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On General Theory of Risk Management and Decision Support Systems

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On General Theory of Risk Management and Decision Support Systems

by

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Abstract. The paper extends the risk management theory based on the two factors utility concept introduced by Kulikowski [5, 6]. The extension concerns, in particular, the rate of return for financial as well as the human capital. The simple success – failure model is used for construction of the utility function. The utility function depends on the subjective probability, which is an explicite function of the objective probability of success. It depends also on the fear of risk quelling parameter. Using that function one can show when the utility of an unfair economic gamble or lottery can be accepted or rejected. Such an approach enables one to take into account the behavioral aspects in the decision support problems.

Key words: risk management, decision support, utility, subjective probability, allocation of capital, diversification, gambles, lotteries, bankruptcy.

1. Introduction. One can observe in recent years an increasing interest in the methodology of computerized decision support in the presence of risk. It concerns, in particular, the area of financial and human capital investments, innovations and knowledge management, which are regarded as economic growth inducing factors.

The optimization of decision support is hampered by the fact that the explicite form of the utility (i.e. the goal function) of the decision maker is generally unknown and it depends on the objective and subjective factors such as emotions (fear of bankruptcy), economic status etc.

That drawback can be eliminated by the two factors utility function proposed by Kulikowski [5,6], where the risk of the planned activity is expressed by the probability of success (p) and failure $(1-p)$. Using that concept one can define the minimum admissible probability $\bar{p}$ (such that $p \geq \bar{p}$) and propose the procedure for identification (scaling) of utility subjective parameters. Then one can construct, in the explicite form, the utility function for an individual decision maker. That function enables one to solve the optimization and management problems for risky activities.

The proposed methodology can be regarded as a continuation of the concepts which stem from the classical works of Bernoulli and later on – Von Neumann and Morgenstern [13] who have proved, in the axiomatic way, the existance of utility (scale) function for gambles (including economic behaviour). That result was generalized by Savage [11], who has introduced the concept of the subjective probability $s(p)$, while Tversky [12] has shown (experimentaly) that peoples decision to accept the gamble can be explained by using the utility function $U(x) = sx^{\beta}$, $\beta \in [0,1]$. In that model, however, the relation $s(p)$ was

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unknown. The explicite relation between the subjective (s) and objective (p) probabilities has been constructed using the concept of two factors utility function by Kulikowski [5, 6]. Using that concept one can solve effectively many problems connected with allocation of capital investments etc [7, 8, 9].

In the present paper some generalizations and extensions of that approach is also described.

2. Generalized rate of return and utility. Human activity can be viewed as a sequence of actions enabling one to achieve certain goals (get food, shelter, money, social position etc). In order to perform an action one has to engage his labour and/or financial resources. It is assumed that people evaluate the expected success and failure of each planned action using a utility scale. The utility values enable one to rank the perceived actions and choose the best one from the set of alternatives.

In order to construct a formal model of utility it is necessary to introduce the notion of rate of return, which is a measure of the effectiveness of resources engaged in an action. The financial capital P_f as well as qualified (skilled) labour, called human capital P_h, engaged in an action in time interval $\Delta t = 1$; can be expressed in monetary terms. The numerical value of P_h is determined by the average wage on the labour market. In the case of a productive activity one can assume that the market value of product created by an action ($\widetilde{P}_m$) requires engagement of P_h units of human and P_f units of financial capital. Then the expected rate of return on f & h capital can be defined as

$$\widetilde{R}=\frac{\widetilde{P}_m - P_h - P_f}{P_h + P_f} \tag{1}$$

The concept (1) can be regarded as a generalization of the rate of return, used in financial analysis (where P_h is usually neglected). When e.g. one invests $P_f = P$ in a share and sells it is a year getting $\widetilde{P}_m$ his rate of return becomes $\widetilde{P}_m / P - 1$. In the similar situation when the person with human capital valued ω gets employed (so his $P_f = 0$) and he earns in the form of wages, bonuses etc. $\widetilde{P}_m$, his rate of return becomes $\widetilde{P}_m / \omega - 1$.

The concept of rate of return can be used by an entrepreneur to evaluate different projects (actions) requiring financial capital investments, as well as – the human capital employed, which contribute to the success of his business activities.

Each employee can be characterized by an efficiency indicator, such as number of items produced or – clients served in given time etc. To evaluate the research workers a commonly used simple indicator is the number of papers published in good professional journals per year or the number of inventions, patents etc. It is possible, as well, to use indicators which are expressed by weighted sum of publications, patents, educational and administrative activities. Assuming that ω is proportional to the standard value of the indicator N, i.e. $\omega = cN$, where $c = const$, one can express the rate of return of the research worker, who has achieved $\widetilde{N}$ indicator value by $\widetilde{N} / N - 1$. It should be observed that values of $\widetilde{R}$, regarded ex post, are real numbers and when one regards them in the ex ante sense they are random variables, which characterize the outcomes in a gamble (in which one can achieve a success or a failure). Since a century many scientists, notably Irving Fisher [4], believed that people base their choices among gambles on the expected values $\overline{R} = E\{\widetilde{R}\}$ as well as on variances $\overline{V} = E(\widetilde{R} - R)^2$ of the random variable $\widetilde{R}$.

In other words constructing a simple utility scale for evaluation of the gambles people ignore the higher (than second) moments of probability density function of $\widetilde{R}$. Using such a scale the utilities of gambles with different p.d.f., but constant $\overline{R}$ and $\overline{V}$, have equal values and the gambles are regarded as utility – equivalent.

Then, in order to construct the utility scale one can introduce a simple two-point p.d.f. [Bern (p)] gamble which has two outcome:

a. success: $P_r(\widetilde{R} = R_u) = p$, b. failure $P_r(\widetilde{R} = 0) = 1 - p$;

which can be called the reference (model) gamble.

The expected rate of return of that gamble is:

(2) $$\overline{R} = pR_u$$

and the variance

(3) $$\overline{V} = p(1-p)R_u.$$

When p is unknown one can estimate it using as an estimator the relative frequency k/n, where k is the number of successful outcomes in n rounds of the gamble.

The unknown model parameters p, R_u in the general case of sequential activities can be also estimated using the historical observations of $\widetilde{R}_t$ $(t = -1, -2, \ldots -T)$ and computing

(4) $$R = \frac{1}{T}\sum_{t=-1}^{-T} \widetilde{R}_t, \quad V = \frac{1}{T-1}\sum_{t=-1}^{-T} [\widetilde{R}_t - R]^2,$$

where T - observation interval.

Then assuming $\overline{R} = R$, $\overline{V} = V$ one can derive the estimators of R_u and p:

(5) $$R_u = R\left[1 + \frac{V}{R^2}\right], \quad p = \frac{1}{1 + V/R^2}, \quad \sigma = \sqrt{V}.$$

The main idea of the present approach is based on the assumption that when constructing a utility scale people introduce a simple, utility equivalent Bern (p) - model. In other words they replace the real gamble with, generally, unknown p.d.f. by the model gamble with p and R_u estimated on the basis of historical observations.

As an example consider a shop owner whose daily returns (profits) R_t are observed for a year and the estimators R, V and R_u, p are computed. If e.g. $R = 0.15$, $\sigma = 0.1$, one gets $p = 0.692$, $R_u = 0.217$.

In the model gamble the shop owner gets each successful day the constant profit R_u and in the failure day he gets zero profit. The average profit $R = pR_u$, where p is the frequency of profitable days in all the working days of the year. When p declines (increases) the risk $V = p(1-p)R_u^2$ increases (declines) while the average profit R declines (increases).

Since R and V in both (real and utility – equivalent) gambles are equal the shop owner accepts the utility equivalent model, unless he feels that it is necessary to take into account the higher (than second) moments of p.d.f. of $\widetilde{R}_t$.

It is possible also to regard Bern (p) - model as a coding device which transforms the $\tilde{R}_t$ into the train of pulses with constant amplitude. The human brain processes the information included in the train in a way similar to the computers.

According to that concept one feels happy when the frequency of pulses is high and he feels depressed when that frequency drops below an admissible level. One does not like also when the volatility of frequency of pulses, interpreted as risk, is increasing.

Proposing the concrete utility scale for the actions characterized by given p and R_u (or σ and R) the following factors, specifying success, should be taken into account:

a) The average, long term profit $Z = PR$, where $P = P_f + R_h$ is the f & h capital engaged in the activities.

b) The short term, "worse case" profit $Y = Z - \kappa P\sigma$, where κ is the subjective weight attached to the risk $P\sigma$. The profit is manifested within the worse cases subintervals, when the risk $\sigma = \sqrt{p(1-p)}$, increases due to the decline of probability of success p.

It should be noted that in financial analysis, where P is an investment, $\kappa P\sigma$ is regarded as the monetary value of risk and denoted VaR. Using that notation one can call Y the value at safety (VaS). Value at safety is complementary notion to the value at risk, namely $VaS = Z - VaR$. When one reduces the investment risk VaR declines while VaS increases.

The utility of an investment is assumed to be an increasing function of both factors Z, Y. Since the factors are expressed in monetary terms it is convenient to scale the utility, as well, in monetary terms. Then the proposed utility should be an increasing homogeneous (of degree one) function of Z and Y, i.e.

(6) $$U = F(Z,Y)$$

Such a function, called also "constant return to scale", does not change the monetary value when one changes the monetary units of both factors (e.g. when one changes \$1 to 100c).

In practical applications it is important to deal with a concrete F functions. The most suitable form of F, proposed in [9] is the Cob-Douglas function i.e. $U = (Z)^{\beta}(Y)^{1-\beta}$, where $\beta \in [0,1]$ is a given number. Introducing the notion of index of safety

(7) $$S = 1 - \kappa\sigma / R$$

one can write

(8) $$U = (PR)^{\beta}\left[PR(1-\kappa\frac{\sigma}{R})\right]^{1-\beta} = PRS^{1-\beta}.$$

It is possible to observe that one can change the relative importance of Z, Y factors by changing the model subjective parameter β. When e.g. $\beta = 1$, $U = U_1 = Z$ so utility does not depend on safety (or risk), while for $\beta = 0$ one gets the full impact of safety S on utility: $U = U_0 = ZS$. In other words assuming $\beta = 1$ one can ignore the risk completely. Such a situation may happen when one is convinced that "winning the war", which is the strategic goal, is much more important than "lose a battle" which is the tactical goal. There are as well the abnormal situations when people under the influence of alcohol ignore the fear of failure (e.g. the soldiers when they are going to attack the enemy).

So far one did not pay attention to the impact of the size of expected return Z, which can be regarded as the source of future consumption, on the utility value. Since the more money one has, the less he values each additional increment (or – in other words – the utility of any additional dollar diminishes with an increase of capital) it is natural to assume (as the long-term goal) the value Zx, where x is the relative value of investment P to the total resources value $\overline{P}$ (or wealth) of the investor: $x = P/\overline{P}$. Then, in the case when $0 \le x \le 1$, the utility becomes

(9) $$U = F[Zx, Y] = PRS^{1-\beta}x^{\beta}.$$

Introducing the notion of subjective probability

(10) $$s = pS^{1-\beta}$$

one can write (9) in the subjective expected utility (SEU) form

(11) $$U = sPR_u x^{\beta}$$

When $s = p$ the utility $U = pPR_u x^{\beta}$, called expected utility, assumes the form postulated by axiomatic approach of Von Neumann and Morgenstern [13]. Since some people remained unconvinced by the Von Neumann & Morgenstern axioms Savage [10] presented another axiomatic theory, where he introduced the notion of subjective probability $s(p)$. However, he did not show how the subjective probability for a concrete gamble can be derived.

In the present approach one postulates the general form of subjective probability (10), where S (see (10)) includes the subjective parameter κ, describing generally the fear of failure or crises.

In order to find κ for a concrete crises such as the bankruptcy of a firm it is necessary to define the least admissible probability of success $p = \overline{p}$. The earnings at that probability $PR_u\overline{p}$ should cover the net minimum liabilities $L_m - A$, where A savings or working capital used as buffer stock to mitigate the volatility of liabilities.

Then the minimum admissible probability characterizing the worse situation

$$\overline{p} = \frac{\lambda}{R_u}, \qquad \lambda = \frac{L_m - A}{P} = \text{liability index.}$$

The utility of the firm at worse situation (and $x = 1$):

$$U_0 = P\overline{p}R_u S_0^{1-\beta}, \qquad S_0 = 1 - \kappa\frac{\sigma}{R} = 1 - \kappa\sqrt{\frac{1-\overline{p}}{\overline{p}}}$$

should not drop below the risk free investment of capital P, described by index of safety $S = 1$, i.e.

$$U_F = PR_F;$$

where R_F = rate of return of risk free investments (such as government bonds).

Equalizing U_0 to U_F one gets

(12) $$S_0 = \left(\frac{R_F}{\overline{p}R_u}\right)^{\frac{1}{1-\beta}} = (R_F/\lambda)^{\frac{1}{1-\beta}}$$

Then

(13) $$\kappa = (1 - S_0)\sqrt{\frac{1-\overline{p}}{\overline{p}}}$$

One can observe that the fear of bankruptcy κ is growing when the liability index λ (and $\overline{p}$) increases. In such a situation S and consequency the subjective probability of success, as well as the utility of the activity of the firm declines.

It should be noted that the present definition of the admissible probability of success $\overline{p}$ enables one to determine the decision maker's responsibility for the decision taken. Some times he is charged and suited for transgressing the admissible risk level, resulting in the firm bankruptcy. Since the law makers did not define the admissible level of risk, the best way to prove the decision maker is not quilty is to show that all his decisions were taken for the success probability $p > \bar{p}$.

The present financial bankruptcy model of κ can be also adopted to investments in human capital. Consider e.g. a research institute with the annual budget P spent mostly on wages of research workers. The minimum liabiblities L_m are determined by wages and costs connected with failed contracts. The rate of return R_u is determined by (4), (5) with $\tilde{R}_t$ expressed in terms of efficiency indicator (e.g. the number of publications per year). The risk free return R_F corresponds to risk free activity (e.g. the administrative activity or, bearing no risk, low quality publications). In the case when λ is growing and/or R_u declining, κ is increasing and the utility of research institute activity goes down. In order to increase the utility the success probability p should increase. That situation can be achieved by undertaking new, innovative activities (projects), which render large rate of return (R_u) and success probability $(p > \bar{p})$.

It should be noted that κ is generally a subjective parameter of an individual. It may reflect the individual liabilities and responsibilities as well as the economic status of the subject. For that reason the firm owner, the CEO and an employee can, generally, exhibit different κ when exposed to the same threat of bankruptcy.

Taking into account (7), (12), (13) one can find $\kappa(/R_F)$ and $s(p)$:

(14) $$\kappa(\lambda/R_F) = \left[1-(R_F/\lambda)^{\frac{1}{1-\beta}}\right]\sqrt{\frac{\lambda/R_F}{a-\lambda/R_F}}, \qquad a = R_u/R_F$$

(15) $$s(p) = p\left[1-\kappa(\lambda/R_F)\sqrt{\frac{1-p}{p}}\right]^{1-\beta}$$

where

$$p > \bar{p} = \frac{\lambda/R_F}{a}.$$

When for a given challenge, characterized by $\{p,> R_u, p\}$ and $\beta = \frac{1}{2}$

1. $\lambda/R_F < 1$ then $s(p) > p$,
2. $\lambda/R_F = 1$ then $s(p) = p$
3. $\lambda/R_F > 1$ then $s(p) < p$.

When $\beta = 1$, $s(p) = p$.

The plots of $\kappa(\lambda/R_F)$, $s(p)$ for $R_u/R_F = const$, $\beta = const$ are shown in Fig. 1, 2. One can observe that for small burden liabilities (expenses) $\lambda/R_F < 1$ and subjective probability s is bigger than p. When λ/R_F becomes > 1, $s(p) < p$ (The continuous change of κ is shown in Fig. 2 by dots). Such a property of the present model is supported by the experimental research results conducted with power utility $U = sx^\beta$, by A. Tversky [12], who concludes "most of the subjects over estimated low probabilities and underestimate high ones". That result, found also in many other experimental psychological studies, explaines the acceptance of the unfair (i.e. with negative expected returns) gambles, when κ is negative

$(S_0 > 1)$. Indeed, consider as an example a lottery, where one can win $\$G$ with probability p after paying $\$c$ for the right to play.

Let the expected value of the lottery

$$EV = pG - c < 0$$

and the utility of the lottery

$$U = cpR_u S^{1-\beta}, \qquad R_u = G/c - 1.$$

Suppose the players liability index λ / R_F is small so

$$\kappa = \left[1 - (R_F/\lambda)^{\frac{1}{1-\beta}}\right]\sqrt{\bar{q}} < 0, \qquad \bar{q} = \frac{\bar{p}}{1-\bar{p}} = \frac{\lambda / R_F}{a - \lambda / R_F}, \qquad a = R_u / R_F.$$

Then for small p (but $p \geq \bar{p} = \lambda / R_u$) one gets $S > 1$ and

(16) $$U > cpR_u = p[G-c] > EV$$

EXAMPLE

Let $G/c = 100$, $p = 0.0091$ so $EV = -0.09c$. Assume also $\beta = 0.5$, $R_F = 0.1$; $\lambda = 0.05$; so $\bar{p} = \lambda / R_u = 0.05/99 = 0.0005$, and $\sqrt{\bar{q}} = 0.0225$. Then $\kappa = (1-2^2)0.0225 = -0.0674$ and $S = 1 + 0.0674\sqrt{0.0091^{-1} - 1} = 1.703$.

The utility of the lottery becomes $U = 0.0091 \cdot 99\sqrt{1.703}c = 1.176c$.

The participation in the lottery is generally an arbitrary decision of the individual player. He can reject the offer to participate when he feels that it is unfair or he can accept it when he feels the offer is rational.

In the present model the decision of participation in a lottery or a game depends on the value of β, which can be called the emotions (fear) quelling parameter. Increasing (decreasing) β one quells (stimulates) the impact of emotions, manifested by rationality (i.e. the value at safety PRS) on the "fair policy" expressed by expectations (PR).

It is possible to observe that different attitudes towards fairness and rationality exist in business and social institutions. In the case of risk free investments fairness is the sole principle so when you invest $\$I$ in the risk free government bonds the government is obligated to pay you back the fair amount $\$I(1+R_F)$.

In the case of a social care or economic aid program one is not obliged to pay back the money received so the acceptance of the fairness principle is not necessary. The supporters of economic aid programs believe however that the aid is rational. They argue that the economic aid stimulates economic growth and helps the recipients of the aid to survive the crises (worse case). They believe also that after recovery the recipients of the aid will help other people who are in trouble so there is a slight probability that society will get a return. The rational policy helps people to establish extensive cooperative relations. In the fair world (i.e. the world where everybody is engaged in the fair games only) there would be probably no casinos, but at the same time there would be no insurance agencies, no social care systems, no charitable organizations etc.

It is also interesting to observe that the public opinion admires and supports those individuals who have achieved a hard success (starting with low level of safety S and objective probability p) and encourages others to undertake the challenges. Due to that encouragement the number of dedicated, successful people (i.e. inventors, national heroes, Nobel-prize winners etc) is growing and the civilization is moving ahead.

There is an important problem to determine the recommended degree of rationality (i.e. the value of β) in concrete, normal situations. There are, of course, the abnormal situations, when e.g. one has to quell the fear in order to attack the strong enemy threatening his existence (survival).

According to the physiological evidence, see e.g. Le Doux [2] the decision to quell the fear is taken in the center of human brain by quick passage of information, processed by the thalamus, to the amygdala. In the meanwhile the cortex receives information from thalamus and with more sophistication and more time relays a support or quell amygdala fearfull response.

The response is also conditioned by the information on an early subjects experience collected partly in hippocampus (declarative memory) and amygdala pathway (emotional learning). These findings support the concept of regarding β as emotions (fear of bankruptcy) quelling factor.

In the normal situation (as early as 18th century) $\beta = 0.5$ was proposed as a prototype for Everyman's utility parameter ($U(x) = \sqrt{x}$, for $x \geq 0$) and many psychologists, e.g. Stevens [11] have defended it on the basis of experimental evidence. That value of β can be regarded as a compromise between the extreme positions such as full fairness ($\beta = 1$) or full rationality ($\beta = 0$) in acceptance of lotteries or liberal and egalitarian policy in the case of allocation of financial and human capital (as will be shown in the section 3).

As a result the support of an activity depends on the challenge (characterized by R_u, p) and the two subjective factors, i.e. the awarness of the subjects economic situation (characterized by κ) and the impact of etical, political etc views (characterized by the moderate value of $\beta = 0.5$). The value $\beta = 0.5$ can be therefore recomended in normal situations, when one employes the model (11) in order to support the economic decisions.

3. Optimum capital allocation, diversification and valuation. At the planning stage the decision maker has to decide which of the activities should he continued and should he start the new innovative projects. In other words the decision maker has to decide which activities with the known R_i, S_i, $i = 1,\ldots,m$ should be accepted.

It can be assumed that each activity will be accepted if the utility $U_i(x_i)$ is not less the risk free activity $U_F(x_i)$ i.e. $U_i(x_i) \geq U_F(x_i)$, where

$$U_i(x_i) = PR_i S_i^{1-\beta} x_i^{\beta}, \qquad U_F(x_i) = PR_F x_i^{\beta}, \qquad x_i = P_i / P, \qquad \forall i$$

P_i - capital invested in i-th activity, P-total capital invested.

Then by $U_i(x_i) \geq U_F(x_i)$ one gets the following acceptance condition

(17) $$R_i \geq R_F : S_i^{1-\beta}, \qquad \forall i$$

Using the acceptance condition a part of traditional activities can be replaced by the new (innovative) activities with higher $R_i S_i^{1-\beta}$.

Now the main problem facing the decision maker is to find such strategy of allocation of capital resources $\hat{x} \equiv \{\hat{x}_1,\ldots,\hat{x}_n\}$ $\sum_i^m \hat{x}_i = 1$, which renders the maximum utility of concatenated activities. From the point of view of measurement theory it is convenient to ensure the additivity of measurement (i.e. utility) scale see e.g. [1]. For that purpose assume that the activities are statistically independent and that concatenated, under optimum strategy utility $U(\hat{x}) = PRS^{1-\beta}$ is the sum of $U_i(\hat{x}_i) = PR_iS_i^{1-\beta}\hat{x}_i^{\beta}$ $\forall i$, where the resulting $R = \sum_{i=1}^{m} \hat{x}_i R_i$, while S can be derived from the relation

$$S = \left\{ \frac{1}{R} \sum_{i=1}^{m} R_i S_i^{1-\beta} \hat{x}_i^{\beta} \right\}^{\frac{1}{1-\beta}}. \tag{18}$$

According to that procedure the decision maker has to solve the problem

$$\max_{x\in\Omega} U(x) = U(\hat{x}), \qquad \Omega = \left\{ x_i \,|\, \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0 \right\}, \qquad n \leq m,$$

where the functional $U(x) = \sum_{i=1}^{m} U_i(x_i)$ can be written in the form

$$U(y) = \sum_{i=1}^{n} a_i y_i, \qquad a_i = PR_iS_i^{1-\beta}, \qquad y_i = x_i^{\beta}, \qquad \forall i. \tag{19}$$

Using the Hölder inequality one gets

$$U(y) \leq \| a \| \| y \| = \| a \| \tag{20}$$

where

$$\| a \| = \left\{ \sum_{i=1}^{n} | a_i |^r \right\}^{1/r}, \qquad \| y \| = \left\{ \sum_{i=1}^{n} | y_i |^{1/\beta} \right\}^{\beta} = \left\{ \sum_{i=1}^{n} x_i \right\}^{\beta} = 1, \qquad r = \frac{1}{1-\beta}.$$

One gets the equality sign in (20) for $a_i^r = const\ y_i^{1/\beta} = const\ x_i$, i.e.

$$x_i = \hat{x}_i = \frac{a_i^r}{\sum_{j=1}^{n} a_j^r}, \quad \forall i, \tag{21}$$

The resulting rate of return $R = \sum_{i=1}^{n} R_i \hat{x}_i$ for $\beta = 1/2$ becomes

$$R = \sum_{i=1}^{n} R_i \frac{a_i^2}{\sum_i a_i^2} = \sum_{i=1}^{n} R_i^3 S_i : \sum_{i=1}^{n} R_i^2 S_i \tag{22}$$

while S by (18) yields

$$S = \left\{ \frac{1}{R} \sum_{i=1}^{n} R_i \sqrt{S_i} a_i : \sqrt{\sum_j a_j^2} \right\}^{1/2} = \left[\sum_j R_j^2 S_j \right]^3 : \left[\sum_i R_i^3 S_i \right]^2 \tag{23}$$

Then the resulting utility becomes

$$U = PR\sqrt{S} = P \left[\sum_i R_i^2 S_i \right]^{1/2} \tag{24}$$

One can observe that the capital allocation according to (21) depends on the metrics of l_β space, i.e. on the β parameter. In the case of small emotions quelling factor e.g. $\beta = 0$ one gets $\hat{x}_i = \frac{a_i}{\sum_{j=1} a_j} \forall i$ while for $\beta \to 1$, $\gamma \to \infty$ and the total capital is allocated to the activity with maximum $a_i = PR_i = \max_{1 \le j \le n} \{PR_j\}$.

One can observe also here the impact of allocation of capital policy on the fear quelling factor β. In the case of on egalitarian policy one can assume $\beta \approx 0$, while in the case of the effectiveness promoting (liberal) policy one should assume $\beta \approx 1$.

EXAMPLE

Consider 3 activities with $R_1 = 0.5$; $S_1 = 0.6$; $R_2 = 0.4$; $S_2 = 0.7$; $R_3 = 0.3$, $S_3 = 0.8$ and $\beta = 0.5$.

The resulting utility by (24) becomes

$$U = P[0.5^2 \cdot 0.6 + 0.4^2 \cdot 0.7 + 0.3^2 \cdot 0.8]^{1/2} = 0.578P.$$

That utility is bigger than the utilities corresponding to each activity taken alone (with $\hat{x}_i = 1$); i.e.

$$U_1(1) = 0.5\sqrt{0.6} = 0.387, \quad U_2(1) = 0.4\sqrt{0.7} = 0.335, \quad U_3(1) = 0.3\sqrt{0.8} = 0.264;$$

due to the diversification of activities and higher resulting safety S.

In the case of $\beta = 1$ one quells the fear bankruptcy and does not benefit by diversification. In that case the maximum utility one obtained when the total capital P is invested in the most effective activity $(R_1 = 0.5)$ i.e. the utility $U_1 = 0.5P$.

It should be noted that in the case when the activities are correlated one can find the best portfolio of activities using a technique proposed by Elton & Gruber [3], which enables one to solve the problem

$$\max_{x_i \in \Omega} \frac{R(x_1, x_2, \ldots, x_n)}{\sigma(x_1, x_2, \ldots, x_n)} = \frac{\hat{R}}{\hat{\sigma}}, \qquad \Omega = \left\{ x_i \,|\, \sum_{i=1}^{n} x_i, \; x_i \ge 0 \; \forall i \right\}$$

Then one can use $\hat{R}/\hat{\sigma}$ to derive the probability of success by (5): $p = \frac{1}{1 + (\hat{\sigma}/\hat{R})^2}$ and find $s = 1 - \kappa\sqrt{\frac{1-p}{p}}$, and $u = P\hat{R}S^{1-\beta}$.

The next important problem concerns the evaluation of the present value (PV) of the stream of cash flow (PR) resulting from the financial & human capital engagement within the given (T) time interval.

Using the discounting concept one should find first of all the appropriate discount rate k, which can be used for evaluation of the present value

$$PV = \sum_{t=1}^{T} \frac{PR}{(1+k)^t},$$

of continued (within $[0, T]$ interval) risky activities with given R, S.

For that purpose assume that the present value of the interests cR_F (from risk free investment of $\$c$) and the risky investment of $\$K$ with the rate of return R and discount k are equal:

$$\frac{cR_F}{1+R_F}=K\frac{R}{1+k}$$

so

(25) $$K=\frac{R_F}{R}\frac{1+k}{1+R_F}c.$$

The utilities for both investments

$$U_c=PR_F\left(\frac{C}{P}\right)^{\beta},\quad U_k=PRS^{1-\beta}\left(\frac{K}{P}\right)^{\beta},\quad P=C+K$$

should be equal so $U_c=U_k$ and

(26) $$K=(R_F/R)^{1/\beta}:S^{\gamma},\quad \gamma=1/\beta-1$$

Equalizing (25), (26) one gets

(27) $$k=(1+R_F)[R_F/SR]^{\gamma}-1.$$

When e.g. $\beta=0.5$, $S=0.5$, $R_F=0.1$, $R=0.2$ one gets

$$k=(1+R_F)R_F:SR-1=0.1.$$

As the final remark it is necessary to note that the present methodology, based on the utility concept (9), (11), has been successfully applied to many practical problems connected with ranking, acceptance of projects, allocation of capital and valuation of activities, described in the papers [5-9]. Since there is a growing interest to the risk and safety management in many areas of human activity still a lot of problems remains to be solved.

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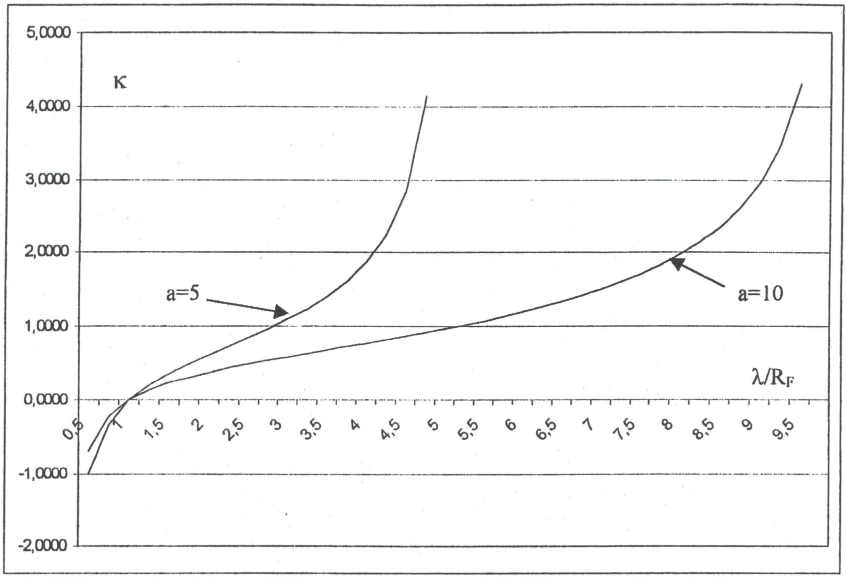

Fig. 1. The plots of $\kappa(\lambda/R_F)$ for $a=R_u/R_F=$const.

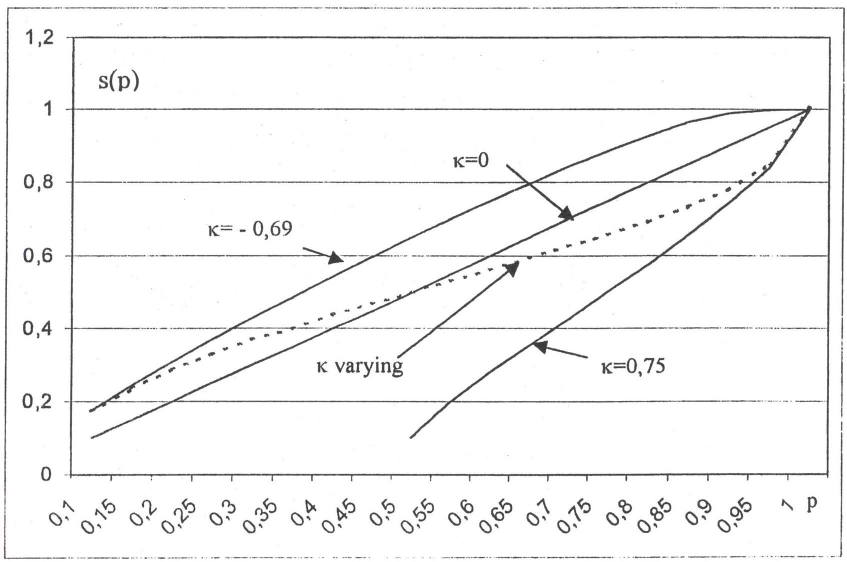

Fig. 2. The plots of s(p) for κ=const and κ varying (dotted).