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Unique global solvability in two-dimensional nonlinear thermoelasticity

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Abstract. The paper is concerned with initial-boundary value problem in two-dimensional nonlinear thermoelasticity which arises as a mathematical model of shape memory materials. The problem has the form of viscoelasticity system with capillarity coupled with heat conduction equation with mechanical dissipation.
The corresponding elastic energy is a nonconvex multiple-well function of strain, with the shape changing qualitatively with temperature. Under assumption on the growth of this energy with respect to temperature we prove global in time existence and uniqueness of solutions for large data. The existence proof is based on parabolic decomposition of the elasticity system and application of the Leray-Schauder fixed point theorem. The main part of the proof consists in deriving Hölder a priori estimates by succesive improvement of energy estimates.
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1. Introduction

The paper is concerned with the existence and uniqueness of global smooth solutions to the initial-boundary value problem for the equations of two-dimensional $(2-D)$ nonlinear thermoelasticity. The problem arises as a mathematical model of the dynamical behaviour of the body made of shape memory material (see [8], [9]). If has the form of viscoelasticity system with capillarity coupled with heat conduction equation with mechanical dissipation.

From physical point of view the characteristic feature of the model under consideraton is that it accounts for an internal, small scale structure of solid behaviour, namely martensitic microstructure of shape memory material. This feature is due to description by means of Ginzburg-Landau type free energy which introduces an energy penalty for the production of interfacial surface area and thereby influences the microstructure behaviour. In our case the energy penalty involves the first order strain gradient term which leads to the fourth order term, usually referred to as capillarity, in elasticity system. From mathematical point of view this term, in connection with mechanical viscosity, admits parabolic decomposition of the elasticity system, and consequently application of the parabolic theory in the existence proof. For mathematical reasons the analysis is restricted to $2-D$ case.

1.1. Problem

$$
\begin{aligned}
&\boldsymbol{u}_{tt} - \nu Q\boldsymbol{u}_t + \frac{\varkappa_0}{4}Q^2\boldsymbol{u} = \nabla\cdot F_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\theta) + \boldsymbol{b} && \text{in } \Omega^T = \Omega\times(0,T),\\
&\boldsymbol{u}\big|_{t=0} = \boldsymbol{u}_0,\quad \boldsymbol{u}_t\big|_{t=0} = \boldsymbol{u}_1 && \text{in } \Omega,\\
&\boldsymbol{u} = 0,\quad Q\boldsymbol{u} = 0 && \text{on } S^T = S\times(0,T),
\end{aligned}
\tag{1.1}
$$

$$
(1.2)\qquad \begin{aligned}
&c_0(\varepsilon,\theta)\theta_t - k_0\Delta\theta = \theta F_{,\theta\varepsilon}(\varepsilon,\theta):\varepsilon_t + \nu(A\varepsilon_t):\varepsilon_t + g && \text{in } \Omega^T,\\
&\theta\big|_{t=0} = \theta_0 && \text{in } \Omega,\\
&n\cdot\nabla\theta = 0 && \text{on } S^T,
\end{aligned}
$$

where

$$
(1.3)\qquad c_0(\varepsilon,\theta) = c_v - \theta F_{,\theta\theta}(\varepsilon,\theta).
$$

Here $\Omega\subset\mathbb{R}^n$, $n=2$, is a bounded domain with a smooth boundary S, representing the material points of a solid body with constant density ($\rho=1$).
The field $u:\Omega^T\to\mathbb{R}^2$ is the displacement and $\theta:\Omega^T\to\mathbb{R}_+$ is the absolute temperature.
The second order tensors

$$
\varepsilon = \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \quad\text{and}\quad \varepsilon_t = \varepsilon(u_t) = \frac{1}{2}(\nabla u_t + (\nabla u_t)^T)
$$

denote respectively the linearized strain and the strain rate.
The fourth order tensor $A=(A_{ijkl})$ represents the elasticity tensor given by

$$
(1.4)\qquad \varepsilon(u)\mapsto A\varepsilon(u) = \lambda tr\varepsilon(u)I + 2\mu\varepsilon(u),
$$

where $I=(\delta_{ij})$ is the unit matrix and λ,μ are Lamé constants specified in assumption (A2).
Moreover, Q stands for the second order differential operator of linearized elasticity defined by

$$
(1.5)\qquad u\mapsto Qu = \nabla\cdot(A\varepsilon(u)) = \mu\Delta u + (\lambda+\mu)\nabla(\nabla\cdot u).
$$

Correspondingly, the operator $Q^2=QQ$ is given by

$$
u\to Q^2u = \mu\Delta(Qu) + (\lambda+\mu)\nabla(\nabla\cdot(Qu)).
$$

The other quantities in (1.1), (1.2) have the following meaning: $F(\varepsilon,\theta)$ — the elastic energy, $c_0(\varepsilon,\theta)$ — the specific heat coefficient, b,g — external body forces and heat sources, $c_v, k_0, \nu, \varkappa_0$ — positive numbers representing respectively thermal specific heat, heat conductivity, viscosity and interfacial energy coefficient.

1.2. Physical principles

System (1.1), (1.2) expresses balance laws of linear momentum and energy

$$
(1.6)\qquad u_{tt} - \nabla\cdot\sigma = b,
$$

$$
(1.7)\qquad e_t + \nabla\cdot q - \sigma:\varepsilon_t = g,
$$

where σ, e, q are the stress tensor, the internal energy and the energy flux.
The constitutive equations for σ and q correspond to the Ginzburg-Landau functional with density given by

$$F_0(\varepsilon, \nabla\varepsilon, \theta) = -c_v\theta\log\theta + F(\varepsilon, \theta) + \frac{\varkappa_0}{8}|Qu|^2, \tag{1.8}$$

where the three terms represent respectively the thermal energy, the elastic energy and so-called capillary energy of the transition layers between different phases.
Typically for phase transitions, the graph of $F(\varepsilon, \theta)$ is a multiple-well in ε with the shape changing qualitatively with temperature θ. The representative of $F(\varepsilon, \theta)$ is the Falk-Konopka model (see references in [9]) in the form of sixth order polynomial in strain components, generalizing the well-known $1-D$ expression

$$F(\varepsilon, \theta) = \alpha_1(\theta - \theta_c)\varepsilon^2 - \alpha_2\varepsilon^4 + \alpha_3\varepsilon^6, \tag{1.9}$$

where $\alpha_i > 0$ are constant parameters and $\theta_c > 0$ is a critical temperature.
The strain gradient term in (1.8) accounts for nonlocal spatial effects which are of importance accross interfaces between different phases where large spatial variations of strain components appear.
The thermodynamically admissible constitutive equations for σ and q which lead to (1.1), (1.2), are given by (see [8], [9]):

$$\sigma = \frac{\delta F_0}{\delta\varepsilon} + \sigma^v, \qquad q = q_0 + p, \tag{1.10}$$

where $\frac{\delta F_0}{\delta\varepsilon}$ denotes the first variation of F_0 with respect to ε,

$$\frac{\delta F_0}{\delta\varepsilon} = F_{0,\varepsilon}(\varepsilon, \nabla\varepsilon, \theta) - \nabla\cdot F_{,\nabla\varepsilon}(\varepsilon, \nabla\varepsilon, \theta) = F_{,\varepsilon}(\varepsilon, \theta) - \frac{\varkappa_0}{4}A\varepsilon(Qu),$$

σ^v is the viscous stress by Hooke's law

$$\sigma^v = \nu A\varepsilon_t,$$

q_0 is the heat flux by Fourier's law

$$q_0 = -k_0\nabla\theta,$$

and p is a nonstationary flux associated with evolving nonzero-width phase interfaces, given by

$$p = -\varepsilon_t F_{0,\nabla\varepsilon}(\varepsilon, \nabla\varepsilon, \theta) = -\frac{\varkappa_0}{4}\varepsilon_t A(\nabla\varepsilon A) = -\frac{\varkappa_0}{4}\varepsilon_t(AQu).$$

According to thermodynamical relations, the internal energy e, the entropy η and the specific heat coefficient c_0 are linked to the free energy F_0 by

$$e = F_0 + \theta\eta, \qquad \eta = -F_{0,\theta}, \tag{1.11}$$

and

$$c_0 = e_{,\theta} = \theta\eta_{,\theta} = -\theta F_{0,\theta\theta}. \tag{1.12}$$

In case of free energy (1.8) this yields expresion (1.3).
We recall also that model (1.1), (1.2) is thermodynamically consistent in the sense that for solutions of (1.1), (1.2) the Clausius-Duhem inequality is satisfied (see [8]):

$$\eta_t + \nabla\cdot\left(\frac{q_0}{\theta}\right) = \varepsilon_t : \left(\frac{\sigma^v}{\theta}\right) + \nabla\left(\frac{1}{\theta}\right)\cdot q_0 + \frac{g}{\theta} \geq \frac{g}{\theta}. \tag{1.13}$$

1.3. Motivation of the paper. Basic ideas

Problem (1.1), (1.2) represents a multidimensional analog of the well-known $1-D$ Falk's model of phase transitions which has been the subject of intensive mathematical studies, see e.g. references in the monograph of Brokate and Sprekels [1], also [11].
In $3-D$ case problem (1.1), (1.2) has been considered in [9] under structural simplification of energy equation $(1.2)_1$. This simplification consisted in neglecting the nonlinear elastic contribution $-\theta F_{,\theta\theta}(\varepsilon,\theta)$ in the specific heat coefficient by setting

$$c_0(\varepsilon,\theta) = c_v = \text{const} > 0.$$

The main motivation of the present work was to eliminate the above structural assumption. For mathematical reasons related to imbeddings theorems we restrict here considerations to $2-D$ case.
Extension of results to $3-D$ case requires different approach and will be addressed in a separate paper.
The idea of the proof of existence of solutions in the present paper is similar to that in [9]. It is based on parabolic decomposition of $(1.1)_1$ and the application of the Leray-Schauder fixed point theorem. The elasticity system $(1.1)_1$ admits the decomposition into the following two parabolic systems:

$$\begin{aligned} & \boldsymbol{w}_t - \beta \boldsymbol{Q}\boldsymbol{w} = \nabla\cdot F_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\theta) + \boldsymbol{b} && \text{in } \Omega^T, \\ & \boldsymbol{w}\big|_{t=0} = \boldsymbol{u}_1 - \alpha \boldsymbol{Q}\boldsymbol{u}_0 && \text{in } \Omega, \\ & \boldsymbol{w} = 0 && \text{on } S^T, \end{aligned} \tag{1.14}$$

$$\begin{aligned} & \boldsymbol{u}_t - \alpha \boldsymbol{Q}\boldsymbol{u} = \boldsymbol{w} && \text{in } \Omega^T, \\ & \boldsymbol{u}\big|_{t=0} = \boldsymbol{u}_o && \text{in } \Omega, \\ & \boldsymbol{u} = 0 && \text{on } S^T, \end{aligned} \tag{1.15}$$

where α,β are numbers satisfying

$$\alpha + \beta = \nu, \qquad \alpha\beta = \frac{\varkappa_0}{4}. \tag{1.16}$$

Further on we assume the condition

$$0 < \sqrt{\varkappa_0} \le \nu \tag{1.17}$$

which assures that $\alpha, \beta \in \mathbb{R}_+$.

We point out that, similarly as in [9], we are unable to admit linear dependence of $F(\varepsilon, \theta)$ on θ, but assuming the growth

$$|F(\varepsilon,\theta)| \le c(1+\theta^s|\varepsilon|^{a_1}) \quad \text{with} \quad 0 < s < \frac{7}{8}, \quad 0 < a_1 < \infty,$$

we can, on the contrary to [9], to consider the arising nonlinearity in energy equation. This is possible thanks to proving Hölder bounds for temperature which are necessary to apply the classical parabolic theory. We point out also that in our case the conditon on s is less restrictive than the corresponding $s < \frac{1}{2}$ in [9].

1.4. Content of the paper

The assumptions and main theorems on existence and uniqueness of global in time solutions to problem (1.1), (1.2) are stated in Section 2.
The proof of a priori bounds for solutions to (1.1), (1.2) is, for clarity, partitioned into the distinct pieces contained in Sections 3 and 4.
Section 3 contains the proof of positivity of temperature, energy estimates and recursive improvement of energy estimates by application of regularity theory of parabolic systems.
Section 4 completes derivation of a priori estimates. It contains the proof of the key lemma providing in two-dimensional case Hölder bounds on temperature. Moreover, it contains additional a priori estimates obtained by application of theory of parabolic systems.
The proof of existence of solutions to (1.1), (1.2) is presented in Section 5. It consists in constructing a solution map and checking the assumptions of the Leray-Schauder fixed point theorem. The estimates derived in Section 3, 4 provide a priori bounds for a fixed point of the solution map. The proof extends the arguments of [9] to the case of system with non-constant coefficient $c_0(\varepsilon, \theta)$.
In Section 6 we present the proof of uniqueness of solutions to problem (1.1), (1.2) which re-establishes the arguments of [9].
In Appendix we collect the needed results on regularity of solutions to parabolic systems.

1.5. Notation

We denote

$$\begin{aligned}
&f_{,i} = \frac{\partial f}{\partial x_i}, \quad i = 1,\dots,n, \quad f_t = \frac{\partial f}{\partial t}, \quad \varepsilon = (\varepsilon_{ij})_{i,j=1,\dots,n},\\
&\varepsilon^a = (\varepsilon^a_{ij}), \quad i,j = 1,\dots,n,\\
&F_{,\varepsilon}(\varepsilon,\theta) = \frac{\partial F(\varepsilon,\theta)}{\partial \varepsilon} = \left(\frac{\partial F(\varepsilon,\theta)}{\partial \varepsilon_{ij}}\right)_{i,j=1,\dots,n},\\
&F_{,\theta}(\varepsilon,\theta) = \frac{\partial F(\varepsilon,\theta)}{\partial \theta}.
\end{aligned}$$

For simplicity, whenever there is no danger of confusion, we omit the arguments (ε, θ) of function $f(\varepsilon, \theta)$. Also specification of tensor indices is omitted. Vector - and tensor-valued mappings are denoted by bold letters.
The summation convention over repeated indices is used and the following notation: for vectors $\boldsymbol{a} = (a_i)$, $\tilde{\boldsymbol{a}} = (\tilde{a}_i)$ and tensors $\mathcal{B} = (B_{ij})$, $\tilde{\mathcal{B}} = (\tilde{B}_{ij})$, $\mathcal{C} = (C_{ijk})$, $\tilde{\mathcal{C}} = (\tilde{C}_{ijk})$ we write

$$\begin{aligned} &\boldsymbol{a} \cdot \tilde{\boldsymbol{a}} = a_i \tilde{a}_i, && \mathcal{B} : \tilde{\mathcal{B}} = B_{ij}\tilde{B}_{ij}, && \mathcal{C} : \tilde{\mathcal{C}} = C_{ijk}\tilde{C}_{ijk}, \\ &\boldsymbol{a}\mathcal{B} = (a_i B_{ij}), && \mathcal{B}\boldsymbol{a} = (B_{ij} a_j), && \boldsymbol{a}\mathcal{C} = (a_i C_{ijk}), \\ &\mathcal{C}\boldsymbol{a} = (C_{ijk} a_k), && \mathcal{B}\mathcal{C} = (B_{ij} C_{ijk}), && \mathcal{C}\mathcal{B} = (C_{ijk} B_{jk}), \\ &|\boldsymbol{a}| = (a_i a_i)^{1/2}, && |\mathcal{B}| = (B_{ij} B_{ij})^{1/2}, && |\mathcal{C}| = (C_{ijk} C_{ijk})^{1/2}, \end{aligned}$$

and analogously for higher order tensors.
∇ and $\nabla\cdot$ denote the gradient and the divergence operators. For $\boldsymbol{u}(x) = (u_i(x))$, $\nabla(\nabla \cdot \boldsymbol{u}) = \text{grad div}\boldsymbol{u}$, $\Delta \boldsymbol{u}$ denotes the component-wise Laplacian.
For the divergence of a tensor field $\boldsymbol{\varepsilon}(x) = (\varepsilon_{ij}(x))$ the convention of the contraction over the last index is used, i.e.,

$$\nabla \cdot \boldsymbol{\varepsilon}(x) = (\varepsilon_{ij,j}(x))_{i=1,\dots,n}.$$

We use Sobolev spaces notation of [5].
Throughout the paper c will denote a generic constant which, whenever not specified differently, depends on the data of problem (1.1), (1.2) and on T^a, $a \in \mathbb{R}_+$.

2. Statement of results

2.1. Assumptions

Throughout the paper we make the following assumptions:
(A1) Domain $\Omega \subset \mathbb{R}^2$ with the boundary S of class C^5. The C^5-regularity is needed in estimate (3.41).
(A2) Operator $\boldsymbol{Q}$. The Lamé coefficients satisfy

$$\mu > 0, \qquad n\lambda + 2\mu > 0 \quad (n = 2).$$

This assures the following properties:
(i) Coercivity and boundedness of $\boldsymbol{A}$, i.e.,

$$\underline{c}|\boldsymbol{\varepsilon}|^2 \le (\boldsymbol{A}\boldsymbol{\varepsilon}) : \boldsymbol{\varepsilon} \le \bar{c}|\boldsymbol{\varepsilon}|^2 \tag{2.1}$$

where $\underline{c} = 2\min\{\lambda + \mu, \mu\}$, $\bar{c} = 2\max\{\lambda + \mu, \mu\}$;
(ii) Strong ellipticity of $\boldsymbol{Q}$.
Thanks to this property and boundary condition $\boldsymbol{u} = 0$ on S, the result of Nečas [7] applies

$$\|\boldsymbol{u}\|_{W_2^2(\Omega)} \le c\|\boldsymbol{Q}\boldsymbol{u}\|_{L_2(\Omega)}. \tag{2.2}$$

(iii) Parabolicity in general (Solonnikov) sense of systems of type (1.14) (see [9], Sec. 7).
(A3) Elastic energy $F(\varepsilon,\theta) : \mathcal{S}^2 \times [0,\infty) \to \mathbb{R}$ is of class C^3, where $\mathcal{S}^2$ denotes the set of symmetric second order tnesors in $\mathbb{R}^2$. We assume that

$$F(\varepsilon,\theta) = F_1(\varepsilon,\theta) + F_2(\varepsilon)$$

with F_1, F_2 satisfying:
(A3-1) $F_1(\varepsilon,\cdot)$ is concave with respect to θ, i.e.,

$$F_{1,\theta\theta}(\varepsilon,\theta) \le 0 \quad \text{for} \quad (\varepsilon,\theta) \in \mathcal{S}^2 \times [0,\infty),$$

such that $F_1(\varepsilon,\cdot)$ is linear in θ over certain interval $[0,\theta_1)$, $\theta_1 = \text{const} > 0$, $F_1(\varepsilon,0) = 0$; for $\theta \ge \theta_1$ and $|\varepsilon_{ij}| \ge \varepsilon_1 = \text{const} > 0$ $(i,j = 1,2)$, $F_1(\varepsilon,\theta)$ is a polynomial satisfying

$$|F_1(\varepsilon,\theta)| \le c + c\theta^s|\varepsilon|^{a_1},$$

where c is a positive constant and

$$0 < s < s' < \frac{7}{8}, \qquad 0 < a_1 < \infty;$$

(A3-2) $0 \le F_2(\varepsilon)$ for $\varepsilon \in \mathcal{S}^2$,
and for $|\varepsilon_{ij}| \ge \varepsilon_1$, $F_1(\varepsilon)$ is a polynomial satisfying

$$|F_2(\varepsilon)| \le c + c|\varepsilon|^{a_2} \quad \text{with} \quad 0 < a_2 < \infty.$$

We note that (A3-1) implies the following bounds for the specific heat coefficient:

$$0 < c_v \le c_0(\varepsilon,\theta) \quad \text{for} \quad (\varepsilon,\theta) \in \mathcal{S}^2 \times [0,\infty), \tag{2.3}$$

$$\begin{aligned} &|c_0(\varepsilon,\theta)| + |c_{0,\theta}(\varepsilon,\theta)| \le c|\varepsilon|^{a_1}, \\ &|c_{0,\varepsilon}(\varepsilon,\theta)| \le c|\varepsilon|^{a_1-1} \quad \text{for} \quad (\varepsilon,\theta) \in \mathcal{S}^2 \times [0,\infty) \end{aligned} \tag{2.4}$$

with a constant c depending on θ_1 and s.
Further, it follows from (A3-1) that

$$E_1(\varepsilon,\theta) \equiv F_1(\varepsilon,\theta) - \theta F_{1,\theta}(\varepsilon,\theta) \ge 0 \quad \text{for} \quad (\varepsilon,\theta) \in \mathcal{S}^2 \times [0,\infty), \tag{2.5}$$

because

$$E_1(\varepsilon,0) = 0 \quad \text{and} \quad E_{1,\theta}(\varepsilon,\theta) = -\theta F_{1,\theta\theta}(\varepsilon,\theta) \ge 0.$$

We note also that (A3-2) and (2.5) imply

$$(F_1(\varepsilon,\theta) - \theta F_{1,\theta}(\varepsilon,\theta)) + F_2(\varepsilon) \ge 0 \quad \text{for} \quad (\varepsilon,\theta) \in \mathcal{S}^2 \times [0,\infty), \tag{2.6}$$

what means that the elastic part of internal energy is nonnegative.

(A4) Data. Source terms satisfy

$$b \in L_p(\Omega^T) \cap W_2^{1,1/2}(\Omega^T), \quad 4 < p < \infty,$$
$$g \in L_q(\Omega^T) \cap W_2^1(\Omega^T), \quad 4 < q < \infty, \quad \text{and} \quad g \geq 0 \ \text{ a.e. in } \Omega^T.$$

Initial data satisfy

$$u_0 \in W_p^{4-2/p}(\Omega) \cap W_2^4(\Omega), \quad u_1 \in W_p^{2-2/p}(\Omega) \cap W_2^2(\Omega), \quad 4 < p < \infty,$$
$$\theta_0 \in W_q^{2-2/q}(\Omega) \cap W_2^2(\Omega), \quad 4 < q < \infty, \quad \text{and} \quad \theta_* \equiv \min_{\Omega} \theta_0 > 0,$$
$$\theta_t\big|_{t=0} \in W_2^1(\Omega), \quad \text{where } \theta_t \text{ is calculated from equation}(1.2)_1.$$

Moreover, the initial data are supposed to satisfy compatibility conditions for the classical solvability of parabolic problems.

Later on we make use of the following implications of (A4):

$$\theta_0 \in C^{1,\alpha_0}(\Omega) \quad \text{with} \quad 0 < \alpha_0 < 1 - \frac{4}{q},$$

and

$$\varepsilon_0 \equiv \varepsilon(u_0) \in W_p^{3-2/p}(\Omega), \quad 4 < p < \infty,$$

so, by imbedding, $\varepsilon_0 \in C^{2,\alpha_0'}(\Omega)$, $0 < \alpha_0' < 1 - \frac{4}{p}$.

2.2. Statement of the main theorems

Theorem 2.1. *(Global existence) Let assumptions (A1)–(A4) be satisfied and the coefficients $\varkappa_0, \nu$ fulfil condition*

$$0 < \sqrt{\varkappa_0} \leq \nu.$$

Then for any $T > 0$ there exists a solution (u, θ) to problem (1.1), (1.2) in the space

$$V(p,q) \equiv \{(u,\theta); u \in W_p^{4,2}(\Omega^T), \ \theta \in W_q^{2,1}(\Omega^T), \ 4 < p \leq q < \infty\} \tag{2.7}$$

such that

$$\|u\|_{W_p^{4,2}(\Omega^T)} \leq c, \quad \|\theta\|_{W_q^{2,1}(\Omega^T)} \leq c, \tag{2.8}$$

with a constant c depending on the data and on T^a, $a \in \mathbb{R}_+$. Moreover, there exists a positive finite number ω such that

$$\theta \geq \theta_* \exp(-\omega t) \quad \text{in } \Omega^T. \tag{2.9}$$

Theorem 2.2. *(Uniqueness) Let the assumptions of Theorem 2.1 be satisfied and, in addition, suppose that*

$$F(\varepsilon,\theta) : S^2 \times [0,\infty) \ \text{ is of class } C^4 \ \text{ and } g \in L_\infty(\Omega^T). \tag{A5}$$

Then the solution (u, θ) to problem (1.1), (1.2) is unique.

The proofs of Theorem 2.1 and 2.2 appear in Section 5 and 6.

3. A priori estimates

In this and next section we derive a priori estimates for solutions of problem (1.1), (1.2).
Let us assume problem (1.1), (1.2) has a solution $(\boldsymbol{u},\theta)\in V(p,q)$ (see (2.7)). Our goal is to obtain a priori estimates of the form

$$\|\boldsymbol{u}\|_{W_p^{1,2}(\Omega^T)}\le c,\quad \|\theta\|_{W_q^{2,1}(\Omega^T)}\le c.$$

Such argumentation is related to finding a priori bounds for a fixed point of a solution map in applying Leray-Schauder fixed point theorem (see Section 5). The idea of the proof of a priori estimates consists in recursive improvement of energy estimates by using imbedding theorems and regularity theory of parabolic systems.

Before establishing energy estimates we have to prove that the absolute temperature is nonnegative. More precisely, for solutions in the class $V(p,q)$ we prove that temperature stays positive what is in accordance with the third law of thermodynamics.
Throughout we assume that (A1)–(A4) are satisfied.

Lemma 3.1. *Let*

$$\theta_*\equiv\min_{\Omega}\theta_0(x)>0,\quad g\ge 0\ \text{ in }\ \Omega^T,$$

and $(\boldsymbol{u},\theta)\in V(p,q)$ be a solution to problem (1.1), (1.2). Then there exists a positive finite number ω satisfying

$$[g+\nu(\boldsymbol{A}\boldsymbol{\varepsilon}_t):\boldsymbol{\varepsilon}_t]\exp(\omega t)+[\omega c_0(\boldsymbol{\varepsilon},\theta)+F_{,\theta\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\theta):\boldsymbol{\varepsilon}_t]\theta_*\ge 0\quad\text{in}\quad\Omega^T,\tag{3.1}$$

such that

$$\theta\ge\theta_*\exp(-\omega t)\quad\text{in}\quad\Omega^T.$$

Proof. Introducing the new function

$$\tilde\theta=\theta\exp(\omega t),\quad \omega>0,$$

problem (1.2) takes the form

$$\begin{aligned}
&c_0(\boldsymbol{\varepsilon},\theta)\tilde\theta_t-k_0\Delta\tilde\theta-(\omega c_0(\boldsymbol{\varepsilon},\theta)+g_1)\tilde\theta=(g+g_2)\exp(\omega t) &&\text{in }\ \Omega^T,\\
&\tilde\theta\big|_{t=0}=\theta_0 &&\text{in }\ \Omega,\\
&\boldsymbol{n}\cdot\nabla\tilde\theta=0 &&\text{on }\ S^T,
\end{aligned}\tag{3.2}$$

where for simplicity we denoted

$$g_1\equiv F_{,\theta\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\theta):\boldsymbol{\varepsilon}_t,\quad g_2\equiv\nu(\boldsymbol{A}\boldsymbol{\varepsilon}_t):\boldsymbol{\varepsilon}_t\ge 0.$$

Multiplying $(3.2)_1$ by $(\tilde{\theta}-\theta_*)_- \equiv \min\{(\tilde{\theta}-\theta_*),0\}$, and integrating over Ω yields

$$\frac{1}{2}\int\limits_\Omega c_0\frac{d}{dt}(\tilde{\theta}-\theta_*)_-^2\,dx + k_0\int\limits_\Omega |\nabla(\tilde{\theta}-\theta_*)_-|^2dx - \int\limits_\Omega(\omega c_0+g_1)(\tilde{\theta}-\theta_*)_-^2\,dx$$
$$= \int\limits_\Omega[(g+g_2)\exp(\omega t)+(\omega c_0+g_1)\theta_*](\tilde{\theta}-\theta_*)_-dx.$$

Hence, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_\Omega c_0(\tilde{\theta}-\theta_*)_-^2\,dx + k_0\int\limits_\Omega|\nabla(\tilde{\theta}-\theta_*)_-|^2dx\\
&= \int\limits_\Omega[(g+g_2)\exp(\omega t)+(\omega c_0+g_1)\theta_*](\tilde{\theta}-\theta_*)_-dx\\
&+\frac{1}{2}\int\limits_\Omega[c_{0,\boldsymbol{\varepsilon}}:\boldsymbol{\varepsilon}_t+2(\omega c_0+g_1)](\tilde{\theta}-\theta_*)_-^2\,dx\\
&+\frac{1}{2}\int\limits_\Omega c_{0,\theta}\theta_t(\tilde{\theta}-\theta_*)_-^2\,dx \equiv I_1+I_2+I_3.
\end{aligned}\tag{3.3}$$

Now we note that, by imbeddings, $(\boldsymbol{u},\theta)\in V(p,q)$ implies that $\boldsymbol{\varepsilon},\theta$ and $\boldsymbol{\varepsilon}_t$ are Hölder continuous in Ω^T.
Consequently, we can choose a positive finite number ω such that (3.1) is satisfied. Then

$$I_1\le 0.$$

Moreover, since the function

$$h(t)\equiv \sup\nolimits_\Omega \frac{1}{c_0}[c_{0,\boldsymbol{\varepsilon}}:\boldsymbol{\varepsilon}_t+2(\omega c_0+g_1)]$$

is continuous in $(0,T)$, we have

$$I_2\le\frac{1}{2}h(t)\int\limits_\Omega c_0(\tilde{\theta}-\theta_*)_-^2\,dx.$$

Further, the integral I_3 is estimated by

$$I_3\le c\|\theta_t\|_{L_q(\Omega)}\|(\tilde{\theta}-\theta_*)_-\|^2_{L_{2q'}(\Omega)}\equiv I_4,$$

where $\frac{1}{q}+\frac{1}{q'}=1$. Now, using interpolation inequality (see [2])

$$\|(\tilde{\theta}-\theta_*)_-\|_{L_{2q'}(\Omega)}\le\delta^{1-\varkappa}\|\nabla(\tilde{\theta}-\theta_*)_-\|_{L_2(\Omega)}+c\delta^{-\varkappa}\|(\tilde{\theta}-\theta_*)_-\|_{L_2(\Omega)}$$

with parameters $\delta, \varkappa$ satisfying conditions

$$\delta > 0, \quad \varkappa = 1 - \frac{1}{q'} < 1,$$

it follows that

$$\begin{aligned} I_4 &\le c\delta^{\frac{2}{q'}} \|\theta_t\|_{L_q(\Omega)} \|\nabla(\tilde{\theta} - \theta_*)_-\|^2_{L_2(\Omega)} + c\delta^{\frac{2}{q'}-2} \|\theta_t\|_{L_q(\Omega)} \|(\tilde{\theta} - \theta_*)_-\|^2_{L_2(\Omega)} \\ &\le \delta_1 \|\nabla(\tilde{\theta} - \theta_*)_-\|^2_{L_2(\Omega)} + c\delta_1^{1-q'} \|\theta_t\|^{q'}_{L_q(\Omega)} \|(\tilde{\theta} - \theta_*)_-\|^2_{L_2(\Omega)}, \end{aligned}$$

where

$$\delta_1 = \delta^{\frac{2}{q'}} \|\theta_t\|_{L_q(\Omega)}$$

is chosen sufficiently small, so δ_1-term can be absorbed by the left-hand side of (3.3). Combining estimates on I_k, it follows from (3.3) that

$$\frac{d}{dt} \int\limits_\Omega c_0(\tilde{\theta} - \theta_*)^2_- \, dx \le \left(\frac{c}{c_v} \|\theta_t\|_{L_q(\Omega)} + h(t) \right) \int\limits_\Omega c_0(\tilde{\theta} - \theta_*)^2_- \, dx.$$

Consequently, since $\theta_t \in L_q(\Omega^T)$, $h \in L_1(0,T)$, we conclude that

$$\begin{aligned} &\int\limits_\Omega c_0(\tilde{\theta} - \theta_*)^2_- \, dx \le \exp\left[\int\limits_0^t \left(\frac{c}{c_v} \|\theta_{t'}\|_{L_q(\Omega)} + h(t') \right) dt' \right] \cdot \\ &\cdot \int\limits_\Omega c_0(\varepsilon_0, \theta_0)(\theta_0 - \theta_*)^2_- \, dx = 0 \quad \text{for} \quad t \in (0,T). \end{aligned}$$

Hence, in view of $c_0 > c_v > 0$, it follows that $\tilde{\theta} \ge \theta_*$ in Ω^T. This shows the assertion. □

Now, using Lemma 3.1 and estimate (2.6), we establish physical integral estimates on thermal, kinetic, elastic and capillary energy.

Lemma 3.2. *Let*

$$\begin{aligned} &\boldsymbol{u}_0 \in \boldsymbol{W}_2^2(\Omega), \quad \boldsymbol{u}_1 \in \boldsymbol{L}_2(\Omega), \quad \theta_0 \in L_1(\Omega), \\ &(F_1(\varepsilon_0, \theta_0) - \theta_0 F_{1,\theta}(\varepsilon_0, \theta_0)) + F_2(\varepsilon_0) \in L_1(\Omega), \\ &\boldsymbol{b} \in \boldsymbol{L}_{2,1}(\Omega^T), \quad g \in L_1(\Omega^T). \end{aligned}$$

Assume that $\theta \ge 0$ a.e. in Ω^T and the energy bound (2.6) holds. Then a solution $(\boldsymbol{u}, \theta)$ of (1.1), (1.2) satisfies the estimate

$$\begin{aligned} &c_v \|\theta\|_{L_\infty(0,T;L_1(\Omega))} + \frac{1}{4} \|\boldsymbol{u}_t\|^2_{L_\infty(0,T;L_2(\Omega))} \\ &+ \|(F_1(\varepsilon, \theta) - \theta F_{1,\theta}(\varepsilon, \theta)) + F_2(\varepsilon)\|_{L_\infty(0,T;L_1(\Omega))} + \frac{\varkappa_0}{8} \|Q\boldsymbol{u}\|^2_{L_\infty(0,T;L_2(\Omega))} \le c \end{aligned} \tag{3.4}$$

with the constant c given by

$$c = c_v\|\theta_0\|_{L_1(\Omega)} + \frac{1}{2}\|\boldsymbol{u}_1\|^2_{L_2(\Omega)} + \|(F_1(\boldsymbol{\varepsilon}_0,\theta_0) - \theta_0 F_{1,\theta}(\boldsymbol{\varepsilon}_0,\theta_0)) + F_2(\boldsymbol{\varepsilon}_0)\|_{L_1(\Omega)}$$
$$+ \frac{\varkappa_0}{8}\|Q\boldsymbol{u}_0\|^2_{L_2(\Omega)} + \|\boldsymbol{b}\|^2_{L_{2,1}(\Omega^T)} + \|g\|_{L_1(\Omega^T)}.$$

Proof. Multiplying $(1.1)_1$ by $\boldsymbol{u}_t$, integrating over Ω and using the identities

(3.5)
$$-\nu\int_\Omega (Q\boldsymbol{u}_t)\cdot\boldsymbol{u}_t dx = \nu\int_\Omega (A\boldsymbol{\varepsilon}_t) : \boldsymbol{\varepsilon}_t dx,$$
$$\frac{\varkappa_0}{4}\int_\Omega (Q^2\boldsymbol{u})\cdot\boldsymbol{u}_t dx = \frac{\varkappa_0}{4}\int_\Omega (Q\boldsymbol{u})\cdot(Q\boldsymbol{u}_t)dx = \frac{\varkappa_0}{8}\frac{d}{dt}\int_\Omega |Q\boldsymbol{u}|^2 dx,$$
$$\int_\Omega (\nabla\cdot F_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\theta))\cdot\boldsymbol{u}_t dx = -\int_\Omega F_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\theta) : \boldsymbol{\varepsilon}_t dx = -\frac{d}{dt}\int_\Omega F_2(\boldsymbol{\varepsilon})dx - \int_\Omega F_{1,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\theta) : \boldsymbol{\varepsilon}_t dx,$$

we get

$$(3.6)\quad \begin{aligned} &\frac{d}{dt}\int_\Omega \left[\frac{1}{2}|\boldsymbol{u}_t|^2 + \frac{\varkappa_0}{8}|Q\boldsymbol{u}|^2 + F_2(\boldsymbol{\varepsilon})\right] dx + \nu\int_\Omega (A\boldsymbol{\varepsilon}_t) : \boldsymbol{\varepsilon}_t dx \\ &= -\int_\Omega F_{1,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\theta) : \boldsymbol{\varepsilon}_t dx + \int_\Omega \boldsymbol{b}\cdot\boldsymbol{u}_t dx. \end{aligned}$$

Integrating $(1.2)_1$ over Ω and using boundary condition $(1.2)_3$ gives

$$(3.7)\quad \int_\Omega (c_v - \theta F_{1,\theta\theta}(\boldsymbol{\varepsilon},\theta))\theta_t dx = \int_\Omega \theta F_{,\theta\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\theta) : \boldsymbol{\varepsilon}_t dx + \nu\int_\Omega (A\boldsymbol{\varepsilon}_t) : \boldsymbol{\varepsilon}_t dx + \int_\Omega g dx.$$

Recalling (2.5) we have

$$-\theta F_{1,\theta\theta}(\boldsymbol{\varepsilon},\theta) = E_{1,\theta}(\boldsymbol{\varepsilon},\theta) = (F_1(\boldsymbol{\varepsilon},\theta) - \theta F_{1,\theta}(\boldsymbol{\varepsilon},\theta))_{,\theta},$$

so that the integral on the left-hand side of (3.7) can be rearranged as follows

$$(3.8)\quad \begin{aligned} \int_\Omega (c_v - \theta F_{1,\theta\theta}(\boldsymbol{\varepsilon},\theta))\theta_t dx &= \frac{d}{dt}\int_\Omega (c_v\theta + E_1(\boldsymbol{\varepsilon},\theta))dx - \int_\Omega E_{1,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\theta) : \boldsymbol{\varepsilon}_t dx \\ &= \frac{d}{dt}\int_\Omega (c_v\theta + E_1(\boldsymbol{\varepsilon},\theta))dx - \int_\Omega F_{1,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\theta) : \boldsymbol{\varepsilon}_t dx + \int_\Omega \theta F_{1,\theta\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\theta) : \boldsymbol{\varepsilon}_t dx. \end{aligned}$$

Using (3.8) in (3.7) and next adding the result to (3.6) gives the identity

$$(3.9)\quad \frac{d}{dt}\int_\Omega \left[c_v\theta + \frac{1}{2}|\boldsymbol{u}_t|^2 + (E_1(\boldsymbol{\varepsilon},\theta) + F_2(\boldsymbol{\varepsilon})) + \frac{\varkappa_0}{8}|Q\boldsymbol{u}|^2\right] dx = \int_\Omega g dx + \int_\Omega \boldsymbol{b}\cdot\boldsymbol{u}_t dx.$$

Now, integrating (3.9) over $[0,t]$, $0 \le t \le T$, and using the estimate

$$
\begin{aligned}
(3.10)\qquad & \left|\int_0^t \int_\Omega b \cdot u_{t'} dx dt'\right| \le \operatorname{ess\,sup}_{0\le t' \le t} \left(\int_\Omega |u_{t'}|^2 dx\right)^{1/2} \int_0^t \left(\int_\Omega |b|^2 dx\right)^{1/2} dt' \\
& \le \frac{1}{4} \operatorname{ess\,sup}_{0\le t'\le t} \|u_{t'}\|^2_{L^2(\Omega)} + \|b\|^2_{L_{2,1}(\Omega^T)}
\end{aligned}
$$

we arrive at the assertion. □

By property (2.2) of the operator Q it follows from (3.4) that

$$
(3.11)\qquad \|u\|_{L_\infty(0,T;W_2^2(\Omega))} \le c.
$$

Consequently,

$$
(3.12)\qquad \|\varepsilon\|_{L_\infty(0,T;W_2^1(\Omega))} \le c,
$$

so, by imbedding,

$$
(3.13)\qquad \|\varepsilon\|_{L_\infty(0,T;L_\sigma(\Omega))} \le c, \qquad 1 < \sigma < \infty.
$$

Now, using regularity properties of parabolic systems, we obtain additional estimates for ε and θ.

Assuming

$$
F_{,\varepsilon}(\varepsilon,\theta) \in L_p(\Omega^T), \quad b \in L_p(\Omega^T), \quad u_0 \in W_p^{3-2/p}(\Omega),
$$
$$
Q(u_1 - \alpha Q u_0) \in L_p(\Omega), \quad 1 < p < \infty,
$$

with the help of Lemmas A2 and A3 (see Appendix), it follows

$$
\begin{aligned}
& \|\varepsilon\|_{W_p^{2,1}(\Omega^T)} \le c\|u\|_{W_p^{3,3/2}(\Omega^T)} \\
(3.14)\qquad & \le c(\|w\|_{W_p^{1,1/2}(\Omega^T)} + \|u_0\|_{W_p^{3-2/p}(\Omega)}) \\
& \le c(\|F_{,\varepsilon}(\varepsilon,\theta)\|_{L_p(\Omega^T)} + \|b\|_{L_p(\Omega^T)} + \|u_1 - \alpha Q u_0\|_{W_p^{2-2/p}(\Omega)} + \|u_0\|_{W_p^{3-2/p}(\Omega)}).
\end{aligned}
$$

Moreover, recalling assumption (A3-1) and bounds (3.13), we estimate

$$
\begin{aligned}
\|F_{,\varepsilon}(\varepsilon,\theta)\|_{L_p(\Omega^T)} & \le c\left(1 + \int_{\Omega^T} \theta^{ps}|\varepsilon|^{p(a_1-1)} dx dt + \int_{\Omega^T} |\varepsilon|^{p(a_2-1)} dx dt\right)^{\frac{1}{p}} \\
& \le c\left[1 + \left(\int_{\Omega^T} \theta^{ps\lambda_1} dx dt\right)^{\frac{1}{p\lambda_1}} \left(\int_{\Omega^T} |\varepsilon|^{p(a_1-1)\lambda_2} dx dt\right)^{\frac{1}{p\lambda_2}}\right] \\
& \le c\left[1 + \left(\int_{\Omega^T} \theta^{ps'} dx dt\right)^{\frac{s}{ps'}}\right] \le c(1 + \|\theta\|^s_{L_{ps'}(\Omega^T)}),
\end{aligned}
$$

where

$$
\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1 \qquad \text{and} \qquad \lambda_1 = \frac{s'}{s}, \quad s < s'.
$$

Next, using estimate (3.4) and (3.14) we prove

Lemma 3.3. *Suppose that* $0 < s < s' < 7/8$,

$$u_0 \in W^{\frac{9}{4}}_{\frac{8}{3}}(\Omega), \quad Q(u_1 - \alpha Q u_0) \in L_{\frac{8}{3}}(\Omega), \quad \theta_0 \in L_2(\Omega),$$

$$G(\varepsilon_0, \theta_0) \equiv -\theta_0^2 F_{1,\theta}(\varepsilon_0, \theta_0) + 2\theta_0 F_1(\varepsilon_0, \theta_0) - 2\int_0^{\theta_0} F_1(\varepsilon_0, \xi) d\xi \in L_1(\Omega),$$

$$b \in L_{\frac{8}{3}}(\Omega^T), \quad g \in L_{\frac{4}{3}}(\Omega^T).$$

Then there exists a constant $c = c(T)$ *depending on the data and* T^a, $a \in \mathbb{R}_+$ *such that*

$$\|\theta\|_{L_\infty(0,T;L_2(\Omega))} + \|\nabla\theta\|_{L_2(\Omega^T)} \le c. \tag{3.15}$$

Proof. Multiplying $(1.2)_1$ by θ and integrating over Ω yields

$$\begin{aligned} &\frac{c_v}{2}\frac{d}{dt}\int_\Omega \theta^2 dx - \int_\Omega \theta^2 F_{1,\theta\theta}(\varepsilon, \theta)\theta_t dx - \int_\Omega \theta^2 F_{1,\theta\varepsilon}(\varepsilon, \theta) : \varepsilon_t dx + k_0 \int_\Omega |\nabla\theta|^2 dx \\ &= \nu \int_\Omega \theta(A\varepsilon_t) : \varepsilon_t dx + \int_\Omega \theta g dx. \end{aligned} \tag{3.16}$$

We introduce the function

$$G(\varepsilon, \theta) = -\theta^2 F_{1,\theta}(\varepsilon, \theta) + 2\theta F_1(\varepsilon, \theta) - 2\hat{F}_1(\varepsilon, \theta), \tag{3.17}$$

where

$$\hat{F}_1(\varepsilon, \theta) = \int_0^\theta F_1(\varepsilon, \xi) d\xi.$$

We see that $G(\varepsilon, \theta)$ is the primitive of $-\theta^2 F_{1,\theta\theta}(\varepsilon, \theta)$ with respect to θ such that

$$G(\varepsilon, 0) = 0 \quad \text{and} \quad G_{,\theta}(\varepsilon, \theta) = -\theta^2 F_{1,\theta\theta}(\varepsilon, \theta) \ge 0.$$

Therefore

$$G(\varepsilon, \theta) \ge 0 \quad \text{for all} \quad (\varepsilon, \theta) \in S^2 \times [0, \infty). \tag{3.18}$$

From (3.17) it follows that

$$G_{,\varepsilon}(\varepsilon, \theta) = -\theta^2 F_{1,\theta\varepsilon}(\varepsilon, \theta) + 2\theta F_{1,\varepsilon}(\varepsilon, \theta) - 2\hat{F}_{1,\varepsilon}(\varepsilon, \theta). \tag{3.19}$$

In view of (3.17) and (3.19), (3.16) takes on the form

$$\begin{aligned} &\frac{c_v}{2}\frac{d}{dt}\int_\Omega \theta^2 dx + \frac{d}{dt}\int_\Omega G(\varepsilon, \theta) dx + k_0 \int_\Omega |\nabla\theta|^2 dx \\ &= \nu \int_\Omega \theta(A\varepsilon_t) : \varepsilon_t dx + 2\int_\Omega (\theta F_{1,\varepsilon}(\varepsilon, \theta) - \hat{F}_{1,\varepsilon}(\varepsilon, \theta)) : \varepsilon_t dx + \int_\Omega \theta g dx. \end{aligned} \tag{3.20}$$

Integrating (3.20) with respect to time and using (3.18) we obtain

$$
\begin{aligned}
X^2 &\equiv \frac{c_v}{2}\int\limits_{\Omega}\theta^2 dx + k_0\int\limits_{\Omega^t}|\nabla\theta|^2 dxdt' \\
&\le \nu\int\limits_{\Omega^t}\theta(A\varepsilon_{t'}):\varepsilon_{t'}dxdt' + 2\int\limits_{\Omega^t}(\theta F_{1,\varepsilon}(\varepsilon,\theta)-\hat{F}_{1,\varepsilon}(\varepsilon,\theta)):\varepsilon_{t'}dxdt' \qquad (3.21)\\
&+\int\limits_{\Omega^t}\theta g dxdt' + \frac{c_v}{2}\int\limits_{\Omega}\theta_0^2 dx + \int\limits_{\Omega}G(\varepsilon_0,\theta_0)dx, \quad 0\le t\le T.
\end{aligned}
$$

We proceed now to estimate the terms on the right-hand side of (3.21). To this end note that by the imbedding of the space $L_\infty(0,T;L_2(\Omega))\cap L_2(0,T;W_2^1(\Omega))$ in $L_4(\Omega^T)$ we have

$$c\|\theta\|_{L_4(\Omega^T)}\le X.$$

Hence, the first term on the right-hand side of (3.21) can be estimated as follows

$$\nu\int\limits_{\Omega^t}\theta(A\varepsilon_{t'}):\varepsilon_{t'}dxdt' \le c\|\theta\|_{L_4(\Omega^T)}\||\varepsilon_t|^2\|_{L_{\frac{4}{3}}(\Omega^T)} = c\|\theta\|_{L_4(\Omega^T)}\|\varepsilon_t\|^2_{L_{\frac{8}{3}}(\Omega^T)}\equiv Y_1.$$

To find bounds on $\|\varepsilon_t\|_{L_{\frac{8}{3}}(\Omega^T)}$ we use (3.14). We have

$$\|F_{,\varepsilon}(\varepsilon,\theta)\|_{L_{\frac{8}{3}}(\Omega^T)}\le c\left[1+\left(\int\limits_{\Omega^T}\theta^{\frac{8s'}{3}}dxdt\right)^{\frac{3}{8}}\right]\equiv I_1.$$

Further, using interpolation inequality and estimate (3.4) it follows that

$$
\begin{aligned}
I_1 &\le c + c\left(\int_0^T\|\theta\|^{\frac{8s'}{3}}_{L_{\frac{8s'}{3}}(\Omega)}dt\right)^{\frac{3}{8}} \le c + c\left(\int_0^T\left(\|\nabla\theta\|^{\frac{8s'}{3}\vartheta_1}_{L_2(\Omega)}\cdot\|\theta\|^{\frac{8s'}{3}(1-\vartheta_1)}_{L_1(\Omega)} + \|\theta\|^{\frac{8s'}{3}}_{L_1(\Omega)}\right)dt\right)^{\frac{3}{8}} \\
&\le c(T) + c\left(\int_0^T\|\nabla\theta\|^{\frac{8s'}{3}\vartheta_1}_{L_2(\Omega)}dt\right)^{\frac{3}{8}} = c(T) + c\left(\int_0^T\|\nabla\theta\|^{\frac{8s'}{3}-1}_{L_2(\Omega)}dt\right)^{\frac{3}{8}} \equiv I_2,
\end{aligned}
$$

where ϑ_1 satisfies condition

$$\frac{2}{\frac{8s'}{3}} = (1-\vartheta_1)\frac{2}{1}+\vartheta_1\left(\frac{2}{2}-1\right), \quad \text{so} \quad \vartheta_1 = 1-\frac{3}{8s'}.$$

Now we set

$$\frac{8s'}{3}-1=\gamma_1$$

where γ_1 is a positive number determined below.

To make I_2 well-defined we need

$$\gamma_1 > 0, \quad \text{that is } s' > \frac{3}{8}.$$

Therefore, in case $s' < \frac{3}{8}$ we have to modify s' according to the following change in estimation of I_1:

$$I_1 \le c + c\left(\int_0^T |\theta \frac{\theta^\sigma}{\theta^\sigma_{min}} \|_{L_{\frac{8s'}{3}}(\Omega)}^{\frac{8s'}{3}} dt\right)^{\frac{3}{8}}$$

$$= \frac{1}{\theta^{\sigma s'}_{min}}\left(\int_0^T \|\theta\|_{L_{\frac{8}{3}s'(1+\sigma)}}^{\frac{8}{3}s'(1+\sigma)} dt\right)^{\frac{3}{8}},$$

where σ is any positive number, and $\theta_{min} = \theta_* \exp(-\omega T)$ (see Lemma 3.1). Consequently, setting

$$s'' = s'(1+\sigma),$$

it can be always assured that $s'' > \frac{3}{8}$, so $\gamma_1 > 0$.

Then

$$I_2 = c(T) + c\left[\left(\int_0^T \|\nabla\theta\|_{L_2(\Omega)}^{\gamma_1} dt\right)^{\frac{1}{\gamma_1}}\right]^{\frac{3}{8}\gamma_1}.$$

Next, using the estimate

$$\left(\int_0^T \|\nabla\theta\|_{L_2(\Omega)}^{\gamma_1} dt\right)^{\frac{1}{\gamma_1}} \le \left[\left(\int_0^T dt\right)^{\frac{1}{\lambda_1}}\left(\int_0^T \|\nabla\theta\|_{L_2(\Omega)}^{\gamma_1\lambda_2} dt\right)^{\frac{1}{\lambda_2}}\right]^{\frac{1}{\gamma_1}}$$

$$\le \left[T^{1-\frac{\gamma_1}{2}}\left(\int_0^T \|\nabla\theta\|_{L_2(\Omega)}^{2} dt\right)^{\frac{\gamma_1}{2}}\right]^{\frac{1}{\gamma_1}} = T^{\frac{1}{\gamma_1}-\frac{1}{2}}\left(\int_0^T \|\nabla\theta\|_{L_2(\Omega)}^{2} dt\right)^{\frac{1}{2}},$$

where

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1, \quad \gamma_1\lambda_2 = 2, \quad \text{so} \quad \frac{1}{\lambda_1} = 1 - \frac{\gamma_1}{2},$$

we obtain

$$I_2 \le c(T)\left(1 + \|\nabla\theta\|_{L_2(\Omega^T)}^{\frac{3}{8}\gamma_1}\right).$$

Consequently,

$$\|F_{,\varepsilon}(\varepsilon,\theta\|_{L_{\frac{8}{3}}(\Omega^T)} \le c(T)\left(1 + \|\nabla\theta\|_{L_2(\Omega^T)}^{\frac{3}{8}\gamma_1}\right),$$

and

$$(3.22) \qquad \|\varepsilon\|_{W^{2,1}_{\frac{8}{3}}(\Omega^T)} \le c(T)\left(1+\|\nabla\theta\|^{\frac{3}{8}\gamma_1}_{L_2(\Omega^T)}\right).$$

Hence,

$$Y_1 \le c(T)\|\theta\|_{L_4(\Omega^T)}\left(1+\|\nabla\theta\|^{\frac{3}{4}\gamma_1}_{L_2(\Omega)}\right) \le c(T)X(1+X^{\frac{3}{4}\gamma_1}).$$

We assume the condition

$$1+\frac{3}{4}\gamma_1 < 2,$$

implying

$$\gamma_1 < \frac{4}{3}, \quad \text{so} \quad s < s' < \frac{3}{8}\left(\frac{4}{3}+1\right) = \frac{7}{8}.$$

Then, by Young's inequality, it follows that

$$Y_1 \le \varepsilon_1 X^2 + c(\varepsilon_1)c(T).$$

Therefore, for a sufficiently small ε_1, the term $\varepsilon_1 X^2$ can be absorbed by the left-hand side of (3.21).

The second integral on the right-hand side of (3.21) is estimated by

$$2\int_{\Omega^t}(\theta F_{1,\varepsilon}(\varepsilon,\theta) - \hat{F}_{1,\varepsilon}(\varepsilon,\theta)) : \varepsilon_{t'}dxdt' \le c\int_{\Omega^T}(1+\theta^{1+s}|\varepsilon|^{a_1-1})|\varepsilon_t|dxdt$$

$$\le c\left(\int_{\Omega^T}|\varepsilon_t|^{\frac{8}{3}}dxdt\right)^{\frac{3}{8}}\cdot\left(\int_{\Omega^T}(1+\theta^{1+s}|\varepsilon|^{a_1-1})^{\frac{8}{5}}dxdt\right)^{\frac{5}{8}}$$

$$\le c\|\varepsilon_t\|_{L_{\frac{8}{3}}(\Omega^T)}\left[c(T)+\left(\int_{\Omega^T}\theta^{\frac{8}{5}(1+s)\lambda_1}dxdt\right)^{\frac{5}{8\lambda_1}}\left(\int_{\Omega^T}|\varepsilon|^{\frac{8}{5}(a_1-1)\lambda_2}dxdt\right)^{\frac{5}{8\lambda_2}}\right]$$

$$\le c\|\varepsilon_t\|_{L_{\frac{8}{3}}(\Omega^T)}\left[c(T)+\left(\int_{\Omega^T}\theta^{\frac{8}{5}(1+s')}dxdt\right)^{\frac{5}{8}\frac{s+1}{s'+1}}\right] \equiv Y_2,$$

where

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2} = 1 \quad \text{and} \quad \lambda_1 = \frac{s'+1}{s+1}, \quad s < s'.$$

Again, with the help of interpolation inequality, it follows that

$$\left(\int\limits_{\Omega^T} \theta^{\frac{8}{5}(1+s')} dx dt\right)^{\frac{5}{8}\frac{s+1}{s'+1}} \le \left(\int\limits_0^T \|\theta\|^{\frac{8}{5}(1+s')}_{L_{\frac{8}{5}(1+s')}(\Omega)} dt\right)^{\frac{5}{8}} + c$$

$$\le c\left(\int\limits_0^T \left(\|\nabla\theta\|^{\frac{8}{5}(1+s')\vartheta_2}_{L_2(\Omega)} \cdot \|\theta\|^{\frac{8}{5}(1+s')(1-\vartheta_2)}_{L_1(\Omega)} + \|\theta\|^{\frac{8}{5}(1+s')}_{L_1(\Omega)}\right) dt\right)^{\frac{5}{8}} + c$$

$$\le c(T) + c\left(\int\limits_0^T \|\nabla\theta\|^{\frac{8}{5}(1+s')\vartheta_2}_{L_2(\Omega)} dt\right)^{\frac{5}{8}} = c(T) + c\left(\int\limits_0^T \|\nabla\theta\|^{\frac{8}{5}(1+s')-1}_{L_2(\Omega)} dt\right)^{\frac{5}{8}} \equiv I_3,$$

where ϑ_2 satisfies the condition

$$\frac{2}{\frac{8}{5}(1+s')} = (1-\vartheta_2)\frac{2}{1} + \vartheta_2\left(\frac{2}{2} - 1\right), \quad \text{so} \quad \vartheta_2 = 1 - \frac{5}{8(1+s')}.$$

Now we set

$$\frac{8}{5}(1+s') - 1 = \gamma_2,$$

where γ_2 is a positive number determined below. Then, similarly as for the term I_2, we obtain

$$I_3 = c(T) + c\left[\left(\int\limits_0^T \|\nabla\theta\|^{\gamma_2}_{L_2(\Omega)} dt\right)^{\frac{1}{\gamma_2}}\right]^{\frac{5}{8}\gamma_2} \le c(T)\left(1 + \|\nabla\theta\|^{\frac{5}{8}\gamma_2}_{L_2(\Omega^T)}\right).$$

Consequently, using (3.22), it follows that

$$\begin{aligned} Y_2 &\le c(T)\left(1 + \|\nabla\theta\|^{\frac{3}{8}\gamma_1}_{L_2(\Omega^T)}\right)\left(1 + \|\nabla\theta\|^{\frac{5}{8}\gamma_2}_{L_2(\Omega^T)}\right) \\ &\le c(T)(1 + X^{\frac{3}{8}\gamma_1})(1 + X^{\frac{5}{8}\gamma_2}). \end{aligned}$$

Now we assume the condition

$$\frac{3}{8}\gamma_1 + \frac{5}{8}\gamma_2 < 2,$$

equivalent to

$$\frac{3}{8}\left(\frac{8}{3}s' - 1\right) + \frac{5}{8}\left(\frac{8}{5}(1+s') - 1\right) < 2,$$

that is, $s' < 1$. Then

$$Y_2 \le \varepsilon_2 X^2 + c(\varepsilon_2)c(T),$$

so, for sufficiently small ε_2, the term $\varepsilon_2 X^2$ can be absorbed by the left-hand side of (3.21).

Finally, the third integral on the right-hand side of (3.21) is bounded by

$$\int_{\Omega^t} \theta g dx dt' \leq \|\theta\|_{L_4(\Omega^T)} \cdot \|g\|_{L_{\frac{4}{3}}(\Omega^T)} \leq \varepsilon_3 X^2 + c(\varepsilon_3)\|g\|^2_{L_{\frac{4}{3}}(\Omega^T)},$$

so again, with a sufficiently small ε_3, the term $\varepsilon_3 X^2$ can be absorbed by the left-hand side of (3.21).
In this way it follows from (3.21) that

$$X^2 \leq c(T),$$

that is estimate (3.15). $\square$

Now, in view of the bound

(3.23) $$\|\theta\|_{L_4(\Omega^T)} \leq c,$$

it follows from (3.14) that

$$\begin{aligned} \|\varepsilon\|_{W_p^{2,1}(\Omega^T)} &\leq c(1 + \|F_{,\varepsilon}(\varepsilon, \theta)\|_{L_p(\Omega^T)}) \\ &\leq c(1 + \|\theta\|^s_{L_{ps'}(\Omega^T)}) \leq c(1 + \|\theta\|^s_{L_4(\Omega^T)}) \leq c \end{aligned}$$

for $p < 4 \cdot \frac{8}{7} = \frac{32}{7}$, so

(3.24) $$\|\varepsilon\|_{W_4^{2,1}(\Omega^T)} \leq c.$$

By imbedding, ε is Hölder continuous in Ω^T and

(3.25) $$\|\varepsilon\|_{C^{\alpha_1,\alpha_1/2}(\Omega^T)} \leq c, \quad 0 < \alpha_1 < 1.$$

We note that, thanks to (3.25), the bounds (2.4) imply

(3.26) $$|c_0(\varepsilon, \theta)|, \quad |c_{0,\varepsilon}(\varepsilon, \theta)|, \quad |c_{0,\theta}(\varepsilon, \theta)| \leq c \quad \text{in } \Omega^T.$$

We continue to prove additional estimates for temperature.

Lemma 3.4. *Suppose that $g \in L_2(\Omega^T)$, $\nabla\theta_0 \in L_2(\Omega)$, (3.25) holds, and*

$$\|\varepsilon_t\|_{L_4(\Omega^T)} + \|\theta\|_{L_4(\Omega^T)} \leq c.$$

Then there exists $c > 0$ such that

(3.27) $$\|\theta_t\|_{L_2(\Omega^T)} + \|\theta\|_{L_\infty(0,T;W_2^1(\Omega))} \leq c.$$

Proof. Multiplying $(1.2)_1$ by θ_t and integating over Ω^t yields

$$
\begin{aligned}
&c_v \int\limits_{\Omega^t} \theta_{t'}^2 dx dt' + \frac{k_0}{2} \int\limits_0^t \frac{d}{dt'} \int\limits_\Omega |\nabla\theta|^2 dx dt' \\
&\le \int\limits_{\Omega^t} |\theta F_{,\theta\varepsilon}(\varepsilon,\theta)||\varepsilon_{t'}||\theta_{t'}| dx dt' + \bar{c} \int\limits_{\Omega^t} |\varepsilon_{t'}|^2 |\theta_{t'}| dx dt' \\
&+ \int\limits_{\Omega^t} |g||\theta_{t'}| dx dt' \equiv R.
\end{aligned}
\tag{3.28}
$$

Using Young and Hölder inequalities the right-hand side of (3.28) is estimated by

$$
\begin{aligned}
R &\le \delta \int\limits_{\Omega^t} \theta_{t'}^2 dx dt' + c(\delta) \left[\int\limits_{\Omega^t} \theta^{2s} |\varepsilon_{t'}|^2 dx dt' + \int\limits_{\Omega^t} |\varepsilon_{t'}|^4 dx dt' + \int\limits_{\Omega^t} |g|^2 dx dt' \right] \\
&\le \delta \int\limits_{\Omega^t} \theta_{t'}^2 dx dt' + c(\delta) [\|\theta\|_{L_{4s}(\Omega^T)}^{2s} \|\varepsilon_t\|_{L_4(\Omega^T)}^2 + \|\varepsilon_t\|_{L_4(\Omega^T)}^4 + \|g\|_{L_2(\Omega^T)}^2] \\
&\le \delta \int\limits_{\Omega^t} \theta_{t'}^2 dx dt' + c(\delta).
\end{aligned}
$$

Hence, choosing δ sufficiently small, it follows from (3.28) that

$$
\|\theta_t\|_{L_2(\Omega^T)} + \|\nabla\theta\|_{L_\infty(0,T;L_2(\Omega))} \le c
$$

what together with (3.15) shows the assertion. □

By imbedding, it follows from (3.27) that

$$
\|\theta\|_{L_\infty(0,T;L_\sigma(\Omega))} \le c, \quad 1 < \sigma < \infty. \tag{3.29}
$$

Now, with the help of estimates obtained so far, we can apply the classical parabolic theory [5] to the temperature equation $(1.2)_1$ written in the form

$$
c_v\theta_t - k_0\Delta\theta = \theta F_{,\theta\theta}(\varepsilon,\theta)\theta_t + \theta F_{,\theta\varepsilon}(\varepsilon,\theta) : \varepsilon_t + \nu(A\varepsilon_t) : \varepsilon_t + g \quad \text{in } \Omega^T. \tag{3.30}
$$

Recalling (3.24)–(3.27), (3.29), we have

$$
\begin{aligned}
&\|\theta F_{,\theta\theta}(\varepsilon,\theta)\theta_t + \theta F_{,\theta\varepsilon}(\varepsilon,\theta) : \varepsilon_t + \nu(A\varepsilon_t : \varepsilon_t)\|_{L_2(\Omega^T)} \\
&\le c \left(\int\limits_{\Omega^T} (\theta_t^2 + \theta^{2s}|\varepsilon_t|^2 + |\varepsilon_t|^4) dx dt \right)^{1/2} \\
&\le c(\|\theta_t\|_{L_2(\Omega^T)} + \|\theta\|_{L_{4s}(\Omega^T)}^{2s} + \|\varepsilon_t\|_{L_4(\Omega^T)}^2) \le c.
\end{aligned}
$$

Therefore, supposing $g \in L_2(\Omega^T)$, $\theta_0 \in W_2^1(\Omega)$, as well as the compatibility conditions, the classical parabolic theory yields

$$\|\theta\|_{W_2^{2,1}(\Omega^T)} \le c. \tag{3.31}$$

This implies that

$$\|\nabla\theta\|_{W_2^{1,1/2}(\Omega^T)} \le c, \tag{3.32}$$

so, by imbedding,

$$\|\nabla\theta\|_{L_\sigma(\Omega^T)} \le c, \quad 1 < \sigma \le 4. \tag{3.33}$$

We proceed now to improve estimates for $\boldsymbol{u}$.
In view of (3.25) and (3.29) we have

$$\|F_{,\varepsilon}(\varepsilon, \theta\|_{L_p(\Omega^T)} \le c \left[1 + \text{ess sup}_{t\in(0,T)} \left(\int_\Omega \theta^{ps} dx\right)^{1/p}\right] \le c \quad \text{for} \quad 1 < p < \infty. \tag{3.34}$$

Hence, with the help of (3.14) it follows that

$$\|\varepsilon\|_{W_p^{2,1}(\Omega^T)} \le c, \quad 1 < p < \infty. \tag{3.35}$$

Consequently,

$$\|\nabla\varepsilon\|_{W_p^{1,1/2}(\Omega^T)} \le c, \quad 1 < p < \infty, \tag{3.36}$$

so, by imbedding, in case $4 < p < \infty$,

$$\|\nabla\varepsilon\|_{L_\sigma(\Omega^T)} \le c, \quad 1 < \sigma < \infty, \tag{3.37}$$

and

$$\|\nabla\varepsilon\|_{C^{\alpha_2,\alpha_2/2}(\Omega^T)} \le c, \quad 0 < \alpha_2 < 1. \tag{3.38}$$

In view of the above estimates we can prove now

Lemma 3.5. *Suppose bounds (3.24)–(3.26), (3.29), (3.31)–(3.33) and (3.38) are satisfied. Then*

$$\nabla \cdot F_{,\varepsilon}(\varepsilon, \theta) \in W_2^{1,1/2}(\Omega^T)$$

and there exists a constant $c > 0$ such that

$$\|\nabla \cdot F_{,\varepsilon}(\varepsilon, \theta)\|_{W_2^{1,1/2}(\Omega^T)} \le c. \tag{3.39}$$

Proof. In view of the equality

$$\nabla \cdot F_{,\varepsilon}(\varepsilon,\theta) = F_{,\varepsilon\varepsilon}(\varepsilon,\theta)\nabla\varepsilon + F_{,\varepsilon\theta}(\varepsilon,\theta)\nabla\theta, \tag{3.40}$$

using (3.25), (3.29), (3.33), (3.38), we estimate

$$\begin{aligned}
&\|\nabla \cdot F_{,\varepsilon}(\varepsilon,\theta)\|_{L_2(\Omega^T)} \\
&\le c\bigg[\int\limits_{\Omega^T} (\theta^{2s}|\varepsilon|^{2(a_1-2)}|\nabla\varepsilon|^2 + |\varepsilon|^{2(a_2-2)}|\nabla\varepsilon|^2 + \theta^{2(s-1)}|\varepsilon|^{2(a_1-1)}|\nabla\theta|^2)dxdt\bigg]^{1/2} \\
&\le c(\|\theta\|^s_{L_2(\Omega^T)} + \|\nabla\theta\|_{L_2(\Omega^T)}) \le c.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|\nabla(\nabla\cdot(F_{,\varepsilon}(\varepsilon,\theta))\|_{L_2(\Omega^T)} \le c\bigg[\int\limits_{\Omega^T}(|F_{,\varepsilon\varepsilon\varepsilon}(\varepsilon,\theta)|^2|\nabla\varepsilon|^4 + |F_{,\varepsilon\varepsilon\theta}(\varepsilon,\theta)|^2|\nabla\varepsilon|^2|\nabla\theta|^2 \\
&+ |F_{,\varepsilon\varepsilon}(\varepsilon,\theta)|^2|\nabla^2\varepsilon|^2 + |F_{,\varepsilon\theta\theta}(\varepsilon,\theta)|^2|\nabla\theta|^4 + |F_{,\varepsilon\theta}(\varepsilon,\theta)|^2|\nabla^2\theta|^2)dxdt\bigg]^{1/2} \\
&\le c + c\bigg[\int\limits_{\Omega^T}(\theta^{2s} + |\nabla\theta|^2 + \theta^{2s}|\nabla^2\varepsilon|^2 + |\nabla^2\varepsilon|^2 + |\nabla\theta|^4 + |\nabla^2\theta|^2 dxdt\bigg]^{1/2} \\
&\le c + c(\|\theta\|^s_{L_{2s}(\Omega^T)} + \|\nabla\theta\|_{L_2(\Omega^T)} + \|\theta\|^{2s}_{L_{4s}(\Omega^T)} + \|\nabla^2\varepsilon\|^2_{L_4(\Omega^T)} \\
&+ \|\nabla^2\varepsilon\|_{L_2(\Omega^T)} + \|\nabla\theta\|^2_{L_4(\Omega^T)} + \|\nabla^2\theta\|_{L_2(\Omega^T)}) \le c,
\end{aligned}$$

where we have used estimates (3.24)–(3.26), (3.29), (3.31), (3.33) and (3.38). Hence,

$$\|\nabla\cdot F_{,\varepsilon}(\varepsilon,\theta)\|_{L_2(0,T;W^1_2(\Omega))} \le c.$$

Now we show that $\nabla \cdot F_{,\varepsilon} \in L_2(\Omega; W^{1/2}_2(0,T))$. To this end we have to estimate the integrals

$$\begin{aligned}
&\bigg(\int\limits_\Omega\int\limits_0^T\int\limits_0^T \frac{|\theta^s(t)\varepsilon^{a_1-2}(t)\nabla\varepsilon(t) - \theta^s(t')\varepsilon^{a_1-2}(t')\nabla\varepsilon(t')|^2}{|t-t'|^2}dt'dtdx\bigg)^{1/2} \\
&+\bigg(\int\limits_\Omega\int\limits_0^T\int\limits_0^T \frac{|\varepsilon^{a_2-2}(t)\nabla\varepsilon(t) - \varepsilon^{a_2-2}(t')\nabla\varepsilon(t')|^2}{|t-t'|^2}dt'dtdx\bigg)^{1/2} \\
&+\bigg(\int\limits_\Omega\int\limits_0^T\int\limits_0^T \frac{|\theta^{s-1}(t)\varepsilon^{a_1-1}(t)\nabla\theta(t) - \theta^{s-1}(t')\varepsilon^{a_1-1}(t')\nabla\theta(t')|^2}{|t-t'|^2}dt'dtdx\bigg)^{1/2} \\
&\equiv I_1 + I_2 + I_3,
\end{aligned}$$

where for simplicity we have restricted considerations to the maximal growth of $F(\varepsilon,\theta)$, and $\theta>\theta_1$, $\varepsilon_{ij}>\varepsilon_1$ $(i,j=1,2)$. In other cases the arguments are similar. We have

$$
\begin{aligned}
I_1 \le & \left(\int_\Omega\int_0^T\int_0^T \frac{|\theta^s(t)-\theta^s(t')|^2}{|t-t'|^2}|\varepsilon^{a_1-2}(t)|^2|\nabla\varepsilon(t)|^2dt'dtdx\right)^{1/2}\\
&+\left(\int_\Omega\int_0^T\int_0^T |\theta^s(t')|^2|\varepsilon^{a_1-3}(\tilde t)|^2\frac{|\varepsilon(t)-\varepsilon(t')|^2}{|t-t'|^2}|\nabla\varepsilon(t)|^2dt'dtdx\right)^{1/2}\\
&+\left(\int_\Omega\int_0^T\int_0^T |\theta^s(t')|^2|\varepsilon^{a_1-2}(t')|^2\frac{|\nabla\varepsilon(t)-\nabla\varepsilon(t')|^2}{|t-t'|^2}dt'dtdx\right)^{1/2} \equiv I_1^1+I_1^2+I_1^3,
\end{aligned}
$$

where $\tilde t\in(t,t')$. Recalling (3.25), (3.38), we see that

$$
\begin{aligned}
I_1^1 &\le c\left(\int_\Omega\int_0^T\int_0^T |\tilde\theta^{s-1}|^2\frac{|\theta(t)-\theta(t')|^2}{|t-t'|^2}dt'dtdx\right)^{1/2}\\
&\le c\left(\int_\Omega\int_0^T\int_0^T \frac{|\theta(t)-\theta(t')|^2}{|t-t'|^2}dt'dtdx\right)^{1/2},
\end{aligned}
$$

where $\tilde\theta=\theta(\tilde t)$. Since, in view of (3.31), $\theta\in W_2^{1,1/2}(\Omega^T)$, it follows that $I_1^1\le c$. Further, in view of (3.25), (3.29) and (3.38), using Hölder inequality, we have

$$
\begin{aligned}
I_1^2 &\le c\left(\int_\Omega\int_0^T\int_0^T \frac{|\varepsilon(t)-\varepsilon(t')|^q}{|t-t'|^q}dt'dtdx\right)^{1/q}\\
&\le c\sup_{x,t,t'}\frac{|\varepsilon(t)-\varepsilon(t')|}{|t-t'|^\alpha}\cdot\left(\int_\Omega\int_0^T\int_0^T\frac{1}{|t-t'|^{q(1-\alpha)}}dt'dtdx\right)^{1/q}\le c,
\end{aligned}
$$

where $q>2$, $q(1-\alpha)<1$. Similarly,

$$I_1^3\le c \quad\text{and}\quad I_2\le c.$$

It remains to estimate the integral

$$I_3 \leq \left(\int\limits_\Omega \int\limits_0^T \int\limits_0^T \frac{|\theta^{s-1}(t) - \theta^{s-1}(t')|^2 |\varepsilon^{a_1-1}(t)|^2 |\nabla\theta(t)|^2}{|t-t'|^2} dt' dt dx\right)^{1/2}$$

$$+ \left(\int\limits_\Omega \int\limits_0^T \int\limits_0^T \frac{|\theta^{s-1}(t')|^2 |\varepsilon^{a_1-1}(t) - \varepsilon^{a_1-1}(t')|^2 |\nabla\theta(t)|^2}{|t-t'|^2} dt' dt dx\right)^{1/2}$$

$$+ \left(\int\limits_\Omega \int\limits_0^T \int\limits_0^T \frac{|\theta^{s-1}(t')|^2 |\varepsilon^{a_1-1}(t')|^2 |\nabla\theta(t) - \nabla\theta(t')|^2}{|t-t'|^2} dt' dt dx\right)^{1/2}$$

$$\leq c \left(\int\limits_\Omega \int\limits_0^T \int\limits_0^T \frac{|\theta(t) - \theta(t')|^2 |\nabla\theta(t)|^2}{|t-t'|^2} dt' dt dx\right)^{1/2}$$

$$+ c \left(\int\limits_\Omega \int\limits_0^T \int\limits_0^T \frac{|\varepsilon(t) - \varepsilon(t')|^2 |\nabla\theta(t)|^2}{|t-t'|^2} dt' dt dx\right)^{1/2}$$

$$+ c \left(\int\limits_\Omega \int\limits_0^T \int\limits_0^T \frac{|\nabla\theta(t) - \nabla\theta(t')|^2}{|t-t'|^2} dt' dt dx\right)^{1/2} \equiv I_3^1 + I_3^2 + I_3^3,$$

where, in view of (3.32),

$$I_3^3 \leq c.$$

Further,

$$I_3^2 \leq c \sup_{x,t,t'} \frac{|\varepsilon(t) - \varepsilon(t')|}{|t-t'|^\alpha} \left(\int\limits_\Omega \int\limits_0^T \int\limits_0^T \frac{|\nabla\theta(t)|^2}{|t-t'|^{2-2\alpha}} dt' dt dx\right)^{1/2} \leq c,$$

because α can be chosen such that $2 - 2\alpha < 1$. Finally,

$$I_3^1 \leq c \left(\int\limits_\Omega \int\limits_0^T \int\limits_0^T \frac{|\theta(t) - \theta(t')|^4}{|t-t'|^{4\mu_1}} dt' dt dx\right)^{1/4} \cdot \left(\int\limits_\Omega \int\limits_0^T \int\limits_0^T \frac{|\nabla\theta(t)|^4}{|t-t'|^{4\mu_2}} dt' dt dx\right)^{1/4},$$

where $\mu_1 + \mu_2 = 1$. Assuming that $4\mu_2 < 1$ and recalling (3.33), we see that the second factor is bounded. Then $\mu_1 > \frac{3}{4}$, so the first factor equals to

$$\left(\int\limits_\Omega \int\limits_0^T \int\limits_0^T \frac{|\theta(t) - \theta(t')|^4}{|t-t'|^{3+\delta}} dt' dt dx\right)^{1/4},$$

with some arbitrary small $\delta > 0$.
Now note that (3.31) implies $\theta \in W_\sigma^{\alpha,\alpha/2}(\Omega^T)$ with $\sigma\alpha \leq 4$. Hence, choosing $\sigma = 4$ and $\alpha < 1$ such that $1 + \sigma\alpha > 3 + \delta$, we see that the above integral is bounded. This shows

$$\|\nabla \cdot F_{,\varepsilon}(\varepsilon, \theta)\|_{L_2(\Omega; W_2^{1/2}(0,T))} \leq c$$

and theoreby completes the proof. □

Now, in view of (3.39), supposing

$$\boldsymbol{u}_0 \in \boldsymbol{W}_2^4(\Omega), \quad \boldsymbol{u}_1 - \alpha \boldsymbol{Q} \boldsymbol{u}_0 \in \boldsymbol{W}_2^2(\Omega), \quad \boldsymbol{b} \in \boldsymbol{W}_2^{1,1/2}(\Omega^T),$$

with the help of Lemma A3 we infer that
(3.41)

$$\begin{aligned} &\|\varepsilon\|_{W_2^{4,2}(\Omega^T)} \leq c\|\boldsymbol{u}\|_{\boldsymbol{W}_2^{5,5/2}(\Omega^T)} \\ &\leq c(\|\boldsymbol{w}\|_{\boldsymbol{W}_2^{3,3/2}(\Omega^T)} + \|\boldsymbol{u}_0\|_{\boldsymbol{W}_2^4(\Omega)}) \\ &\leq c(\|\nabla \cdot F_{,\varepsilon}(\varepsilon, \theta)\|_{\boldsymbol{W}_2^{1,1/2}(\Omega^T)} + \|\boldsymbol{b}\|_{\boldsymbol{W}_2^{1,1/2}(\Omega^T)} + \|\boldsymbol{u}_1 - \alpha \boldsymbol{Q}\boldsymbol{u}_0\|_{\boldsymbol{W}_2^2(\Omega)} + \|\boldsymbol{u}_0\|_{\boldsymbol{W}_2^4(\Omega)}) \leq c. \end{aligned}$$

The latter estimate implies

$$\begin{aligned} &\|\nabla\varepsilon\|_{W_2^{3,3/2}(\Omega^T)} \leq c, \\ &\|\varepsilon_t\|_{W_2^{2,1}(\Omega^T)} \leq c, \\ &\|\nabla\varepsilon_t\|_{W_2^{1,1/2}(\Omega^T)} \leq c. \end{aligned} \tag{3.42}$$

Hence, by imbeddings,

$$\begin{aligned} &\|\varepsilon_t\|_{L_\sigma(\Omega^T)} \leq c, \quad 1 < \sigma < \infty, \\ &\|\varepsilon_t\|_{L_\infty(0,T;L_\sigma(\Omega))} \leq c\|\varepsilon_t\|_{W_2^{2,1}(\Omega^T)} \leq c, \quad 1 < \sigma < \infty, \end{aligned} \tag{3.43}$$

and

$$\|\nabla\varepsilon_t\|_{L_\sigma(\Omega^T)} \leq c, \quad 1 < \sigma \leq 4. \tag{3.44}$$

4. Additional a priori estimates

With the help of estimates (3.25)–(3.27), (3.38), (3.41)–(3.44) we prove now the key lemma which in two-dimensional case implies Hölder continuity of temperature.

Lemma 4.1. *Assume that*

$$\begin{aligned}
&|\varepsilon| + |\nabla\varepsilon| + |c_{,\varepsilon}(\varepsilon,\theta)| + |c_{0,\theta}(\varepsilon,\theta| \le c \quad \text{in } \Omega^T, \\
&\|\varepsilon_t\|_{L_\infty(0,T;L_\sigma(\Omega))} + \|\varepsilon_{tt}\|_{L_2(\Omega^T)} + \|\nabla\varepsilon_t\|_{L_4(\Omega^T)} \le c, \quad 1<\sigma<\infty, \\
&\|\theta_t\|_{L_2(\Omega^T)} + \|\theta\|_{L_\infty(0,T;W_2^1(\Omega))} \le c, \\
&\nabla g \in L_2(\Omega^T), \quad g_t \in L_2(\Omega^T), \quad b \in L_\sigma(\Omega^T), \\
&u_0 \in W_\sigma^{4-2/\sigma}(\Omega, \quad u_1 \in W_\sigma^{2-2/\sigma}(\Omega), \quad 1<\sigma<\infty, \\
&\theta\big|_{t=0} \in W_2^2(\Omega), \quad \theta_t\big|_{t=0} \in W_2^1(\Omega).
\end{aligned}$$

Then there exists $c>0$ such that

$$\|\nabla^2\theta\|_{L_\infty(0,T;L_2(\Omega))} + \|\nabla\theta_t\|_{L_\infty(0,T;L_2(\Omega))} + \|\theta_{tt}\|_{L_2(\Omega^T)} + \|\nabla\theta_t\|_{L_2(\Omega^T)} \le c, \tag{4.1}$$

Proof. For simplicity we denote the right-hand side of $(1.2)_1$ by

$$f \equiv \theta F_{,\theta\varepsilon}(\varepsilon,\theta) : \varepsilon_t + \nu(A\varepsilon_t) : \varepsilon_t + g. \tag{4.2}$$

Differentiating $(1.2)_1$ with respect to t, multiplying the result by θ_{tt} and integrating over Ω yields

$$\begin{aligned}
&\int_\Omega c_0\theta_{tt}^2 dx + \frac{k_0}{2}\frac{d}{dt}\int_\Omega |\nabla\theta_t|^2 dx = \int_\Omega (-c_{0,t}\theta_t + f_t)\theta_{tt} dx \\
&\le \delta\int_\Omega \theta_{tt}^2 dx + c(\delta)[\int_\Omega |c_{0,\theta}|^2\theta_t^4 dx + \int_\Omega |c_{0,\varepsilon}|^2|\varepsilon_t|^2\theta_t^2 dx + \int_\Omega |f_t|^2 dx].
\end{aligned}$$

Hence, with the help of bounds on $c_{0,\theta}$, $c_{0,\varepsilon}$ and ε_t, it follows that

$$\|\theta_{tt}\|_{L_2(\Omega)}^2 + \frac{d}{dt}\|\nabla\theta_t\|_{L_2(\Omega)}^2 \le c(\|\theta_t\|_{L_4(\Omega)}^4 + \|f_t\|_{L_2(\Omega)}^2 + 1). \tag{4.3}$$

We proceed now to examine the right-hand side of (4.3). Using the interpolation inequality

$$\|\theta_t\|_{L_4(\Omega)} \le c\|\nabla\theta_t\|_{L_2(\Omega)}^{\vartheta_1}\|\theta_t\|_{L_2(\Omega)}^{1-\vartheta_1} + c\|\theta_t\|_{L_2(\Omega)}, \tag{4.4}$$

with ϑ_1 determined by the condition

$$\frac{2}{4} = (1-\vartheta_1)\frac{2}{2} + \vartheta_1\left(\frac{2}{2} - 1\right), \quad \text{so} \quad \vartheta_1 = \frac{1}{2},$$

the first term on the right-hand side of (4.3) is estimated by

$$\|\theta_t\|_{L_4(\Omega)}^4 \le c\|\nabla\theta_t\|_{L_2(\Omega)}^2 \cdot \|\theta_t\|_{L_2(\Omega)}^2 + c\|\theta_t\|_{L_2(\Omega)}^4. \tag{4.5}$$

To examine the second term on the right-hand side of (4.3) we rearrange f_t as follows

$$\begin{aligned} f_t &= \theta F_{,\theta\varepsilon} : \varepsilon_{tt} + \theta_t(F_{,\theta\varepsilon} : \varepsilon_t) + \theta(F_{,\theta\varepsilon\varepsilon}\varepsilon_t) : \varepsilon_t + \theta(F_{,\theta\theta\varepsilon} : \varepsilon_t)\theta_t + 2\nu(A\varepsilon_t) : \varepsilon_{tt} + g_t \\ &= \varepsilon_{tt} : (\theta F_{,\theta\varepsilon} + 2\nu A\varepsilon_t) + \theta_t(F_{,\theta\varepsilon} + \theta F_{,\theta\theta\varepsilon}) : \varepsilon_t + \tilde{f}_t, \end{aligned}$$

where we separated the highest order terms involving ε_{tt} and θ_t, and denoted the remaining by

$$\tilde{f}_t = \theta(F_{,\theta\varepsilon\varepsilon}\varepsilon_t) : \varepsilon_t + g_t.$$

Then, it follows that

$$\begin{aligned} \|f_t\|^2_{L_2(\Omega)} &\le c\int_\Omega |\theta F_{,\theta\varepsilon}|^2|\varepsilon_{tt}|^2dx + c\int_\Omega |\varepsilon_t|^2|\varepsilon_{tt}|^2dx \\ &\quad + c\int_\Omega (|F_{,\theta\varepsilon}|^2 + |\theta F_{,\theta\theta\varepsilon}|^2)|\varepsilon_t|^2|\theta_t|^2dx + c\int_\Omega \tilde{f}_t^2dx \\ &\le c[\|\theta\|^{2s}_{L_\infty(\Omega)}\|\varepsilon_{tt}\|^2_{L_2(\Omega)} + \|\varepsilon_t\|^2_{L_4(\Omega)}\|\theta_t\|^2_{L_4(\Omega)} \\ &\quad + \|\varepsilon_t\|^2_{L_\infty(\Omega)}\|\varepsilon_{tt}\|^2_{L_2(\Omega)} + \|\tilde{f}_t\|^2_{L_2(\Omega)}]. \end{aligned} \tag{4.6}$$

Further, utilizing (4.4) and the estimate

$$\|\theta\|^{2s}_{L_\infty(\Omega)} \le c(\|\theta\|^2_{L_\infty(\Omega)} + 1) \le c(\|\nabla^2\theta\|^2_{L_2(\Omega)} + \|\theta\|^2_{L_2(\Omega)} + 1), \tag{4.7}$$

we infer

$$\begin{aligned} \|f_t\|^2_{L_2(\Omega)} &\le c[\|\nabla^2\theta\|^2_{L_2(\Omega)}\|\varepsilon_{tt}\|^2_{L_2(\Omega)} + (\|\theta\|^2_{L_2(\Omega)} + 1)\|\varepsilon_{tt}\|^2_{L_2(\Omega)} \\ &\quad + \|\nabla\theta_t\|^2_{L_2(\Omega)}\|\varepsilon_t\|^2_{L_4(\Omega)} + \|\theta_t\|^2_{L_2(\Omega)}\|\varepsilon_t\|^2_{L_4(\Omega)} + \|\varepsilon_t\|^2_{L_\infty(\Omega)}\|\varepsilon_{tt}\|^2_{L_2(\Omega)} + \|\tilde{f}_t\|^2_{L_2(\Omega)}]. \end{aligned} \tag{4.8}$$

Consequently, using (4.5) and (4.8) in (4.3) we arrive at

$$\begin{aligned} &\|\theta_{tt}\|^2_{L_2(\Omega)} + \frac{d}{dt}\|\nabla\theta_t\|^2_{L_2(\Omega)} \\ &\le c\Big[\|\nabla\theta_t\|^2_{L_2(\Omega)}(\|\theta_t\|^2_{L_2(\Omega)} + \|\varepsilon_t\|^2_{L_4(\Omega)}) + \|\nabla^2\theta\|^2_{L_2(\Omega)}\|\varepsilon_{tt}\|^2_{L_2(\Omega)} + \|\theta_t\|^4_{L_2(\Omega)} \\ &\quad + (\|\theta\|^2_{L_2(\Omega)} + 1)\|\varepsilon_{tt}\|^2_{L_2(\Omega)} + \|\theta_t\|^2_{L_2(\Omega)}\|\varepsilon_t\|^2_{L_4(\Omega)} + \|\varepsilon_t\|^2_{L_\infty(\Omega)}\|\varepsilon_{tt}\|^2_{L_2(\Omega)} + \|\tilde{f}_t\|^2_{L_2(\Omega)} + 1\Big]. \end{aligned} \tag{4.9}$$

Now, we multiply $(1.2)_1$ by $-\Delta\theta_t$ and integrate over Ω to get

$$\begin{aligned} &\int_\Omega c_0|\nabla\theta_t|^2dx + \frac{k_0}{2}\frac{d}{dt}\int_\Omega |\Delta\theta|^2dx = -\int_\Omega \theta_t\nabla c_0\cdot\nabla\theta_tdx + \int_\Omega \nabla f\cdot\nabla\theta_tdx \\ &\le \delta\int_\Omega |\nabla\theta_t|^2dx + c(\delta)\left[\int_\Omega |\nabla c_0|^2\theta_t^2dx + \int_\Omega |\nabla f|^2dx\right]. \end{aligned} \tag{4.10}$$

Hence, recalling the bounds on $c_{0,\varepsilon}$, $c_{0,\theta}$ and $\nabla\varepsilon$, it follows that

$$(4.11)\qquad \int\limits_\Omega |\nabla\theta_t|^2 dx + \frac{d}{dt}\int\limits_\Omega |\Delta\theta|^2 dx \le c\int\limits_\Omega |\nabla\theta|^2\theta_t^2 dx + c\int\limits_\Omega \theta_t^2 dx + c\int\limits_\Omega |\nabla f|^2 dx.$$

The first term on the right-hand side of (4.11) is handled as follows

$$\int\limits_\Omega |\nabla\theta|^2\theta_t^2 dx \le \|\nabla\theta\|^2_{L_4(\Omega)}\|\theta_t\|^2_{L_4(\Omega)}$$
$$\le c(\|\nabla^2\theta\|_{L_2(\Omega)}\|\nabla\theta\|_{L_2(\Omega)} + \|\nabla\theta\|^2_{L_2(\Omega)})\cdot(\|\nabla\theta_t\|_{L_2(\Omega)}\|\theta_t\|_{L_2(\Omega)} + \|\theta_t\|^2_{L_2(\Omega)}),$$

where we have used Hölder inequality and interpolation inequality (4.4). Consequently, (4.11) leads to

$$(4.12)\qquad \begin{aligned} &\|\nabla\theta_t\|^2_{L_2(\Omega)} + \frac{d}{dt}\|\Delta\theta\|^2_{L_2(\Omega)} \\ &\le c\Big[(\|\nabla^2\theta\|_{L_2(\Omega)}\|\nabla\theta\|_{L_2(\Omega)} + \|\nabla\theta\|^2_{L_2(\Omega)})\cdot(\|\nabla\theta_t\|_{L_2(\Omega)}\|\theta_t\|_{L_2(\Omega)} + \|\theta_t\|^2_{L_2(\Omega)}) \\ &+ \|\theta_t\|^2_{L_2(\Omega)} + \|\nabla f\|^2_{L_2(\Omega)}\Big] \\ &\le c\Big[(\|\nabla^2\theta\|^2_{L_2(\Omega)} + \|\nabla\theta_t\|^2_{L_2(\Omega)})\cdot(\|\nabla\theta\|^2_{L_2(\Omega)} + \|\theta_t\|^2_{L_2(\Omega)}) \\ &+ (\|\nabla\theta\|^2_{L_2(\Omega)} + 1)\|\theta_t\|^2_{L_2(\Omega)} + \|\nabla f\|^2_{L_2(\Omega)}\Big].\end{aligned}$$

Now we add (4.9) and (4.12) to obtain

$$(4.13)\qquad \begin{aligned} &\|\theta_{tt}\|^2_{L_2(\Omega)} + \|\nabla\theta_t\|^2_{L_2(\Omega)} + \frac{d}{dt}(\|\Delta\theta\|^2_{L_2(\Omega)} + \|\nabla\theta_t\|^2_{L_2(\Omega)}) \\ &\le c\Big[(\|\nabla^2\theta\|^2_{L_2(\Omega)} + \|\nabla\theta_t\|^2_{L_2(\Omega)})\cdot(\|\varepsilon_{tt}\|^2_{L_2(\Omega)} + \|\varepsilon_t\|^2_{L_4(\Omega)} + \|\nabla\theta\|^2_{L_2(\Omega)} + \|\theta_t\|^2_{L_2(\Omega)}) \\ &+ \|\theta_t\|^4_{L_2(\Omega)} + (\|\theta\|^2_{L_2(\Omega)} + 1)\|\varepsilon_{tt}\|^2_{L_2(\Omega)} + (\|\varepsilon_t\|^2_{L_4(\Omega)} + \|\nabla\theta\|^2_{L_2(\Omega)} + 1)\|\theta_t\|^2_{L_2(\Omega)} \\ &+ \|\tilde{f}_t\|^2_{L_2(\Omega)} + \|\nabla f\|^2_{L_2(\Omega)} + \|\varepsilon_t\|^2_{L_\infty(\Omega)}\|\varepsilon_{tt}\|^2_{L_2(\Omega)} + 1\Big].\end{aligned}$$

Further, using the inequality

$$\|\theta\|_{W_2^2(\Omega)} \le c(\|\Delta\theta\|_{L_2(\Omega)} + \|\theta\|_{L_2(\Omega)}),$$

where $\boldsymbol{n}\cdot\nabla\theta = 0$ on S, and denoting for simplicity

$$p(t) \equiv \|\varepsilon_{tt}\|^2_{L_2(\Omega)} + \|\varepsilon_t\|^2_{L_4(\Omega)} + \|\nabla\theta\|^2_{L_2(\Omega)} + \|\theta_t\|^2_{L_2(\Omega)},$$

(4.13) yields

$$(4.14)\qquad \begin{aligned} &\|\theta_{tt}\|^2_{L_2(\Omega)} + \|\nabla\theta_t\|^2_{L_2(\Omega)} + \frac{d}{dt}(\|\Delta\theta\|^2_{L_2(\Omega)} + \|\nabla\theta_t\|^2_{L_2(\Omega)}) \\ &\le c_1(\|\Delta\theta\|^2_{L_2(\Omega)} + \|\nabla\theta_t\|^2_{L_2(\Omega)})p(t) + c\Big[(\|\varepsilon_t\|^2_{L_4(\Omega)} + \|\theta\|^2_{W_2^1(\Omega)} + 1)p(t) \\ &+ \|\tilde{f}_t\|^2_{L_2(\Omega)} + \|\nabla f\|^2_{L_2(\Omega)} + \|\varepsilon_t\|^2_{L_\infty(\Omega)}\|\varepsilon_{tt}\|^2_{L_2(\Omega)} + \|\theta_t\|^4_{L_2(\Omega)} + 1\Big],\end{aligned}$$

where we have distinguished a positive constant c_1.
Now we multiply (4.14) by $\exp\left(-c_1\int_0^t p(t')dt'\right)$. As a result it follows that

$$\begin{aligned}
&(\|\theta_{tt}\|^2_{L_2(\Omega)}+\|\nabla\theta_t\|^2_{L_2(\Omega)})\exp\left(-c_1\int\limits_0^t p(t')dt'\right)\\
&+\frac{d}{dt}\left[(\|\Delta\theta\|^2_{L_2(\Omega)}+\|\nabla\theta_t\|^2_{L_2(\Omega)})\exp\left(-c_1\int\limits_0^t p(t')dt'\right)\right]\\
&\le c\Big[(\|\varepsilon_t\|^2_{L_4(\Omega)}+\|\theta\|^2_{W^1_2(\Omega)}+1)p(t)+\|\tilde{f}_t\|^2_{L_2(\Omega)}+\|\nabla f\|^2_{L_2(\Omega)}+1\\
&+\|\varepsilon_t\|^2_{L_\infty(\Omega)}\|\varepsilon_{tt}\|^2_{L_2(\Omega)}+\|\theta_t\|^4_{L_2(\Omega)}\Big]\exp\left(-c_1\int\limits_0^t p(t')dt'\right).
\end{aligned}\tag{4.15}$$

Integration of (4.15) with respect to t leads to

$$\begin{aligned}
&\int\limits_0^t(\|\theta_{t't'}(t')\|^2_{L_2(\Omega)}+\|\nabla\theta_{t'}(t')\|^2_{L_2(\Omega)})\exp\left(-c_1\int\limits_0^{t'} p(t'')dt''\right)dt'\\
&+(\|\Delta\theta(t)\|^2_{L_2(\Omega)}+\|\nabla\theta_t(t)\|^2_{L_2(\Omega)})\exp\left(-c_1\int\limits_0^t p(t')dt'\right)\\
&\le c\int\limits_0^t\Big[(\|\varepsilon_{t'}(t')\|^2_{L_4(\Omega)}+\|\theta(t')\|^2_{W^1_2(\Omega)}+1)p(t')+\|\tilde{f}_{t'}(t')\|^2_{L_2(\Omega)}+\|\nabla f(t')\|^2_{L_2(\Omega)}+1\\
&+\|\varepsilon_{t'}(t')\|^2_{L_\infty(\Omega)}\|\varepsilon_{t't'}(t')\|^2_{L_2(\Omega)}+\|\theta_{t'}(t')\|^4_{L_2(\Omega)}\Big]\exp\left(-c_1\int\limits_0^{t'} p(t'')dt''\right)dt'\\
&+\|\Delta\theta(0)\|^2_{L_2(\Omega)}+\|\nabla\theta_t(0)\|^2_{L_2(\Omega)}.
\end{aligned}$$

From this inequality we conclude that

$$\begin{aligned}
&\exp\left(-c_1\int\limits_0^t p(t')dt'\right)\cdot\\
&\cdot\left[\int\limits_0^t(\|\theta_{t't'}(t')\|^2_{L_2(\Omega)}+\|\nabla\theta_{t'}(t')\|^2_{L_2(\Omega)})dt'+\|\Delta\theta(t)\|^2_{L_2(\Omega)}+\|\nabla\theta_t(t)\|^2_{L_2(\Omega)}\right]\\
&\le c\int\limits_0^t\Big[(\|\varepsilon_{t'}(t')\|^2_{L_4(\Omega)}+\|\theta(t')\|^2_{W^1_2(\Omega)}+1)p(t')+\|\tilde{f}_{t'}(t')\|^2_{L_2(\Omega)}+\|\nabla f(t')\|^2_{L_2(\Omega)}+1\\
&+\|\varepsilon_{t'}(t')\|^2_{L_\infty(\Omega)}\|\varepsilon_{t't'}(t')\|^2_{L_2(\Omega)}+\|\theta_{t'}(t')\|^4_{L_2(\Omega)}\Big]dt'+\|\Delta\theta(0)\|^2_{L_2(\Omega)}+\|\nabla\theta_t(0)\|^2_{L_2(\Omega)}.
\end{aligned}\tag{4.16}$$

Now, recalling the bounds on ε_{tt}, ε_t, θ_t and θ, it follows that

$$
\begin{aligned}
&\int_0^t p(t')dt' \le \|\varepsilon_{tt}\|^2_{L_2(\Omega^T)} + T\|\varepsilon_t\|^2_{L_\infty(0,T;L_4(\Omega))} \\
&+ T\|\nabla\theta\|^4_{L_\infty(0,T;L_2(\Omega))} + \|\theta_t\|^2_{L_2(\Omega^T)} + |\Omega^T| \le c, \\
&\int_0^t (\|\varepsilon_{t'}(t')\|^2_{L_4(\Omega)} + \|\theta(t')\|^2_{W^1_2(\Omega)})p(t')dt' \\
&\le (\|\varepsilon_t\|^2_{L_\infty(0,T;L_4(\Omega))} + \|\theta\|^2_{L_\infty(0,T;W^1_2(\Omega))}) \int_0^t p(t')dt' \le c.
\end{aligned}
\tag{4.17}
$$

Moreover, recalling the bounds on ε, $\nabla\varepsilon$, $\nabla\varepsilon_t$ and assumption on g we obtain

$$
\begin{aligned}
&\int_0^t \|\tilde{f}_{t'}(t')\|^2_{L_2(\Omega)}dt' \le c\int_{\Omega^T} (|\theta F_{,\theta\varepsilon\varepsilon}|^2|\varepsilon_t|^4 + |g_t|^2)dxdt \\
&\le c\|\theta\|_{L_2(\Omega^T)}\|\varepsilon_t\|^4_{L_8(\Omega^T)} + c\|g_t\|^2_{L_2(\Omega^T)} \le c,
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
&\int_0^t \|\nabla f(t')\|^2_{L_2(\Omega)}dt' \le c\int_{\Omega^T} [|\theta F_{,\theta\varepsilon\varepsilon}|^2|\nabla\varepsilon|^2|\varepsilon_t|^2 + |F_{,\theta\varepsilon} + \theta F_{,\theta\theta\varepsilon}|^2|\nabla\theta|^2|\varepsilon_t|^2 \\
&+ |\theta F_{,\theta\varepsilon}|^2|\nabla\varepsilon_t|^2 + |\varepsilon_t|^2|\nabla\varepsilon_t|^2 + |\nabla g|^2]dxdt \\
&\le c(\|\theta\|_{L_2(\Omega^T)}\|\varepsilon_t\|^2_{L_4(\Omega^T)} + \|\nabla\theta\|^2_{L_4(\Omega^T)}\|\varepsilon_t\|^2_{L_4(\Omega^T)} \\
&+ \|\theta\|_{L_2(\Omega^T)}\|\nabla\varepsilon_t\|^2_{L_4(\Omega^T)} + \|\varepsilon_t\|^2_{L_4(\Omega^T)}\|\nabla\varepsilon_t\|^2_{L_4(\Omega^T)} + \|\nabla g\|^2_{L_2(\Omega^T)}) \le c.
\end{aligned}
$$

Inserting (4.17) and (4.18) into (4.16) leads to

$$
\begin{aligned}
&\operatorname{ess\,sup}_{0\le t\le T}(\|\Delta\theta\|^2_{L_2(\Omega)} + \|\nabla\theta_t\|^2_{L_2(\Omega)}) + \|\theta_{tt}\|^2_{L_2(\Omega^T)} + \|\nabla\theta_t\|^2_{L_2(\Omega^T)} \\
&\le c + c\|\varepsilon_t\|^2_{L_\infty(\Omega^T)}\|\varepsilon_{tt}\|^2_{L_2(\Omega^T)} + c\int_0^T \|\theta_{t'}\|^4_{L_2(\Omega)}dt' \\
&\le c + c\|\varepsilon_t\|^2_{L_\infty(\Omega^T)} + c\cdot \operatorname{ess\,sup}_{t\le T}\|\theta_t\|^2_{L_2(\Omega)},
\end{aligned}
\tag{4.19}
$$

where in the last inequality we used $L_2(\Omega^T)$-bound on ε_{tt} and $L_2(\Omega^T)$-bound on θ_t. We proceed now to estimate the terms on the right-hand side of (4.19). First we estimate $\|\theta_t\|_{L_\infty(0,T;L_2(\Omega))}$.

Differentiating $(1.2)_1$ with respect to t, multiplying by θ_t, integrating over Ω and integrating by parts yield

$$\frac{1}{2}\frac{d}{dt}\int_\Omega c_0\theta_t^2dx+k_0\|\nabla\theta_t\|^2_{L_2(\Omega)}=-\frac{1}{2}\int_\Omega c_{0,t}\theta_t^2dx+\int_\Omega(\theta F_{,\theta\varepsilon}:\varepsilon_t)_{,t}\theta_tdx$$
$$+2\nu\int_\Omega(A\varepsilon_t):\varepsilon_{tt}\theta_tdx+\int_\Omega g_t\theta_tdx\equiv I_1+I_2+I_3+I_4.$$

With the help of Hölder, Young and interpolation inequalities we obtain

$$\begin{aligned}|I_1|&\le c\int_\Omega|c_{0,\theta}\theta_t+c_{0,\varepsilon}\varepsilon_t|\theta_t^2dx\le c\int_\Omega|\theta_t|^3dx+c\int_\Omega|\varepsilon_t||\theta_t|^2dx\\&\le c\int_\Omega|\theta_t|^3dx+\delta_1\int_\Omega|\nabla\theta_t|^2dx+c(\delta_1)\int_\Omega\theta_t^2dx,\end{aligned}$$

$$\begin{aligned}|I_2|&\le c\int_\Omega|\varepsilon_t||\theta_t|^2dx+c\int_\Omega\theta^s|\varepsilon_t|^2|\theta_t|dx+c\int_\Omega\theta^s|\varepsilon_{tt}||\theta_t|dx\\&\le c\|\theta_t\|^2_{L_{2+\delta}(\Omega)}+c\|\theta_t\|_{L_{2+\delta}(\Omega)}+c\|\varepsilon_{tt}\|_{L_2(\Omega)}\|\theta_t\|_{L_{2+\delta}(\Omega)}\\&\le\delta_2\|\nabla\theta_t\|^2_{L_2(\Omega)}+c(\delta_2)\|\theta_t\|^2_{L_2(\Omega)}+c\|\varepsilon_{tt}\|^2_{L_2(\Omega)}+c,\\|I_3|&\le c\int_\Omega|\varepsilon_t||\varepsilon_{tt}||\theta_t|dx\le c\|\varepsilon_{tt}\|_{L_2(\Omega)}\|\theta_t\|_{L_{2+\delta}(\Omega)}\\&\le\delta_3\|\nabla\theta_t\|^2_{L_2(\Omega)}+c(\delta_3)\|\theta_t\|^2_{L_2(\Omega)}+c\|\varepsilon_{tt}\|^2_{L_2(\Omega)},\end{aligned}$$

where $\delta>0$. Choosing $\delta_1,\delta_2,\delta_3$ sufficiently small we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega c_0\theta_t^2dx+\frac{k_0}{2}\|\nabla\theta_t\|^2_{L_2(\Omega)}\\&\le c(\|\theta_t\|^3_{L_3(\Omega)}+\|\varepsilon_{tt}\|^2_{L_2(\Omega)}+\|\theta_t\|^2_{L_2(\Omega)}+\|g_t\|^2_{L_2(\Omega)})+c.\end{aligned}$$

Using the interpolation inequality

$$\|\theta_t\|_{L_3(\Omega)}\le c\|\nabla\theta_t\|^{1/3}_{L_2(\Omega)}\|\theta_t\|^{2/3}_{L_2(\Omega)}+c\|\theta_t\|_{L_2(\Omega)}$$

it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega c_0\theta_t^2dx+\frac{k_0}{4}\|\nabla\theta_t\|^2_{L_2(\Omega)}\\&\le c(\|\theta_t\|^4_{L_2(\Omega)}+\|\varepsilon_{tt}\|^2_{L_2(\Omega)}+\|\theta_t\|^2_{L_2(\Omega)}+\|g_t\|^2_{L_2(\Omega)})+c.\end{aligned}$$

Hence, since $c_0 \geq c_v > 0$,

$$\frac{d}{dt}\int\limits_\Omega c_0\theta_t^2 dx - c\|\theta_t\|^2_{L_2(\Omega)}\int\limits_\Omega c_0\theta_t^2 dx \leq c(\|\varepsilon_{tt}\|^2_{L_2(\Omega)} + \|\theta_t\|^2_{L_2(\Omega)} + \|g_t\|^2_{L_2(\Omega)}) + c.$$

Multiplying by $\exp\left(-c\int_0^t \|\theta_{t'}(t')\|^2_{L_2(\Omega)}dt'\right)$ we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\left(\int\limits_\Omega c_0\theta_t^2 dx\right)\exp\left(-c\int\limits_0^t \|\theta_{t'}(t')\|^2_{L_2(\Omega)}dt'\right)\right]\\
&\leq c(\|\varepsilon_{tt}\|^2_{L_2(\Omega)} + \|\theta_t\|^2_{L_2(\Omega)}\\
&+ \|g_t\|^2_{L_2(\Omega)} + 1)\exp\left(-c\int\limits_0^t \|\theta_{t'}(t')\|^2_{L_2(\Omega)}dt'\right).
\end{aligned}$$

Integrating with respect to t yields

$$\begin{aligned}
&\left(\int\limits_\Omega c_0\theta_t^2 dx\right)\exp\left(-c\int\limits_0^t \|\theta_{t'}(t')\|^2_{L_2(\Omega)}dt'\right)\\
&\leq c\int\limits_0^t (\|\varepsilon_{t't'}\|^2_{L_2(\Omega)} + \|\theta_{t'}\|^2_{L_2(\Omega)} + \|g_{t'}\|^2_{L_2(\Omega)} + 1)\exp\left(-\int\limits_0^{t'} \|\theta_{t''}\|^2_{L_2(\Omega)}dt''\right)dt'\\
&+ \int\limits_\Omega c_0(\varepsilon_0,\theta_0)\theta_t^2(0)dx.
\end{aligned}$$

Hence, by assumptions of lemma it follows that

$$\text{ess sup}_t\|\theta_t\|^2_{L_2(\Omega)} \leq c. \tag{4.20}$$

We procced now to estimate $\|\varepsilon_t\|_{L_\infty(\Omega^T)}$. In view of (4.20), recalling that $\|\theta\|_{L_\infty(0,T;W_2^1(\Omega))} \leq c$, (4.19) implies

$$\|\nabla\theta\|_{L_\infty(0,T;W_2^1(\Omega))} \leq c + c\|\varepsilon_t\|_{L_\infty(\Omega^T)}.$$

Hence,

$$\|\nabla\theta\|_{L_\infty(0,T;L_\sigma(\Omega))} \leq c + c\|\varepsilon_t\|_{L_\infty(\Omega^T)}, \quad 1 < \sigma < \infty. \tag{4.21}$$

We return now to the elasticity system and estimate its right-hand side in $L_\sigma(\Omega^T)$-norm. Using (4.21) we get

$$\begin{aligned}
\|\nabla\cdot F_{,\varepsilon}(\varepsilon,\theta)\|_{L_\sigma(\Omega^T)} &\leq c(\|\theta\|^s_{L_\sigma(\Omega^T)} + \|\nabla\theta\|_{L_\sigma(\Omega^T)})\\
&\leq c + c\|\varepsilon_t\|_{L_\infty(\Omega^T)}, \quad 1 < \sigma < \infty.
\end{aligned}$$

Consequently, in view of assumptions on the data, with the help of Lemma A1, it follows that

$$
\begin{aligned}
&\|u\|_{W_\sigma^{4,2}(\Omega^T)} \le c(\|w\|_{W_\sigma^{2,1}(\Omega^T)} + \|u_0\|_{W_\sigma^{4-2/\sigma}(\Omega)}) \\
(4.22)\qquad &\le (\|\nabla\cdot F_{,\varepsilon}(\varepsilon,\theta) + b\|_{L_\sigma(\Omega^T)} + \|u_1 - \alpha Q u_0\|_{W_\sigma^{2-2/\sigma}(\Omega)} + \|u_0\|_{W_\sigma^{4-2/\sigma}(\Omega)}) \\
&\le c + c\|\varepsilon_t\|_{L_\infty(\Omega^T)}, \quad 1<\sigma<\infty.
\end{aligned}
$$

Hence, by imbedding,

$$
\begin{aligned}
(4.23)\qquad \|\varepsilon_t\|_{W_\sigma^{1,1/2}(\Omega^T)} &\le c\|u\|_{W_\sigma^{4,2}(\Omega^T)} \\
&\le c + c\|\varepsilon_t\|_{L_\infty(\Omega^T)}, \quad 1<\sigma<\infty.
\end{aligned}
$$

Next, we utilize in (4.23) the interpolation inequality

$$
(4.24)\qquad \|\varepsilon_t\|_{L_\infty(\Omega^T)} \le \delta\|\varepsilon_t\|_{W_{\sigma_1}^{1,1/2}(\Omega^T)} + c(\delta)\|\varepsilon_t\|_{L_{\sigma_1}(\Omega^T)}
$$

with σ_1 satisfying

$$
\frac{4}{\sigma_1} - \frac{4}{\infty} < 1, \quad \text{so} \quad \sigma_1 > 4.
$$

Then (4.23) implies

$$
(4.25)\qquad \|\varepsilon_t\|_{W_\sigma^{1,1/2}(\Omega^T)} \le c + c\|\varepsilon_t\|_{L_\sigma(\Omega^T)} \le c \quad \text{for} \quad 4<\sigma<\infty,
$$

where in the last inequality we used $L_\sigma(\Omega^T)$-estimate on ε_t. Combining (4.24) and (4.25) leads to

$$
(4.26)\qquad \|\varepsilon_t\|_{L_\infty(\Omega^T)} \le c.
$$

Finally, using (4.26) and (4.20) in (4.19) ne arrive at the assertion. □

We note now that (4.1) implies

$$
(4.27)\qquad \|\theta\|_{W_2^2(\Omega^T)} \le c
$$

where $W_2^2(\Omega^T) = W_2^{2,2}(\Omega^T)$.
Consequently, by imbedding, the Hölder bound follows

$$
(4.28)\qquad \|\theta\|_{C^{\alpha_3}(\Omega^T)} \le c, \quad 0<\alpha_3<\frac{1}{2}.
$$

Moreover, (3.27) and (4.1) imply

$$
(4.29)\qquad \|\nabla\theta\|_{L_\infty(0,T;W_2^1(\Omega))} \le c,
$$

so, by imbedding,

$$
(4.30)\qquad \|\nabla\theta\|_{L_\infty(0,T;L_\sigma(\Omega))} \le c, \quad 1<\sigma<\infty.
$$

Thanks to Hölder continuity of ε and θ, in view of a priori bounds (3.37), (3.43), (4.30), we can obtain final estimates for a solution (u,θ) to (1.1), (1.2) in $V(p,q)$-norm.

Lemma 4.2. *Suppose that* ε, θ *are Hölder continuous in* Ω^T *with*

$$|\varepsilon| + \theta \le c \quad \text{in} \quad \Omega^T,$$

and

$$\|\nabla\varepsilon\|_{L_p(\Omega^T)} + \|\nabla\theta\|_{L_p(\Omega^T)} \le c, \quad 4 < p < \infty,$$
$$\|\varepsilon_t\|_{L_q(\Omega^T)} \le c, \quad 4 < q < \infty.$$

Moreover, suppose

$$\begin{aligned}&\boldsymbol{b} \in \boldsymbol{L}_p(\Omega^T), \quad g \in L_q(\Omega^T),\\ &\boldsymbol{u}_0 \in \boldsymbol{W}_p^{4-2/p}(\Omega), \quad \boldsymbol{u}_1 \in \boldsymbol{W}_p^{2-2/p}(\Omega), \quad \theta_0 \in W_q^{2-2/q}(\Omega), \quad 4 < p < \infty, \quad 4 < q < \infty,\end{aligned}$$

and compatibility conditions. Then

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_p^{4,2}(\Omega^T)} \le c, \quad 4 < p < \infty, \tag{4.31}$$

$$\|\theta\|_{W_q^{2,1}(\Omega^T)} \le c, \quad 4 < q < \infty. \tag{4.32}$$

Proof. In view of assumptions,

$$\|\nabla \cdot F_{,\varepsilon}(\varepsilon,\theta)\|_{L_p(\Omega^T)} \le c \quad \text{for} \quad 4 < p < \infty.$$

Then, similarly as in (4.22), estimate (4.31) follows.
Further, recalling $L_q(\Omega^T)$-bound on ε_t, it follows that the right-hand side of equation $(1.2)_1$ is bounded in $L_q(\Omega^T)$-norm for $4 < q < \infty$. Therefore, in view of Hölder continuity of the coefficient $c_0(\varepsilon,\theta)$, the classical parabolic theory [5] assures the bound (4.32). □

We note also that (4.31), (4.32) imply

$$\begin{aligned}&\|\varepsilon_t\|_{\boldsymbol{W}_p^{1,1/2}(\Omega^T)} \le c, \quad 4 < p < \infty,\\ &\|\nabla\theta\|_{\boldsymbol{W}_q^{1,1/2}(\Omega^T)} \le c, \quad 4 < q < \infty.\end{aligned} \tag{4.33}$$

Hence, by imbeddings, ε_t and $\nabla\theta$ are Hölder continuous in Ω^T and satisfy bounds

$$\|\varepsilon_t\|_{C^{\alpha_4,\alpha_4/2}(\Omega^T)} \le c, \quad 0 < \alpha_4 < 1, \tag{4.34}$$

$$\|\nabla\theta\|_{C^{\alpha_5,\alpha_5/2}(\Omega^T)} \le c, \quad 0 < \alpha_5 < 1. \tag{4.35}$$

5. Proof of Theorem 2.1

The proof is based on the application of the Leray-Schauder fixed point theorem to the system (1.14), (1.15), (1.2) in a similar manner as in [9]. Here we focus on the differences related to non-constant coefficient $c_0(\varepsilon,\theta)$.
First, we extend the definiton of $F_1(\varepsilon,\theta)$ to all values of θ in $\mathbb{R}$ in such a way that it is of class C^3, and that

$$F_{1,\theta\theta}(\varepsilon,\theta) \geq 0 \quad \text{for} \quad (\varepsilon,\theta) \in \mathcal{S}^2 \times (-\infty, 0).$$

We note that such extension preserves the lower bound (2.3) of $c_0(\varepsilon,\theta)$ for all $(\varepsilon,\theta) \in \mathcal{S}^2 \times \mathbb{R}$.
The solution space is

$$V(p,q) \equiv \{(\boldsymbol{u},\theta);\ \boldsymbol{u} \in \boldsymbol{W}_p^{4,2}(\Omega^T),\ \theta \in W_q^{2,1}(\Omega^T),\ 4 < p < \infty,\ 4 < q < \infty\}.$$

The solution map

$$T(\tau,\cdot) : (\bar{\boldsymbol{u}},\bar{\theta}) \in V(p,q) \to (\boldsymbol{u},\theta) \in V(p,q), \quad \tau \in [0,1], \tag{5.1}$$

is defined by means of the following initial-boundary value problems

$$\begin{aligned} &\boldsymbol{w}_t - \beta \boldsymbol{Q}\boldsymbol{w} = \tau[\nabla \cdot \boldsymbol{F}_{,\varepsilon}(\bar{\varepsilon},\bar{\theta}) + \boldsymbol{b}] && \text{in } \Omega^T,\\ &\boldsymbol{w}\big|_{t=0} = \tau[\boldsymbol{u}_1 - \alpha \boldsymbol{Q}\boldsymbol{u}_0] && \text{in } \Omega,\\ &\boldsymbol{w} = 0 && \text{on } S^T, \end{aligned} \tag{5.2}$$

$$\begin{aligned} &\boldsymbol{u}_t - \alpha \boldsymbol{Q}\boldsymbol{u} = \boldsymbol{w} && \text{in } \Omega^T,\\ &\boldsymbol{u}\big|_{t=0} = \tau \boldsymbol{u}_0 && \text{in } \Omega,\\ &\boldsymbol{u} = 0 && \text{on } S^T, \end{aligned} \tag{5.3}$$

$$\begin{aligned} &c_0(\varepsilon,\bar{\theta},\tau)\theta_t - k_0\Delta\theta = \tau[\bar{\theta}F_{,\theta\varepsilon}(\varepsilon,\bar{\theta}) : \varepsilon_t + \nu(A\varepsilon_t) : \varepsilon_t + g] && \text{in } \Omega^T,\\ &\theta\big|_{t=0} = \tau\theta_0 && \text{in } \Omega,\\ &\boldsymbol{n}\cdot\nabla\theta = 0 && \text{on } S^T, \end{aligned} \tag{5.4}$$

where

$$c_0(\varepsilon,\bar{\theta},\tau) = c_v - \tau\bar{\theta}F_{,\theta\theta}(\varepsilon,\bar{\theta}).$$

We employ the following formulation of the Leray-Schauder fixed point theorem (see [3]):

Theorem 5.1. *Let $\mathcal{B}$ be a Banach space. Assume that $T : [0,1] \times \mathcal{B} \to \mathcal{B}$ is a map with the following properties:*
(i) For any fixed $\tau \in [0,1]$ the map $T(\tau,\cdot) : \mathcal{B} \to \mathcal{B}$ is completely continuous.
(ii) For every bounded subset $\mathcal{C}$ of $\mathcal{B}$, the family of maps $T(\cdot,\chi) : [0,1] \to \mathcal{B}$, $\chi \in \mathcal{C}$, is uniformly equicontinuous.

(iii) There is a bounded subset $\mathcal{C}$ of $\mathcal{B}$ such that any fixed point in $\mathcal{B}$ of $T(\tau,\cdot)$, $0 \le \tau \le 1$, is contained in $\mathcal{C}$.

(iv) $T(0,\cdot)$ has precisely one fixed point in $\mathcal{B}$.

Then $T(1,\cdot)$ has at least one fixed point in $\mathcal{B}$.

We verify the properties (i)–(iv) for the soluton map (5.1).
The property (iii) results from Lemma 4.2.
The property (ii) follows by direct comparison of two solutions $(\boldsymbol{w},\boldsymbol{u},\theta)$ and $(\tilde{\boldsymbol{w}},\tilde{\boldsymbol{u}},\tilde{\theta})$ to problem (5.2)–(5.4) corresponding to parameters τ and $\tilde{\tau}$, respectively. In fact, the application of parabolic theory [10] gives the estimates

$$(5.5)\qquad \|\boldsymbol{w}-\tilde{\boldsymbol{w}}\|_{\boldsymbol{W}_p^{2,1}(\Omega^T)} \le c|\tau-\tilde{\tau}|, \quad \|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{\boldsymbol{W}_p^{4,2}(\Omega^T)} \le c|\tau-\tilde{\tau}|, \quad 4<p<\infty.$$

To estimate the difference $\eta = \theta - \tilde{\theta}$, we note that η satisfies the following problem:

$$(5.6)\qquad \begin{aligned} & c_0(\varepsilon,\bar{\theta},\tau)\eta_t - k_0\Delta\eta = (\tau-\tilde{\tau})\bar{\theta}F_{,\theta\theta}(\varepsilon,\bar{\theta})\tilde{\theta}_t + (\tau-\tilde{\tau})P && \\ & \quad + \tilde{\tau}(P-\tilde{P}) + \tilde{\tau}\bar{\theta}\tilde{\theta}_t(F_{,\theta\theta}(\varepsilon,\bar{\theta}) - F_{,\theta\theta}(\tilde{\varepsilon},\bar{\theta})) && \text{in } \Omega^T, \\ & \eta\big|_{t=0} = (\tau-\tilde{\tau})\theta_0 && \text{in } \Omega, \\ & \boldsymbol{n}\cdot\nabla\eta = 0 && \text{on } S^T, \end{aligned}$$

where

$$P = \bar{\theta}F_{,\theta\varepsilon}(\varepsilon,\bar{\theta}) : \varepsilon_t + \nu(\boldsymbol{A}\varepsilon_t) : \varepsilon_t + g, \qquad \tilde{P} = \bar{\theta}F_{,\theta\varepsilon}(\tilde{\varepsilon},\bar{\theta}) : \tilde{\varepsilon}_t + \nu(\boldsymbol{A}\tilde{\varepsilon}_t) : \tilde{\varepsilon}_t + g.$$

Thanks to (5.5), $L_q(\Omega^T)$ — norm of the right-hand side of $(5.6)_1$ is bounded by $c|\tau-\tilde{\tau}|$. In consequence, since the coefficient $c_0(\varepsilon,\bar{\theta},\tau)$ is Hölder continuous, the classical parabolic theory [5] implies

$$(5.7)\qquad \|\theta-\tilde{\theta}\|_{W_q^{2,1}(\Omega^T)} \le c|\tau-\tilde{\tau}|, \quad 4<q<\infty,$$

what concludes (ii).
In view of the regularity of problem (5.2)–(5.4), the property (iv) is obvious.
The property (i) follows by showing that for any fixed $\tau\in[0,1]$, $T(\tau,\cdot)$ maps the bounded subsets into precompact subsets in $V(p,q)$. Let $(\bar{\boldsymbol{u}}^n,\bar{\theta}^n)$ be a bounded sequence in $V(p,q)$ such that for $n\to\infty$

$$\begin{aligned} \bar{\boldsymbol{u}}^n &\to \bar{\boldsymbol{u}} \ \text{ weakly in } \boldsymbol{W}_p^{4,2}(\Omega^T), \quad 4<p<\infty, \\ \bar{\theta}^n &\to \bar{\theta} \ \text{ weakly in } W_q^{2,1}(\Omega^T), \quad 4<q<\infty. \end{aligned}$$

Our aim is to show that for the values of $T(\tau,\cdot)$

$$(5.8)\qquad (\boldsymbol{u}^n,\theta^n) = T(\tau,\bar{\boldsymbol{u}}^n,\bar{\theta}^n)$$

the following convergences hold for $n\to\infty$

$$(5.9)\qquad \boldsymbol{u}^n \to \boldsymbol{u} \ \text{ strongly in } \boldsymbol{W}_p^{4,2}(\Omega^T), \quad 4<p<\infty,$$

$$\theta^n \to \theta \text{ strongly in } W_q^{2,1}(\Omega^T), \quad 4 < q < \infty, \tag{5.10}$$

where

$$(\boldsymbol{u}, \theta) = T(\tau, \bar{\boldsymbol{u}}, \bar{\theta}). \tag{5.11}$$

With the help of compact imbeddings [2] it follows that for $n \to \infty$

$$\begin{aligned} &\bar{\boldsymbol{u}}^n \to \bar{\boldsymbol{u}} \text{ strongly in } \boldsymbol{W}_p^{3,3/2}(\Omega^T), \quad 4 < p < \infty, \\ &\bar{\theta}^n \to \bar{\theta} \text{ strongly in } W_q^{1,1/2}(\Omega^T), \quad 4 < q < \infty. \end{aligned} \tag{5.12}$$

This implies that

$$\begin{aligned} &\bar{\boldsymbol{\varepsilon}}^n \to \bar{\boldsymbol{\varepsilon}} && \text{strongly in } C^{\alpha_1,\alpha_1/2}(\Omega^T), \quad 0 < \alpha_1 < 1, \\ &\nabla\bar{\boldsymbol{\varepsilon}}^n \to \nabla\bar{\boldsymbol{\varepsilon}} && \text{strongly in } C^{\alpha_2,\alpha_2/2}(\Omega^T), \quad 0 < \alpha_2 < 1, \\ &\bar{\theta}^n \to \bar{\theta} && \text{strongly in } C^{\alpha_3,\alpha_3/2}(\Omega^T), \quad 0 < \alpha_3 < 1, \end{aligned} \tag{5.13}$$

where

$$\bar{\boldsymbol{\varepsilon}}^n = \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}^n), \quad \bar{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}).$$

Recalling equality (3.40), thanks to above convergences, it follows that for $n \to \infty$

$$\nabla \cdot \boldsymbol{F}_{,\boldsymbol{\varepsilon}}(\bar{\boldsymbol{\varepsilon}}^n, \bar{\theta}^n) \to \nabla \cdot \boldsymbol{F}_{,\boldsymbol{\varepsilon}}(\bar{\boldsymbol{\varepsilon}}, \bar{\theta}) \text{ strongly in } \boldsymbol{L}_p(\Omega^T) \text{ for } 4 < p \le q < \infty. \tag{5.14}$$

Consequently, by theory of parabolic systems [10],

$$\boldsymbol{w}^n \to \boldsymbol{w} \text{ strongly in } \boldsymbol{W}_p^{2,1}(\Omega^T), \quad 4 < p < \infty,$$

what, in turn, implies convergence (5.9).
Further, we note that by (5.9),

$$\begin{aligned} &\boldsymbol{\varepsilon}^n \to \boldsymbol{\varepsilon} \text{ strongly in } C^{\alpha_1,\alpha_1/2}(\Omega^T), \quad 0 < \alpha_1 < 1, \\ &\boldsymbol{\varepsilon}_t^n \to \boldsymbol{\varepsilon}_t \text{ strongly in } \boldsymbol{L}_p(\Omega^T), \quad 1 < p < \infty, \text{ and} \\ &\qquad\quad \text{strongly in } C^{\alpha_4,\alpha_4/2}(\Omega^T), \quad 0 < \alpha_4 < 1 - \frac{4}{p}. \end{aligned} \tag{5.15}$$

To prove convergence (5.10) we consider the difference $\eta^n = \theta^n - \theta$. We have

$$\begin{aligned} &c_0(\boldsymbol{\varepsilon}, \bar{\theta}, \tau)\eta_t^n - k_0\Delta\eta^n = (P^n(\boldsymbol{\varepsilon}^n, \bar{\theta}^n, \tau) - P(\boldsymbol{\varepsilon}, \bar{\theta}, \tau)) \\ &\qquad - (c_0(\boldsymbol{\varepsilon}^n, \bar{\theta}^n, \tau) - c_0(\boldsymbol{\varepsilon}, \bar{\theta}, \tau))\theta_t^n && \text{in } \Omega^T, \\ &\eta^n\big|_{t=0} = 0 && \text{in } \Omega, \\ &\boldsymbol{n} \cdot \nabla\eta^n = 0 && \text{on } S^T, \end{aligned} \tag{5.16}$$

where

$$P^n(\varepsilon^n,\bar\theta^n,\tau)=\tau[\bar\theta^n F_{,\theta\varepsilon}(\varepsilon^n,\bar\theta^n):\varepsilon_t^n+\nu(A\varepsilon_t^n):\varepsilon_t^n+g],$$
$$P(\varepsilon,\bar\theta,\tau)=\tau[\bar\theta F_{,\theta\varepsilon}(\varepsilon,\bar\theta):\varepsilon_t+\nu(A\varepsilon_t):\varepsilon_t+g],$$

and $\varepsilon^n=\varepsilon(u^n)$, $\varepsilon=\varepsilon(u)$.
In view of Hölder continuity of the coefficient $c_0(\varepsilon,\bar\theta,\tau)$, in order to prove that for $n\to\infty$

$$\eta^n\to 0\ \text{ strongly in }\ W_q^{2,1}(\Omega^T),$$

it is sufficient, by virtue of the classical parabolic theory, to show that the right-hand side of $(5.16)_1$ converges to 0 in $L_q(\Omega^T)$-norm. For the first term we have

$$\begin{aligned}&\int\limits_{\Omega^T}|P^n(\varepsilon^n,\bar\theta^n,\tau)-P(\varepsilon,\bar\theta,\tau)|^q dxdt\\
&\le c\bigg(\int\limits_{\Omega^T}|\bar\theta^n-\bar\theta|^q|F_{,\theta\varepsilon}(\varepsilon^n,\bar\theta^n)|^q|\varepsilon_t^n|^q dxdt+\int\limits_{\Omega^T}\bar\theta^q(|\varepsilon^n-\varepsilon|^q+|\bar\theta^n-\bar\theta|^q)|\varepsilon_t^n|^q dxdt\\
&+\int\limits_{\Omega^T}\bar\theta^q|F_{,\theta\varepsilon}(\varepsilon,\bar\theta)|^q|\varepsilon_t^n-\varepsilon_t|^q dxdt+\int\limits_{\Omega^T}|\varepsilon_t^n-\varepsilon_t|^q(|\varepsilon_t^n|^q+|\varepsilon_t|^q)dxdt\bigg)\to 0\quad\text{as }\ n\to\infty,\end{aligned}$$

where we used the uniform with respect to n Hölder bounds on ε^n, ε_t^n, $\bar\theta^n$ and the convergences (5.13), (5.15). For the second term on the right-hand side of $(5.16)_1$, recalling convergences (5.13), (5.15), we have

$$\begin{aligned}&\int\limits_{\Omega^T}|(c_0(\varepsilon^n,\bar\theta^n,\tau)-c_0(\varepsilon,\bar\theta,\tau))\theta_t^n|^q dxdt\\
&\le \sup\nolimits_{\Omega^T}|c_0(\varepsilon^n,\bar\theta^n,\tau)-c_0(\varepsilon,\bar\theta,\tau)|^q\|\theta_t^n\|^q_{L_q(\Omega^T)}\to 0\quad\text{as }\ n\to\infty.\end{aligned}$$

This shows (5.10) and thereby the complete continuity of $T(\tau,\cdot)$.

Summarizing, we have shown that the solution map (5.1) satisfies assumptions (i)–(iv) of the Leray-Schauder fixed point theorem. Thus, $T(1,\cdot)$ has at least one fixed point in $V(p,q)$ which is equivalent to a solution (u,θ) of problem (1.1), (1.2) in $V(p,q)$.
Recalling bounds (4.31), (4.32) and Lemma 3.1 the proof Theorem 2.1 is completed. □

6. Proof of Theorem 2.2

Let (u,θ) and $(\tilde u,\tilde\theta)$ be two solutions of (1.1), (1.2) corresponding to the same data. We denote

$$v=u-\tilde u,\qquad \eta=\theta-\tilde\theta.$$

Next, to simplify notation, we set

$$\varepsilon=\varepsilon(u),\quad \varepsilon_t=\varepsilon(u_t),\quad F_{,\varepsilon}=F_{,\varepsilon}(\varepsilon,\theta),\quad F_{,\theta\varepsilon}=F_{,\theta\varepsilon}(\varepsilon,\theta),$$

$$c_0 = c_0(\boldsymbol{\varepsilon}, \theta), \quad \gamma_0 = \gamma_0(\boldsymbol{\varepsilon}, \theta),$$

and respectively $\tilde{\boldsymbol{\varepsilon}}$, $\tilde{\boldsymbol{\varepsilon}}_t$, $\tilde{F}_{,\boldsymbol{\varepsilon}}$, $\tilde{F}_{,\theta,\boldsymbol{\varepsilon}}$, $\tilde{c}_0$, $\tilde{\gamma}_0$ for the functions of $(\tilde{\boldsymbol{u}}, \tilde{\theta})$. Further, it is convenient to rewrite equation $(1.2)_1$ in the form

$$\theta_t - k_0\gamma_0\Delta\theta = \gamma_0\theta F_{,\theta\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \theta) : \boldsymbol{\varepsilon}_t + \nu\gamma_0(\boldsymbol{A}\boldsymbol{\varepsilon}_t) : \boldsymbol{\varepsilon}_t + \gamma_0 g \quad \text{in } \Omega^T, \tag{6.1}$$

where

$$\gamma_0 = \gamma_0(\boldsymbol{\varepsilon}, \theta) \equiv \frac{1}{c_0(\boldsymbol{\varepsilon}, \theta)}.$$

We note also that

$$\frac{1}{c_0^*} \le \gamma_0 \le \frac{1}{c_v} \quad \text{in } \Omega^T, \tag{6.2}$$

where

$$c_0^* \equiv \max_{\Omega^T} c_0(\boldsymbol{\varepsilon}, \theta).$$

Subtracting the corresponding equations we see that $(\boldsymbol{v}, \eta)$ satisfy the following problems:

$$\begin{aligned}
&\boldsymbol{v}_{tt} - \nu\boldsymbol{Q}\boldsymbol{v}_t + \frac{\varkappa_0}{4}\boldsymbol{Q}^2\boldsymbol{v} = \nabla\cdot(F_{,\boldsymbol{\varepsilon}} - \tilde{F}_{,\boldsymbol{\varepsilon}}) && \text{in } \Omega^T,\\
&\boldsymbol{v}\big|_{t=0} = 0, \quad \boldsymbol{v}_t\big|_{t=0} = 0 && \text{in } \Omega,\\
&\boldsymbol{v} = 0, \quad \boldsymbol{Q}\boldsymbol{v} = 0 && \text{on } S^T,
\end{aligned} \tag{6.3}$$

$$\begin{aligned}
&\eta_t - k_0\gamma_0\Delta\eta = \gamma_0\theta F_{,\theta\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}_t - \tilde{\gamma}_0\tilde{\theta}\tilde{F}_{,\theta\boldsymbol{\varepsilon}} : \tilde{\boldsymbol{\varepsilon}}_t\\
&\qquad + \nu\gamma_0(\boldsymbol{A}\boldsymbol{\varepsilon}_t) : \boldsymbol{\varepsilon}_t - \nu\tilde{\gamma}_0(\boldsymbol{A}\tilde{\boldsymbol{\varepsilon}}_t) : \tilde{\boldsymbol{\varepsilon}}_t + (\gamma_0 - \tilde{\gamma}_0)g\\
&\qquad + k_0(\gamma_0 - \tilde{\gamma}_0)\Delta\tilde{\theta} \equiv R_1 + R_2 + R_3 + R_4 && \text{in } \Omega^T,\\
&\eta\big|_{t=0} = 0 && \text{in } \Omega,\\
&\boldsymbol{n}\cdot\nabla\eta = 0 && \text{on } S^T.
\end{aligned} \tag{6.4}$$

Multiplying $(6.3)_1$ by $\boldsymbol{v}_t$ and integrating over Ω^t yields

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\boldsymbol{v}_t|^2 dx + \frac{\varkappa_0}{8}\int_\Omega |\boldsymbol{Q}\boldsymbol{v}|^2 dx + \nu\int_{\Omega^t}(\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_{t'})) : \boldsymbol{\varepsilon}(\boldsymbol{v}_{t'})dxdt'\\
&= -\int_{\Omega^t}(F_{,\boldsymbol{\varepsilon}} - \tilde{F}_{,\boldsymbol{\varepsilon}}) : \boldsymbol{\varepsilon}(\boldsymbol{v}_{t'})dxdt'.
\end{aligned} \tag{6.5}$$

Next, adding to (6.5) the identity

$$\frac{1}{2}\int_\Omega |\boldsymbol{\varepsilon}(\boldsymbol{v})|^2 dx = \int_{\Omega^t} \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v}_{t'})dxdt', \tag{6.6}$$

valid thanks to initial condition $(6.3)_2$, recalling (2.1), and using Young inequality we get

$$\begin{aligned}
(6.7)\qquad & \int\limits_{\Omega}\left(\frac{1}{2}|v_t|^2+\frac{1}{2}|\varepsilon(v)|^2+\frac{\varkappa_0}{8}|Qv|^2\right)dx+\nu\underline{c}\int\limits_{\Omega^t}|\varepsilon(v_{t'})|^2dxdt'\\
&\le\delta\int\limits_{\Omega^t}|\varepsilon(v_{t'})|^2dxdt'+c(\delta)\int\limits_{\Omega^t}(|F_{,\varepsilon}-\tilde F_{,\varepsilon}|^2+|\varepsilon(v)|^2)dxdt'.
\end{aligned}$$

Hence, using the estimate

$$(6.8)\qquad |F_{,\varepsilon}-\tilde F_{,\varepsilon}|\le c(|\varepsilon(v)|+|\eta|),$$

which follows from uniform bounds on ε and θ in Ω^T, and choosing δ appropriately, we obtain

$$(6.9)\qquad \int\limits_{\Omega}(|v_t|^2+|\varepsilon(v)|^2+|Qv|^2)dx+\int\limits_{\Omega^t}|\varepsilon(v_{t'})|^2dxdt'\le c\int\limits_{\Omega^t}(|\varepsilon(v)|^2+|\eta|^2)dxdt'.$$

Consequently, with the help of Gronwall's inequality, we arrive at the estimate

(6.10)

$$\|v_t\|_{L_\infty(0,T;L_2(\Omega))}+\|\varepsilon(v)\|_{L_\infty(0,T;L_2(\Omega))}+\|Qv\|_{L_\infty(0,T;L_2(\Omega))}+\|\varepsilon(v_t)\|_{L_2(\Omega^T)}\le c\|\eta\|_{L_2(\Omega^T)}.$$

By virtue of (2.2) it follows from (6.10) that

$$(6.11)\qquad \|v\|_{L_\infty(0,T;W_2^2(\Omega))}\le c\|\eta\|_{L_2(\Omega^T)}.$$

Now we multiply $(6.4)_1$ by η and integrate over Ω^t to get, after integration by parts,

$$(6.12)\qquad \frac{1}{2}\int\limits_{\Omega}\eta^2dx+k_0\int\limits_{\Omega^t}\gamma_0|\nabla\eta|^2dxdt'=-k_0\int\limits_{\Omega^t}\eta\nabla\eta\cdot\nabla\gamma_0dxdt'+\sum_{i=1}^{4}\int\limits_{\Omega^t}R_i\eta dxdt'.$$

Hence, by (6.2),

$$(6.13)\qquad \frac{1}{2}\int\limits_{\Omega}\eta^2dx+\frac{k_0}{c_0^*}\int\limits_{\Omega^t}|\nabla\eta|^2dxdt'\le-k_0\int\limits_{\Omega^t}\eta\nabla\eta\cdot\nabla\gamma_0dxdt'+\sum_{i=1}^{4}\int\limits_{\Omega^t}R_i\eta dxdt'.$$

We proceed now to estimate the terms on the right-hand side of (6.13). Note that by virtue of Hölder estimates on ε, θ, $\nabla\varepsilon$ and $\nabla\theta$ in Ω^T, we have

$$(6.14)\qquad |\nabla\gamma_0|\le\frac{1}{c_0^2}(|c_{0,\varepsilon}||\nabla\varepsilon|+|c_{0,\theta}||\nabla\theta|)\le c\quad\text{in}\ \ \Omega^T.$$

Consequently, the first term on the right-hand side of (6.13) is, with the help of Young's inequality, estimated by

$$(6.15)\qquad k_0\int\limits_{\Omega^t}|\eta||\nabla\eta||\nabla\gamma_0|dxdt'\le\delta_1\int\limits_{\Omega^t}|\nabla\eta|^2dxdt'+c(\delta_1)\int\limits_{\Omega^t}\eta^2dxdt'.$$

Further, thanks to the uniform bounds on ε, θ, ε_t, γ_0 and $\tilde{\varepsilon}$, $\tilde{\theta}$, $\tilde{\varepsilon}_t$, $\tilde{\gamma}_0$ in Ω^T, we have

$$|\gamma_0 - \tilde{\gamma}_0| \le c(|\varepsilon(\boldsymbol{v})| + |\eta|), \tag{6.16}$$

and

$$|R_1|, |R_2| \le c(|\varepsilon(\boldsymbol{v})| + |\eta| + |\varepsilon(\boldsymbol{v}_t)|), \quad |R_3| \le c(|\varepsilon(\boldsymbol{v})| + |\eta|), \tag{6.17}$$

where in the last bound we used assumption $g \in L_\infty(\Omega^T)$. Hence, by virtue of (6.10) we have

$$\sum_{i=1}^{3} \int_{\Omega^t} |R_i||\eta| dx dt' \le c \int_{\Omega^T} \eta^2 dx dt. \tag{6.18}$$

The R_4-term is first integrated by part

$$\int_{\Omega^t} \eta R_4 dx dt' = k_0 \int_{\Omega^t} (\gamma_0 - \tilde{\gamma}_0) \nabla\tilde{\theta} \cdot \nabla\eta dx dt' + k_0 \int_{\Omega^t} \eta \nabla\tilde{\theta} \cdot \nabla(\gamma_0 - \tilde{\gamma}_0) dx dt'. \tag{6.19}$$

Utilizing (6.16), the uniform bound on $\nabla\tilde{\theta}$ and (6.10), the first term on the right-hand side of (6.19) is estimated by

$$|k_0 \int_{\Omega^t} (\gamma_0 - \tilde{\gamma}_0) \nabla\tilde{\theta} \cdot \nabla\eta dx dt'| \le \delta_2 \int_{\Omega} |\nabla\eta|^2 dx dt' + c(\delta_2) \int_{\Omega^t} \eta^2 dx dt'. \tag{6.20}$$

Similarly, in view of the bounds

$$|\gamma_{0,\varepsilon} - \tilde{\gamma}_{0,\tilde{\varepsilon}}| + |\gamma_{0,\theta} - \tilde{\gamma}_{0,\tilde{\theta}}| \le c(|\varepsilon(\boldsymbol{v})| + |\eta|), \tag{6.21}$$

which follow thanks to assumption (A5), utilizing the uniform bounds on ε, θ, $\nabla\varepsilon$, $\nabla\theta$ and $\gamma_{0,\varepsilon}$, $\gamma_{0,\theta}$, we have

$$\begin{aligned} |\nabla(\gamma_0 - \tilde{\gamma}_0| &\le |\nabla\varepsilon||\gamma_{0,\varepsilon} - \tilde{\gamma}_{0,\tilde{\varepsilon}}| + |\nabla\theta||\gamma_{0,\theta} - \tilde{\gamma}_{0,\tilde{\theta}}| + |\tilde{\gamma}_{0,\tilde{\varepsilon}}||\nabla\varepsilon(\boldsymbol{v})| + |\tilde{\gamma}_{0,\tilde{\theta}}||\nabla\eta| \\ &\le c(|\varepsilon(\boldsymbol{v})| + |\nabla\varepsilon(\boldsymbol{v})| + |\eta| + |\nabla\eta|). \end{aligned} \tag{6.22}$$

Consequently, the second term on the right-hand side of (6.19) is estimated by

$$\begin{aligned} &|k_0 \int_{\Omega^t} \eta \nabla\tilde{\theta} \cdot \nabla(\gamma_0 - \tilde{\gamma}_0) dx dt'| \\ &\le \delta_3 \int_{\Omega^t} (|\varepsilon(\boldsymbol{v})|^2 + |\nabla\varepsilon(\boldsymbol{v})|^2 + \eta^2 + |\nabla\eta|^2) dx dt' + c(\delta_3) \int_{\Omega^t} \eta^2 dx dt' \\ &\le \delta_3 \int_{\Omega^t} (\eta^2 + |\nabla\eta|^2) dx dt' + c(\delta_3) \int_{\Omega^t} \eta^2 dx dt, \end{aligned} \tag{6.23}$$

where in the last inequality we applied the bound (6.11). Finally, combining (6.13), (6.15), (6.18), (6.20), and choosing constants δ_i appropriately, we arrive at

$$\int_\Omega \eta^2 dx + \int_{\Omega^t} |\nabla\eta|^2 dx dt' \le c \int_{\Omega^t} \eta^2 dx dt, \quad t \le T. \tag{6.24}$$

Hence, by Gronwall's inequality, $\eta = 0$ in Ω^T.
Simultaneously, by inequality (6.11), it follows that $v = 0$ in Ω^T. This completes the proof. □

Appendix

Regularity results for parabolic systems

We collect here the results on regularity of solutions to parabolic systems of the form (1.14), (1.15) which are used in the existence proof.
Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be a bounded domain. First we recall the classical result due to Solonnikov [10]:

Lemma A.1. *Let us consider the problem*

$$(A1) \qquad \begin{aligned} &\boldsymbol{w}_t - \boldsymbol{Q}\boldsymbol{w} = \boldsymbol{f} && \text{in } \Omega^T, \\ &\boldsymbol{w}\big|_{t=0} = \boldsymbol{w}_0 && \text{in } \Omega, \\ &\boldsymbol{w} = 0 && \text{on } S^T. \end{aligned}$$

Assume $\boldsymbol{f} \in \boldsymbol{W}_p^{2k,k}(\Omega^T)$, $\boldsymbol{w}_0 \in \boldsymbol{W}_p^{2k-2/p}(\Omega)$, $1 < p < \infty$, $S \in C^{2k+2}$, $k \in \mathbb{N}$, and corresponding compatibility conditions. Then there exists a unique solution to (A1) such that $\boldsymbol{w} \in \boldsymbol{W}_p^{2k+2,k+1}(\Omega^T)$ and

$$(A2) \qquad \|\boldsymbol{w}\|_{\boldsymbol{W}_p^{2k+2,k+1}(\Omega^T)} \le c(\|\boldsymbol{f}\|_{\boldsymbol{W}_p^{2k,k}(\Omega^T)} + \|\boldsymbol{w}_0\|_{\boldsymbol{W}_p^{2k-2/p}(\Omega)})$$

with constant c depending on Ω, T and S.

The next result is a generalization of the Friedman-Nečas lemma [4] to the case of parabolic systems.

Lemma A.2. *Let us consider the problem*

$$(A3) \qquad \begin{aligned} &\boldsymbol{w}_t - \boldsymbol{Q}\boldsymbol{w} = \nabla\cdot\boldsymbol{\sigma} + \boldsymbol{b} && \text{in } \Omega^T, \\ &\boldsymbol{w}\big|_{t=0} = \boldsymbol{w}_0 && \text{in } \Omega, \\ &\boldsymbol{w} = 0 && \text{on } S^T, \end{aligned}$$

where $\sigma = (\sigma_{ij})_{i,j=1,\dots,n}$, $b = (b_i)_{i=1,\dots,n}$.

Assume $\sigma \in L_p(\Omega^T)$, $b \in L_p(\Omega^T)$, $1 < p < \infty$, $w_0 \in W_p^{2-2/p}(\Omega)$, $S \in C^2$. Then there exists a unique solution to (A3) such that $w \in W_p^{1,1/2}(\Omega^T)$ and

$$(A4) \qquad \|w\|_{W_p^{1,1/2}(\Omega^T)} \le c(\|\sigma\|_{L_p(\Omega^T)} + \|b\|_{L_p(\Omega^T)} + \|w_0\|_{W_p^{2-2/p}(\Omega)})$$

with a constant c depending on Ω, T and p.

Proof. We follow the arguments of [4] where the single equation is considered. First assume that $\Omega_* = \{x_n > 0\}$. Let u^i, $i = 0, 1, \cdots, n$ be the solutions of the following auxiliary BVP's:

$$(A4) \qquad \begin{aligned} & w_t^i - Qw^i = h^i && \text{in } \Omega_*^T = \{x_n > 0\} \times (0,T), \\ & w^i\big|_{t=0} = \delta^{i0} w_0 && \text{in } \{x_n > 0\}, \\ & w^i = 0 && \text{on } \{x_n = 0\} \times (0,T) \text{ for } 0 \le i \le n-1, \\ & \frac{\partial w^n}{\partial n} = 0 && \text{on } \{x_n = 0\} \times (0,T) \text{ for } i = n, \end{aligned}$$

where $h^0 = b$, $h^i = (\sigma_{ki})_{k=1,\cdots,n}$ for $i = 1, \cdots, n$, and δ^{ij}, $i, j = 0, 1, \cdots, n$ is the Kronecker delta. Then

$$w = w^0 + \sum_{i=1}^n \frac{\partial w^i}{\partial x_i}$$

satisfies

$$\begin{aligned} & w_t - Qw = \nabla \cdot \sigma + b && \text{in } \{x_n > 0\} \times (0,T), \\ & w\big|_{t=0} = w_0 && \text{in } \{x_n > 0\}, \\ & w = 0 && \text{on } \{x_n = 0\} \times (0,T). \end{aligned}$$

By virtue of Lemma A.1 we have the existence and uniqueness of solutions to problems (A4) for $i = 0, 1, \cdots, n$, and the estimates

$$\|w^i\|_{W_p^{2,1}(\Omega_*^T)} \le c(\|h^i\|_{L_p(\Omega_*^T)} + \delta^{i0}\|w_0\|_{W_p^{2-2/p}(\Omega_*)}).$$

Hence

$$\|w\|_{W_p^{1,1/2}(\Omega_*^T)} \le c(\sum_{i=0}^n \|h^i\|_{L_p(\Omega_*^T)} + \|w_0\|_{W_p^{2-2/p}(\Omega_*)}).$$

By the regularizer technique we conclude the existence of solution to (A3) in Ω^T and estimate (A4). $\square$

In case the right-hand side of (A1) belongs to spaces with fractional derivatives with respect to time the following lemma can be proved by using methods from [6].

Lemma A.3. *Let us consider problem (A1). Assume* $f \in W_p^{k,k/2}(\Omega^T)$, $w_0 \in W_p^{k+2-2/p}(\Omega)$, $k \in \mathbb{N}$, $1 < p < \infty$, $S \in C^{k+2}$, *and appropriate compatibility conditions.*

Then there exists a unique solution to problem (A1) such that $w \in W_p^{k+2,k/2+1}(\Omega^T)$ and

$$\|w\|_{W_p^{k+2,k/2+1}(\Omega^T)} \le c(\|f\|_{W_p^{k,k/2}(\Omega^T)} + \|w_0\|_{W_p^{k+2-2/p}(\Omega)})$$

with a constant c depending on Ω, T and S.

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