\infty$.

First we establish that $\theta\in L_r(\Omega^T)$ for $1<r<\infty$. In $2-D$ case this is assured by estimate (4.34). In $3-D$ we have the following

Lemma 5.1. *Let function $F_1(\varepsilon,\theta)$ be defined as in Example in Section 2. Moreover, suppose that estimates (4.24), (4.35) hold, and*

$$0<s<\frac{n+1}{2n},\qquad g\in L_{\frac{n+2}{2}}(\Omega^T),$$

$$\theta_0\in L_r(\Omega^T),\qquad 1<r<\infty,$$

$$G(\varepsilon_0,\theta_0;r)\equiv-\theta_0^{r+1}F_{1,\theta}(\varepsilon_0,\theta_0)+(r+1)\theta_0^rF_1(\varepsilon_0,\theta_0)-r(r+1)\int\limits_0^{\theta_0}\xi^{r-1}F_1(\varepsilon_0,\xi)d\xi\in L_1(\Omega).$$

Then $\theta\in L_r(\Omega^T)$ for $1<r<\infty$, and

$$\|\theta\|_{L_{\frac{(n+2)r}{n}}(\Omega^T)}\le c(r), \tag{5.1}$$

where

$$c(r)\equiv c(T)^{\frac{1}{r}}r^{\frac{1}{r}}r^{\frac{1}{1-s}}\to\infty\quad\text{as } r\to\infty.$$

Proof. Without loss of generality we can assume that $\theta\ge\theta_2$ a.e. in Ω^T. Multiplying equation $(1.2)_1$ by θ^r, $r>1$, and integrating over Ω we get

$$\begin{aligned}&c_v\int\limits_\Omega\theta^r\theta_tdx-\int\limits_\Omega(\theta^{r+1}F_{1,\theta\theta}\theta_t+\theta^{r+1}F_{1,\theta\varepsilon}:\varepsilon_t)dx\\&+rk_0\int\limits_\Omega\theta^{r-1}|\nabla\theta|^2dx=\nu\int\limits_\Omega\theta^r(A\varepsilon_t):\varepsilon_tdx+\int\limits_\Omega\theta^rgdx.\end{aligned}\tag{5.2}$$

Similarly as in the proof of Lemma 4.3, we introduce the function

$$G(\varepsilon,\theta;r)=-\theta^{r+1}F_{1,\theta}(\varepsilon,\theta)+(r+1)\theta^rF_1(\varepsilon,\theta)-r(r+1)\hat F_1(\varepsilon,\theta;r),\tag{5.3}$$

where

$$\hat F_1(\varepsilon,\theta;r)=\int\limits_0^\theta\xi^{r-1}F_1(\varepsilon,\xi)d\xi.$$

Clearly, we have

$$G(\varepsilon,0;r)=0,\qquad G_{,\theta}(\varepsilon,\theta;r)=-\theta^{r+1}F_{1,\theta\theta}(\varepsilon,\theta)\geq 0, \tag{5.4}$$

so that

$$G(\varepsilon,\theta;r)\geq 0 \tag{5.5}$$

for all arguments. Besides,

$$G_{,\varepsilon}(\varepsilon,\theta;r)=-\theta^{r+1}F_{1,\theta\varepsilon}(\varepsilon,\theta)+(r+1)\theta^{r}F_{1,\varepsilon}(\varepsilon,\theta)-r(r+1)\hat{F}_{1,\varepsilon}(\varepsilon,\theta;r). \tag{5.6}$$

In view of (5.4) and (5.6) we can express identity (5.2) in the form

$$\begin{aligned}&\frac{c_v}{r+1}\frac{d}{dt}\int\limits_\Omega \theta^{r+1}dx+\frac{d}{dt}\int\limits_\Omega G(\varepsilon,\theta;r)dx+\frac{4rk_0}{(r+1)^2}\int\limits_\Omega|\nabla\theta^{\frac{r+1}{2}}|^2dx\\ &=(r+1)\int\limits_\Omega[\theta^rF_{1,\varepsilon}(\varepsilon,\theta)-r\hat{F}_{1,\varepsilon}(\varepsilon,\theta;r)]:\varepsilon_t dx+\nu\int\limits_\Omega\theta^r(A\varepsilon_t):\varepsilon_t dx+\int\limits_\Omega\theta^r g dx.\end{aligned} \tag{5.7}$$

Integrating (5.7) with respect to time, using (5.5) and noting that $2r/(r+1)>1$ for $r>1$, we obtain

$$\begin{aligned}\frac{c}{r+1}X^2(t)&\leq\frac{c_v}{r+1}\int\limits_\Omega(\theta^{\frac{r+1}{2}})^2dx+\frac{2k_0}{r+1}\int\limits_{\Omega^t}|\nabla\theta^{\frac{r+1}{2}}|^2dxdt'\\ &\leq(r+1)\int\limits_{\Omega^t}[\theta^rF_{1,\varepsilon}(\varepsilon,\theta)-r\hat{F}_{1,\varepsilon}(\varepsilon,\theta;r)]:\varepsilon_{t'}dxdt'\\ &+\nu\int\limits_{\Omega^t}\theta^r(A\varepsilon_{t'}):\varepsilon_{t'}dxdt'+\int\limits_{\Omega^t}\theta^r g dxdt'+\frac{c_v}{r+1}\int\limits_\Omega\theta_0^{r+1}dx\\ &+\int\limits_\Omega G(\varepsilon_0,\theta_0;r)dx,\qquad 0\leq t\leq T,\end{aligned} \tag{5.8}$$

where

$$X(t)\equiv\left(\|\theta^{\frac{r+1}{2}}\|^2_{L_\infty(0,t;L_2(\Omega))}+\|\nabla\theta^{\frac{r+1}{2}}\|^2_{L_2(\Omega^t)}\right)^{\frac{1}{2}}.$$

In view of the imbedding of the space $V_2(\Omega^t)$ in $L_{2(n+2)/n}(\Omega^t)$,

$$X^2(t)\geq c\|\theta^{\frac{r+1}{2}}\|^2_{V_2(\Omega^t)}\geq c\|\theta^{\frac{r+1}{2}}\|^2_{L_{\frac{2(n+2)}{n}}(\Omega^t)}=c\|\theta\|^{r+1}_{L_{\frac{(n+2)(r+1)}{n}}(\Omega^t)}.$$

Therefore, for the left-hand side of (5.8) we have

$$\frac{c}{r+1}\|\theta\|^{r+1}_{L_{\frac{(n+2)(r+1)}{n}}(\Omega^t)}\leq\frac{c}{r+1}X^2(t). \tag{5.10}$$

Having this in mind we proceed now to estimate the integrals on the right-hand side of (5.8). Turning to the first integral, a direct calculation shows that for the assumed form of $F_1(\varepsilon,\theta)$,

$$\begin{aligned}&\theta^r F_{1,\varepsilon}(\varepsilon,\theta) - r\hat{F}_{1,\varepsilon}(\varepsilon,\theta;r)\\ &= \sum_{i=1}^N \left[\frac{s_i}{r+s_i}\theta^{r+s_i} - \frac{r}{r+1}\theta_1^{r+1} - r\int_{\theta_1}^{\theta_2} \xi^{r-1}\varphi_i(\xi)d\xi + \frac{r}{r+s_i}\theta_2^{r+s_i}\right]\tilde{F}_{2i,\varepsilon}(\varepsilon)\\ &\qquad \text{for} \quad \theta_2 \le \theta < \infty.\end{aligned}$$

Therefore, using estimate (4.24) and the bounds $(r+1)s_i/(r+s_i) < 2$, $s_i < s$,

$$r\int_{\theta_1}^{\theta_2} \xi^{r-1}\varphi_i(\xi)d\xi \le (\theta_2^r - \theta_1^r)\theta_2^{s_i},$$

it follows that

$$\begin{aligned}&(r+1)\int_{\Omega^t} \left[\theta^r F_{1,\varepsilon}(\varepsilon,\theta) - r\hat{F}_{1,\varepsilon}(\varepsilon,\theta;r)\right] : \varepsilon_t' dx dt'\\ &\le c(T)\int_{\Omega^T} \theta^{r+s}|\varepsilon_t| dx dt + c^r \equiv R + c^r.\end{aligned} \tag{5.11}$$

Next, by means of Hölder inequality,

$$R \le c(T)\|\theta\|^{r+s}_{L_{\frac{(n+2)(r+s)}{n}}(\Omega^T)} \|\varepsilon_t\|_{L_{\frac{n+2}{2}}(\Omega^T)}.$$

By virtue of (4.35), since

$$\frac{n+2}{2} < \frac{q_n(n+2)}{n} < \frac{q_n(n+2)}{ns},$$

it follows that $\|\varepsilon_t\|_{L_{(n+2)/2}(\Omega^T)} \le c(T)$. Hence, applying Young's inequality

$$ab \le \frac{\varepsilon_0^{\lambda_1}}{\lambda_1}a^{\lambda_1} + \frac{1}{\lambda_2 \varepsilon_0^{\lambda_2}} b^{\lambda_2} \tag{5.12}$$

with $\varepsilon_0 = (\frac{\varepsilon}{r+1})^{1/\lambda_1}$, $\varepsilon = \text{const} > 0$, $\lambda_1 = \frac{r+1}{r+s}$, $\lambda_2 = \frac{r+1}{1-s}$, we obtain

$$\begin{aligned}R &\le c(T)\left(\frac{r+s}{r+1}\right)\frac{\varepsilon}{r+1}\|\theta\|^{r+1}_{L_{\frac{(n+2)(r+s)}{n}}(\Omega^T)} + c(T)\left(\frac{1-s}{r+1}\right)\left(\frac{r+1}{\varepsilon}\right)^{\frac{r+s}{1-s}}\\ &\le c(T)\frac{\varepsilon}{r+1}\|\theta\|^{r+1}_{L_{\frac{(n+2)(r+1)}{n}}(\Omega^T)} + c(T)\left(\frac{r+1}{\varepsilon}\right)^{\frac{r+s}{1-s}}.\end{aligned}$$

Consequently, in view of (5.10), the ε-term in the above inequality can be absorbed by the left-hand side of (5.8).

For the second integral on the right-hand side of (5.8) we get

$$\nu \int_{\Omega^t} \theta^r (A\varepsilon_{t'}) : \varepsilon_{t'} dx dt' \le c \|\theta\|^r_{L_{\frac{(n+2)r}{n}}(\Omega^T)} \|\varepsilon_t\|^2_{L_{n+2}(\Omega^T)} \le c(T) \|\theta\|^r_{L_{\frac{(n+2)r}{n}}(\Omega^T)}$$

$$\le c(T)\left(\frac{r}{r+1}\right)\frac{\varepsilon}{r+1}\|\theta\|^{r+1}_{L_{\frac{(n+2)r}{n}}(\Omega^T)} + c(T)\left(\frac{1}{r+1}\right)\left(\frac{r+1}{\varepsilon}\right)^r$$

$$\le c(T)\frac{\varepsilon}{r+1}\|\theta\|^{r+1}_{L_{\frac{(n+2)(r+1)}{n}}(\Omega^T)} + c(T)\left(\frac{r+1}{\varepsilon}\right)^r,$$

where on the way we have used estimate (4.35) and inequality (5.12) with $\lambda_1 = (r+1)/r$, $\lambda_2 = r+1$, $\varepsilon_0 = (\varepsilon/(r+1))^{\frac{1}{\lambda_1}}$. Again, the first-term in the above inequality can be absorbed by the left-hand side of (5.8).

The third integral on the right-hand side of (5.8) is treated similarly. Namely,

$$\int_{\Omega^t} \theta^r g dx dt' \le \|\theta\|^r_{L_{\frac{(n+2)r}{n}}(\Omega^T)} \|g\|_{L_{\frac{n+2}{2}}(\Omega^T)} \le c\frac{\varepsilon}{r+1}\|\theta\|^{r+1}_{L_{\frac{(n+2)(r+1)}{n}}(\Omega^T)} + c\left(\frac{r+1}{\varepsilon}\right)^r.$$

Combining the above estimates it follows that

$$\frac{c}{r+1}\|\theta\|^{r+1}_{L_{\frac{(n+2)(r+1)}{n}}(\Omega^T)} \le c(T)c^{\frac{r+1}{1-\epsilon}}(r+1)^{\frac{r+1}{1-\epsilon}}$$

what provides the assertion. □

We indicate the consequences of the above lemma concerning regularity of ε. Recalling estimate (4.25), in view of (5.1), it follows that

$$\|\varepsilon\|_{W^{2,1}_p(\Omega^T)} \le c(T,p) \quad \text{for} \quad 1 < p < \infty, \tag{5.13}$$

where $c(T,p) \to \infty$ as $p \to \infty$.

In the next lemma, with the help of estimate (4.40), we prove $L_\infty(\Omega^T)$ - norm bound on θ.

Lemma 5.2. *Let function $F_1(\varepsilon,\theta)$ be defined as in Example in Section 2. Moreover, suppose that*

$$\theta_0 \in L_\infty(\Omega), \quad g \in L_1(0,T;L_\infty(\Omega)), \quad \text{and} \quad \|\varepsilon_t\|_{L_2(0,T;L_\infty(\Omega))} \le c(T).$$

Then the following estimate holds

$$\begin{aligned} \|\theta\|_{L_\infty(\Omega^T)} &\le c\exp\left(c(T)T^{\frac{1}{2}}\|\varepsilon_t\|_{L_2(0,T;L_\infty(\Omega))}\right)\cdot \\ &\cdot\left(\|\varepsilon_t\|^2_{L_2(0,T;L_\infty(\Omega))} + \|g\|_{L_1(0,T;L_\infty(\Omega))} + \|\theta_0\|_{L_\infty(\Omega)}\right) \le c(T). \end{aligned} \tag{5.14}$$

Proof. Without loss of generality we assume that $\theta \geq \theta_2$ a.e. in Ω^T. We proceed as in Lemma 5.1 by multiplying $(1.2)_1$ by $\theta^r, r > 1$, and integrating over Ω. As a result we arrive at the identity (5.7). Now the idea is to find additional information from the term $G(\varepsilon, \theta; r)$. A direct calculation shows that

$$G(\varepsilon,\theta;r) = \theta^{r+1}\alpha(\varepsilon,\theta;r) \quad \text{for} \quad \theta_2 \leq \theta < \infty, \tag{5.15}$$

where

$$\alpha(\varepsilon,\theta;r) = \sum_{i=1}^{N}\left[\frac{s_i(1-s_i)}{(r+s_i)\theta^{1-s_i}} + \left(-r\theta_1^{r+1} - r(r+1)\int_{\theta_1}^{\theta_2}\zeta^{r-1}\varphi_i(\zeta)d\zeta + \frac{r(r+1)}{r+s_i}\theta_2^{r+s_i}\right)\frac{1}{\theta^{r+1}}\right]\tilde{F}_{2i}(\varepsilon).$$

We note that by virtue of (5.5), $\alpha(\varepsilon,\theta;r) \geq 0$ for all arguments.
Then, recalling estimate (5.11), we conclude from identity (5.7) that

$$\begin{aligned} &\frac{c_v}{r+1}\frac{d}{dt}\int_\Omega \theta^{r+1}\left(1 + \frac{r+1}{c_v}\alpha(\varepsilon,\theta;r)\right)dx \\ &\leq c(T)\int_\Omega \theta^{r+s}|\varepsilon_t|dx + c\int_\Omega \theta^r|\varepsilon_t|^2dx + \int_\Omega \theta^r|g|dx. \end{aligned} \tag{5.16}$$

Let us introduce now the new function

$$\theta' \equiv \theta\left(1 + \frac{r+1}{c_v}\alpha(\varepsilon,\theta;r)\right)^{\frac{1}{r+1}}. \tag{5.17}$$

We note that $\theta' \geq \theta$. We can also see that

$$\theta' \to \bar{\alpha}\theta \quad \text{as} \quad r \to \infty, \tag{5.18}$$

where $\bar{\alpha}$ is some constant from the interval $[1,2]$. The latter convergence follows from the following estimates

$$\left(1 + \frac{r+1}{c_v}\alpha(\varepsilon,\theta;r)\right)^{\frac{1}{r+1}} \leq 1 + \left(\frac{\alpha(\varepsilon,\theta;r)}{c_v}\right)^{\frac{1}{r+1}}(r+1)^{\frac{1}{r+1}}$$

and

$$\left(\frac{\alpha(\varepsilon,\theta;r)}{c_v}\right)^{\frac{1}{r+1}} \leq \left[c\left(\frac{1}{r} + r + 1\right)\right]^{\frac{1}{r+1}} \leq c^{\frac{1}{r+1}}\left[\left(\frac{1}{r}\right)^{\frac{1}{r+1}} + (r+1)^{\frac{1}{r+1}}\right],$$

which in view of the convergences

$$\left(\frac{1}{r}\right)^{\frac{1}{r+1}} \to 1, \qquad (r+1)^{\frac{1}{r+1}} \to 1,$$

imply (5.18). Then, with the help of Hölder inequality, (5.16) yields

$$\frac{c_v}{r+1}\frac{d}{dt}\int_\Omega (\theta')^{r+1}dx \le c(T)\|\theta\|^{r+s}_{L_{r+1}(\Omega)}\|\varepsilon_t\|_{L_{\frac{r+1}{1-s}}(\Omega)}$$
$$+ c\|\theta\|^r_{L_{r+1}(\Omega)}\|\varepsilon_t\|^2_{L_{2(r+1)}(\Omega)} + \|\theta\|^r_{L_{r+1}(\Omega)}\|g\|_{L_{r+1}(\Omega)}.$$

Therefore, in view of the equality

$$\frac{c_v}{r+1}\frac{d}{dt}\int_\Omega (\theta')^{r+1}dx = c_v\|\theta'\|^r_{L_{r+1}(\Omega)}\frac{d}{dt}\|\theta'\|_{L_{r+1}(\Omega)},$$

taking into account that $\theta' \ge \theta$ and $s < 1$, it follows that

$$c_v\frac{d}{dt}\|\theta'\|_{L_{r+1}(\Omega)} \le c(T)\|\theta'\|_{L_{r+1}(\Omega)}\|\varepsilon_t\|_{L_{\frac{r+1}{1-s}}(\Omega)} + c\|\varepsilon_t\|^2_{L_{2(r+1)}(\Omega)} + \|g\|_{L_{r+1}(\Omega)}. \tag{5.19}$$

Now we apply a standard procedure by multiplying both sides of (5.19) by $\exp\left(-c(T)\int_0^t \|\varepsilon_{t'}(t')\|_{L_{\frac{r+1}{1-s}}(\Omega)}dt'\right)$. This leads to the inequality

$$\frac{d}{dt}\left[c_v\|\theta'\|_{L_{r+1}(\Omega)}\exp\left(-c(T)\int_0^t \|\varepsilon_{t'}(t')\|_{L_{\frac{r+1}{1-s}}(\Omega)}dt'\right)\right]$$
$$\le c\left(\|\varepsilon_t\|^2_{L_{2(r+1)}(\Omega)} + \|g\|_{L_{r+1}(\Omega)}\right)\exp\left(-c(T)\int_0^t \|\varepsilon_{t'}(t')\|_{L_{\frac{r+1}{1-s}}(\Omega)}dt'\right),$$

which in turn implies that

$$\begin{aligned}\|\theta'(t)\|_{L_{r+1}(\Omega)} &\le c\exp\left[c(T)t^{\frac12}\left(\int_0^t \|\varepsilon_{t'}(t')\|^2_{L_{\frac{r+1}{1-s}}(\Omega)}dt'\right)^{\frac12}\right]\cdot \\ &\cdot\left[\int_0^t\left(\|\varepsilon_{t'}(t')\|^2_{L_{2(r+1)}(\Omega)} + \|g(t')\|_{L_{r+1}\Omega)}\right)dt' + \|\theta'(0)\|_{L_{r+1}(\Omega)}\right]\end{aligned} \tag{5.20}$$

for all $t \in [0,T]$ and $r > 1$. In view of (5.18) and the $L_2(0,T;L_\infty(\Omega))$-norm estimate on ε_t, we can pass in (5.20) to the limit with $r \to \infty$ to conclude the assertion. □

6. Hölder continuity of temperature. Completing the proof of Theorem 2.1

In this section we shall apply the method of [5], Chap.II.7, to prove that θ is Hölder continuous. The essential for the procedure is $L_\infty(\Omega^T)$-norm estimate on θ, proved in Lemma 5.2, and $L_p(\Omega^T)$-norm estimate (5.13) on ε_t. For reader's convenience we record here the definition of the space $\mathcal{B}_2(\Omega^T, M, \gamma, r, \delta, \varkappa)$ (see [5], Chap. II. 7).
The function $u \in \mathcal{B}_2(\Omega^T, M, \gamma, r, \delta, \varkappa)$, where $\Omega^T = \Omega \times (0,T)$, and $M, \gamma, r, \delta, \varkappa$ are positive numbers, if the following conditions are satisfied:
(i) $u \in V_2^{1,0}(\Omega^T) \equiv C(0,T; L_2(\Omega)) \cap L_2(0,T; W_2^1(\Omega))$,
(ii) $\operatorname{ess\,sup}_{\Omega^T} |u| \le M$,
(iii) the function $w(x,t) = \pm u(x,t)$ satisfies the following inequalities

$$\max_{t_0 \le t \le t_0+\tau} \|(w-k)_+\|^2_{L_2(B_{\rho-\sigma_1\rho}(x_0))} \le \|(w-k)_+(\cdot,t_0)\|^2_{L_2(B_\rho(x_0))}$$
$$+\gamma\left[(\sigma_1\rho)^{-2}\|(w-k)_+\|^2_{L_2(Q(\rho,\tau))} + \mu^{\frac{2}{r}(1+\varkappa)}(k,\rho,\tau)\right],$$

and

$$\|(w-k)_+\|^2_{V_2(Q(\rho-\sigma_1\rho,\tau-\sigma_2\tau))}$$
$$\le \gamma\left\{\left[(\sigma_1\rho)^{-2} + (\sigma_2\tau)^{-1}\right]\|(w-k)_+\|^2_{L_2(Q(\varrho,\tau))} + \mu^{\frac{2}{r}(1+\varkappa)}(k,\varrho,\tau)\right\}.$$

Here the notation is as follows:

$(w-k)_+ = \max\{w-k, 0\}$ – the truncation of w,
$B_\varrho(x_0) = \{x \in \Omega |\ |x - x_0| < \varrho\}$ – a ball in Ω,
$Q(\varrho,\tau) = B_\varrho(x_0) \times (t_0, t_0+\tau) = \{(x,t) \in \Omega^T |\ \ |x-x_0| < \varrho,\ t_0 < t < t_0+\tau\}$ –

a cylinder in Ω^T, where ϱ, τ are arbitray positive numbers, σ_1, σ_2 are arbitrary numbers from the interval (0,1), k is an arbitrary number satisfying condition

$$\operatorname{ess\,sup}_{Q(\varrho,\tau)} w(x,t) - k \le \delta.$$

Moreover,

$$A_{k,\varrho}(t) = \{x \in B_\varrho(x_0) |\ w(x,t) > k\},$$

$$\mu(k,\sigma,\tau) = \int_{t_0}^{t_0+\tau} meas^{\frac{r}{q}} A_{k,\varrho}(t)dt,$$

where positive numbers q and r are linked by the relation

$$\frac{1}{r} + \frac{n}{2q} = \frac{n}{4},$$

with the admissible ranges

$$q \in (2, \frac{2n}{n-2}], \quad r \in [2,\infty) \quad \text{for } n \ge 3,$$
$$q \in (2,\infty), \qquad r \in (2,\infty) \quad \text{for } n = 2,$$
$$q \in (2,\infty], \qquad r \in [4,\infty) \quad \text{for } n = 1.$$

Furthermore,

$$V_2(\Omega^T) \equiv L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;W_2^1(\Omega)).$$

We have the following

Lemma 6.1. *Suppose that*

$$|\varepsilon| \le c(T) \quad \text{in } \Omega^T, \quad \|\varepsilon_t\|_{L_p(\Omega^T)} \le c(T),\ 1<p<\infty,$$
$$\|\theta\|_{L_\infty(\Omega^T)} \le M = c(T),$$
$$|c_0(\varepsilon,\theta)| + |c_{0,\varepsilon}(\varepsilon,\theta)| + |c_{0,\theta}(\varepsilon,\theta)| \le c(T) \quad \text{in} \quad \Omega^T,$$
$$g \in L_p(\Omega^T), \quad 1<p<\infty, \quad \theta_0 \in C^{\alpha_0}(\Omega), \quad 1<\alpha_0<1.$$

Furthermore, let k be a positive number such that

$$k > \sup_\Omega \theta_0(\boldsymbol{x}), \tag{6.1}$$

and

$$M-k<\delta \quad \text{with some} \quad \delta>0.$$

Then

$$\theta \in \mathcal{B}_2(\Omega^T, M, \gamma, r, \delta, \varkappa), \tag{6.2}$$

where

$$r = q = \frac{2(n+2)}{n}, \quad \varkappa \in (0,\frac{2}{n}), \quad \gamma = c(T).$$

Proof. We check that θ satisfies conditions (i)–(iii) in the definition of the space $\mathcal{B}_2(\Omega^T, M, \gamma, r, \delta, \varkappa)$. By virtue of (4.33), $\theta \in W_2^{2,1}(\Omega^T)$. Hence, by the imbedding theorem (see [5], Lemma II.3.4),

$$\theta \in C(0,T;W_q^{2-2/q}(\Omega)), \quad 1<q\le 2,$$

so condition (i) is clearly satisfied.
Furthermore, thanks to Lemma 5.2, condition (ii) is also satisfied with constant $M=c(T)$.

We proceed now to check that θ satisfies the second inequality in condition (iii). To this end, first we recall that by Lemma 4.1, $\theta(\boldsymbol{x},t)>0$ in Ω^T.
Let $Q(\varrho,\tau) = B_\varrho(\boldsymbol{x}_0)\times(t_0,t_0+\tau)$ be an arbitrary cylinder in Ω^T, and $\zeta(\boldsymbol{x},t)$ be a smooth function such that supp $\zeta(\boldsymbol{x},t)\subset Q(\varrho,\tau)$ and $\zeta(\boldsymbol{x},t)=1$ for $(\boldsymbol{x},t)\in Q(\varrho-\sigma_1\varrho,\tau-\sigma_2\tau)$, where $\sigma_1,\sigma_2\in(0,1)$. Moreover, let

$$A_{k,\varrho}(t) = \{\boldsymbol{x}\in B_\varrho(\boldsymbol{x}_0)|\ \theta(\boldsymbol{x},t)>k\}.$$

Multiplying equation $(1.2)_1$ by $\zeta^2(\theta-k)_+$ and integrating over Ω we get

$$\begin{aligned}&\frac{1}{2}\int_\Omega c_0\zeta^2\frac{d}{dt}(\theta-k)_+^2dx + k_0\int_\Omega|\nabla(\theta-k)_+|^2\zeta^2dx\\&+2k_0\int_\Omega\zeta(\theta-k)_+\nabla(\theta-k)_+\cdot\nabla\zeta dx = \int_\Omega f\zeta^2(\theta-k)_+dx,\end{aligned} \tag{6.3}$$

where for simplicity we have denoted the right-hand side of $(1.2)_1$ by f,

$$f \equiv \theta F_{,\theta\varepsilon}(\varepsilon,\theta) : \varepsilon_t + \nu(A\varepsilon_t) : \varepsilon_t + g.$$

The first term on the left-hand side of (6.3) is rearranged as

$$\begin{aligned}
\frac{1}{2}\int_\Omega c_0\zeta^2 \frac{d}{dt}(\theta-k)_+^2 dx &= \frac{1}{2}\frac{d}{dt}\int_\Omega c_0(\theta-k)_+^2\zeta^2 dx \\
&- \frac{1}{2}\int_{A_{k,\varrho}(t)} (c_{0,\varepsilon} : \varepsilon_t)(\theta-k)_+^2\zeta^2 dx - \frac{1}{2}\int_{A_{k,\varrho}(t)} c_{0,\theta}\theta_t(\theta-k)_+^2\zeta^2 dx \\
&- \int_{A_{k,\varrho}(t)} c_0(\theta-k)_+^2\zeta\zeta_t dx.
\end{aligned} \tag{6.4}$$

The third integral on the right-hand side of the above inequality requires special technical treatment because of the presence of θ_t. We rearrange it in a similar way as in Lemma 4.3. To this end we first observe that on the set $A_{k,\varrho}(t)$ it holds

$$c_0(\varepsilon,\theta) = c_0(\varepsilon,(\theta-k)_+ + k),$$

so

$$c_{0,\theta}(\varepsilon,\theta) = c_{0,(\theta-k)_+}(\varepsilon,(\theta-k)_+ + k) \quad \text{on } A_{k,\varrho}(t).$$

Now, restricting considerations to the set $A_{k,\varrho}(t)$, we define the function

$$G(\varepsilon,(\theta-k)_+) = \int_0^{(\theta-k)_+} c_{0,\xi}(\varepsilon,\xi+k)\xi^2 d\xi. \tag{6.5}$$

Clearly, it satisfies the conditions

$$\begin{aligned}
&G(\varepsilon,0) = 0,\\
&G_{,(\theta-k)_+}(\varepsilon,(\theta-k)_+) = c_{0,(\theta-k)_+}(\varepsilon,(\theta-k)_+ + k)(\theta-k)_+^2.
\end{aligned} \tag{6.6}$$

Then the third mentioned above integral transforms as follows (further on for simplicity we omit functions arguments):

$$\begin{aligned}
-\frac{1}{2}\int_{A_{k,\varrho}(t)} c_{0,\theta}\theta_t(\theta-k)_+^2\zeta^2 dx &= -\frac{1}{2}\int_{A_{k,\varrho}(t)} c_{0,(\theta-k)_+}(\theta-k)_+^2\frac{d}{dt}(\theta-k)_+\zeta^2 dx \\
&= -\frac{1}{2}\int_{A_{k,\varrho}(t)} G_{,(\theta-k)_+}\frac{d}{dt}(\theta-k)_+\zeta^2 dx \\
&= -\frac{1}{2}\int_{A_{k,\varrho}(t)} \left(\frac{d}{dt}G\right)\zeta^2 dx + \frac{1}{2}\int_{A_{k,\varrho}(t)} (G_{,\varepsilon} : \varepsilon_t)\zeta^2 dx.
\end{aligned} \tag{6.7}$$

Setting

$$G_+ \equiv \begin{cases} G(\varepsilon, (\theta - k)_+) & \text{for} \quad \theta > k, \\ 0 & \text{for} \quad \theta \le k, \end{cases}$$

we rewrite the first integral in the last equality as

$$\tag{6.8} \begin{aligned} &-\frac{1}{2} \int\limits_{A_{k,\varrho}(t)} \left(\frac{d}{dt} G\right) \zeta^2 dx \\ &= -\frac{1}{2} \int\limits_{\Omega} \left(\frac{d}{dt} G_+\right) \zeta^2 dx = -\frac{1}{2} \frac{d}{dt} \int\limits_{\Omega} G_+ \zeta^2 dx + \int\limits_{\Omega} G_+ \zeta \zeta_t dx. \end{aligned}$$

Summarizing, in view of (6.4), (6.7) and (6.8), identity (6.3) takes the form

$$\tag{6.9} \begin{aligned} &\frac{1}{2} \frac{d}{dt} \int\limits_{\Omega} c_0 (\theta - k)_+^2 \zeta^2 dx + k_0 \int\limits_{\Omega} |\nabla (\theta - k)_+|^2 \zeta^2 dx \\ &= \frac{1}{2} \frac{d}{dt} \int\limits_{\Omega} G_+ \zeta^2 dx - \int\limits_{\Omega} G_+ \zeta \zeta_t dx - \frac{1}{2} \int\limits_{A_{k,\varrho}(t)} (G_{,\varepsilon} : \varepsilon_t) \zeta^2 dx \\ &+ \int\limits_{A_{k,\varrho}(t)} c_0 (\theta - k)_+^2 \zeta \zeta_t dx + \frac{1}{2} \int\limits_{A_{k,\varrho}(t)} (c_{0,\varepsilon} : \varepsilon_t)(\theta - k)_+^2 \zeta^2 dx \\ &- 2k_0 \int\limits_{A_{k,\varrho}(t)} \zeta (\theta - k)_+ \nabla (\theta - k)_+ \cdot \nabla \zeta dx + \int\limits_{A_{k,\varrho}(t)} f \zeta^2 (\theta - k)_+ dx. \end{aligned}$$

Integrating (6.9) with respect to t, and taking into account that $(\theta_0 - k)_+ = 0$, and $G(\varepsilon_0, (\theta_0 - k)_+) = 0$, we obtain

$$\tag{6.10} \begin{aligned} &\frac{c_v}{2} \int\limits_{\Omega} (\theta - k)_+^2 \zeta^2 dx + k_0 \int\limits_{\Omega^t} |\nabla (\theta - k)_+|^2 \zeta^2 dx dt' \\ &\le c \Bigg[\int\limits_{\Omega} |G_+| \zeta^2 dx + \int\limits_{\Omega^t} |G_+| |\zeta_{t'}| dx dt' + \int\limits_{\Omega^t} |G_{+,\varepsilon}| \; |\varepsilon'_t| \zeta^2 dx dt' \\ &+ \int\limits_{\Omega^t} |c_0| (\theta - k)_+^2 \; |\zeta_{t'}| dx dt' + \int\limits_{\Omega^t} |c_{0,\varepsilon}| \; |\varepsilon_{t'}| (\theta - k)_+^2 \zeta^2 dx dt' \\ &+ \int\limits_{\Omega^t} |f| (\theta - k)_+ |\zeta|^2 dx dt' + \int\limits_{\Omega^t} (\theta - k)_+ |\nabla (\theta - k)_+| \; |\zeta| \; |\nabla \zeta| dx dt' \Bigg]. \end{aligned}$$

Now we observe that owing to the boundedness of functions $c_{0,\theta}, c_{0,\theta\varepsilon}$, it follows that

$$\tag{6.11} |G(\varepsilon, (\theta - k)_+)| + |G_{,\varepsilon} (\varepsilon, (\theta - k)_+)| \le c(\theta - k)_+^3.$$

Moreover, by the assumption on k,

$$|G(\varepsilon,(\theta-k)_+)| \le c\delta(\theta-k)_+^2. \tag{6.12}$$

Therefore, choosing δ appropriately, the first integral on the right-hand side of (6.10) can be absorbed by the left-hand side. The last integral on the right-hand side of (6.10) is estimated by use of the Young inequality as follows

$$\begin{aligned}&\int_{\Omega^t}(\theta-k)_+\,|\nabla((\theta-k)_+|\,|\zeta|\,|\nabla\zeta|dxdt'\\ &\le \frac{k_0}{2}\int_{\Omega^t}|\nabla(\theta-k)_+|^2\zeta^2dxdt'+\frac{1}{2k_0}\int_{\Omega^t}(\theta-k)_+^2|\nabla\zeta|^2dxdt',\end{aligned} \tag{6.13}$$

so the first integral on the right-hand side of above inequality is absorbed by the left-hand side of (6.10). Combining (6.11)–(6.13) in (6.10) we arrive at

$$\begin{aligned}&\int_{\Omega}(\theta-k)_+^2\zeta^2dx+\int_{\Omega^t}|\nabla(\theta-k)_+|^2\zeta^2dxdt'\\ &\le c\Bigg[\int_{\Omega^t}(\theta-k)_+^2(\zeta^2+|\nabla\zeta|^2+|\zeta_{t'}|)dxdt'\\ &+\int_{\Omega^t}(|\varepsilon_{t'}|(\theta-k)_+^2+|f|(\theta-k)_+)\zeta^2dxdt'\Bigg]\equiv I_1+I_2.\end{aligned} \tag{6.14}$$

Clearly, the integral I_1 is estimated by

$$I_1\le c\left[(\sigma_1\varrho)^{-2}+(\sigma_2\tau)^{-1}\right]\int_{Q(\varrho,\tau)}(\theta-k)_+^2dxdt'.$$

For the integral I_2, using the boundedness of θ and applying Hölder inequality, we obtain

$$\begin{aligned}I_2&\le c\int_{t_0}^{t_0+\tau}\int_{A_{k,\varrho}(t')}(|\varepsilon_{t'}|+|f|)\zeta^2dxdt'\\ &\le c\left(\int_{t_0}^{t_0+\tau}\int_{A_{k,\varrho}(t')}(|\varepsilon_{t'}|^{\lambda_1}+|f|^{\lambda_1})dxdt'\right)^{\frac{1}{\lambda_1}}\left(\int_{t_0}^{t_0+\tau}meas A_{k,\varrho}(t')dt'\right)^{\frac{1}{\lambda_2}},\end{aligned}$$

with $1/\lambda_1 + 1/\lambda_2 = 1$. We set now

$$\frac{1}{\lambda_2} = \frac{2}{r}(1+\varkappa) \quad \text{and} \quad r = q = \frac{2(n+2)}{n},$$

where $\varkappa$ is arbitrary number from interval $(0, \frac{2}{n})$. Then

$$\lambda_1 = \frac{n+2}{2-n\varkappa}, \quad \lambda_2 = \frac{(n+2)}{n(1+\varkappa)} \in (1,\infty).$$

Clearly, r, q satisfy conditions in definition of $\mathcal{B}_2(\Omega^T, M, \gamma, r, \delta, \varkappa)$. Consequently,

$$I_2 \le \left(\|\varepsilon_t\|_{L_{\lambda_1}(\Omega^T)} + \|f\|_{L_{\lambda_1}(\Omega^T)} \right) \mu^{\frac{1}{\lambda_2}}(k,\varrho,\tau).$$

Taking into account that by assumptions

$$\|f\|_{L_{\lambda_1}(\Omega^T)} \le c \left(\|\varepsilon_t\|_{L_{\lambda_1}(\Omega^T)} + \|\varepsilon_t\|^2_{L_{2\lambda_1}(\Omega^T)} + \|g\|_{L_{\lambda_1}(\Omega^T)} \right) \le c(T),$$

it follows that

$$I_2 \le c(T)\mu^{\frac{2}{r}(1+\varkappa)}(k,\varrho,\tau).$$

Combining estimates on I_1, I_2 in (6.14) leads to

$$\begin{aligned} &\|(\theta-k)_+\|^2_{V_2(Q(\varrho-\sigma_1\varrho,\tau-\sigma_2\tau))} \\ &= \operatorname{ess\,sup}_{t\in[0,T]} \int_\Omega (\theta-k)_+^2 \zeta^2 dx + \int_{\Omega^T} |\nabla(\theta-k)_+|^2 \zeta^2 dx dt \\ &\le c(T) \left\{ \left[(\sigma_1\varrho)^{-2} + (\sigma_2\tau)^{-1} \right] \|(\theta-k)_+\|^2_{L_2(Q(\varrho,\tau))} + \mu^{\frac{2}{r}(1+\varkappa)}(k,\varrho,\tau) \right\}. \end{aligned}$$

Since $\theta > 0$, this shows that the second inequality in condition (iii) is satisfied with constant $\gamma = c(T)$.

The first inequality in (iii) can be proved by multiplying $(1.2)_1$ by $\zeta_0^2(\theta-k)_+$, where $\zeta_0(\boldsymbol{x})$ is a smooth function such that supp $\zeta_0(\boldsymbol{x}) \subset B_\varrho(\boldsymbol{x}_0)$, $\zeta_0(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in B_{\varrho-\sigma_1\varrho}(\boldsymbol{x}_0)$, and next integrating over $\Omega \times (t_0, t_0+\tau)$. In this case, repeating the above arguments, inequality (6.14) is replaced by

$$\begin{aligned} &\int_\Omega (\theta-k)_+^2 \zeta_0^2 dx + \int_{Q(\varrho,\tau)} |\nabla(\theta-k)_+|^2 \zeta_0^2 dx dt \\ &\le c \Bigg[\int_{B_\sigma(\boldsymbol{x}_0)} (\theta(t_0)-k)_+^2 \zeta_0^2 dx + \int_{Q(\varrho,\tau)} (\theta-k)_+^2 (\zeta_0^2 + |\nabla\zeta_0|^2) dx dt \\ &+ \int_{Q(\varrho,\tau)} (|\varepsilon_t|(\theta-k)_+^2 |f|(\theta-k)_+) \zeta_0^2 dx dt \Bigg]. \end{aligned} \tag{6.15}$$

Since the last two integrals on the right-hand side of (6.15) are estimated as above, this leads to the required inequality. The proof is completed. □

By virtue of (6.2) we can apply the imbedding result of [5], Theorem II.7.1, to conclude that θ is Hölder continuous in Ω^T, and

$$\|\theta\|_{C^{\alpha,\alpha/2}(\Omega^T)} \le c(T), \tag{6.16}$$

with Hölder exponent $0 < \alpha < 1$ depending on $M = c(T), \gamma = c(T), r, \delta$ and $\varkappa$.
Thanks to Hölder continuity of ε and θ, in view of a priori bounds (4.36), (5.13), we can obtain the final estimates for a solution $(\boldsymbol{u}, \theta)$ to problem (1.1), (1.2) in $\boldsymbol{V}(p, q)$ - norm and thereby complete the proof of Theorem 2.1.

Lemma 6.2. *Suppose that ε and θ are Hölder continuous in Ω^T, and*

$$\begin{aligned} &|\varepsilon| + |\theta| \le c(T) \quad \text{in} \quad \Omega^T, \\ &\|\nabla\varepsilon\|_{L_\sigma(\Omega^T)} + \|\varepsilon_t\|_{L_\sigma(\Omega^T)} \le c(T) \quad \text{for} \quad 1 < \sigma < \infty. \end{aligned}$$

Moreover, suppose that

$$\begin{aligned} &\boldsymbol{b} \in \boldsymbol{L}_p(\Omega^T), \quad g \in L_q(\Omega^T), \\ &\boldsymbol{u}_0 \in \boldsymbol{W}_p^{4-2/p}(\Omega), \quad \boldsymbol{u}_1 \in \boldsymbol{W}_p^{2-2/p}(\Omega), \quad \theta_0 \in W_q^{2-2/q}(\Omega), \quad n+2 < p, q < \infty, \end{aligned}$$

and compatibility conditions. Then

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_p^{4,2}(\Omega^T)} \le c(T), \quad n+2 < p < \infty, \tag{6.17}$$

$$\|\theta\|_{W_q^{2,1}(\Omega^T)} \le c(T), \quad n+2 < q < \infty, \tag{6.18}$$

Proof. Owing to the bound on ε_t, the right-hand side of equation $(1.2)_1$ is bounded in $L_q(\Omega^T)$ — norm for $1 < q < \infty$. Therefore in view of the Hölder continuity of the coefficient $c_0(\varepsilon, \theta)$, the classical parabolic theory [5] assures bound (6.18). Consequently, by virtue of imbeddings, (6.18) implies that

$$\|\nabla\theta\|_{\boldsymbol{W}_q^{1,1/2}(\Omega^T)} \le c(T),$$

so in case $q \ge n+2$,

$$\|\nabla\theta\|_{L_\sigma(\Omega^T)} \le c(T) \quad \text{for} \quad 1 < \sigma < \infty. \tag{6.19}$$

Hence, recalling estimate (4.37), it follows that

$$\|\nabla \cdot F_{,\varepsilon}(\varepsilon, \theta)\|_{L_p(\Omega^T)} \le c(T), \qquad 1 < p < \infty. \tag{6.20}$$

By the regularity theory of parabolic systems, (6.20) implies (6.17).
Finally, we observe that the bounds $n + 2 < p, q$, by virtue of imbeddings, are compatible with assumptions of the lemma. □

Lemma 6.2 completes the derivation of a priori bounds for a fixed point of the map $T(1, \cdot)$, and thereby proves property (iii) of the Leray-Schauder fixed point theorem.

Summarizing, we have shown that the solution map (3.4) satisfies assumptions (i)-(iv) of the Leray-Schauder theorem.
Thus, $T(1, \cdot)$ has at least one fixed point in $\boldsymbol{V}(p, q)$ which is equivalent to a solution $(\boldsymbol{u}, \theta) \in \boldsymbol{V}(p, q)$ to problem (1.1), (1.2). Now, in view of bounds (6.17), (6.18) and (4.1), the proof of Theorem 2.1 is completed.

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