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Remarks on delay approximations based on feedback

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Abstract

The response of a unity-feedback system with a delay element in the forward path exhibits a periodic component that can be approximated by truncating its harmonic expansion. Rational approximants of the transfer function e^{-Ts} of such element can simply be obtained from this closed-loop approximation. A unifying approach to recent methods based on this criterion [2], [3] is presented, which allows us to point out their respective features. The standard Padé technique and a heuristic method described in [5] are also considered.

1 Introduction and problem statement

In modelling dynamic systems for control purposes, it is often necessary to account for time delays due, e.g., to transport phenomena or distributed-parameter components.

The response of an ideal delay element (delayor) to an input $u(t)$, identically equal to 0 for $t < 0$, is $y(t) = u(t-T)$, $T > 0$, where T indicates the time delay. By denoting with $U(s)$ the Laplace transform of $u(t)$, the Laplace transform of $y(t)$ is $Y(s) = e^{-Ts}U(s)$. Therefore the transfer function of the delayor is the transcendental function e^{-Ts}.

The problem of approximating e^{-Ts} by means of a rational function has a long history (see, e.g., [1]) but is still important from both the computational and the conceptual point of view: a few recent contributions on the subject are quoted in [2]. In many practical applications the physical realizability and the stability of the approximant limit the choice of the approximant to proper rational functions with real coefficients and a Hurwitz denominator. These requirements are satisfied by Blaschke products, i.e., functions of the form:

$$B(s) = \frac{\prod_{i=1}^{n}(s-a_i)}{\prod_{i=1}^{n}(s+a_i)} \qquad \mathbb{R}e\,[a_i] > 0\,. \tag{1.1}$$

This has the desirable property that $|B(j\omega)| = |e^{-jT\omega}| = 1$, $\forall\omega$, and $\arg[B(j\omega)]$ is monotonically decreasing with ω like $\arg[e^{-jT\omega}] = -T\omega$. On the other hand, the step response of a system with transfer function $B(s)$ starts from $+1$ or -1, whereas the step response of an ideal delayor obviously starts from 0.

The most widely adopted method to form a rational approximant of a delay element is based on the Padé technique which does not always guarantee stability (even if biproper Padé models are necessarily stable). Since such a technique leads to the retention of the first Maclaurin expansion coefficients of e^{-Ts}, the resulting approximation is the best in the neighbourhood of $\omega = 0$. In different frequency bands, other types of models may be preferred.

In [3] a unity-feedback system whose forward path consists of a delayor is analysed.

In the case of negative feedback, the unit step response is a piecewise constant function taking on the value 0 for $2kT < t < (2k+1)T$ and the value 1 for $(2k+1)T < t < (2k+2)T$, $k \geq 0$, which can be decomposed into a step of amplitude $\frac{1}{2}$, and a square wave of amplitude $\frac{1}{2}$ starting from $-\frac{1}{2}$ at $t = 0$.

In the case of positive feedback, similar considerations allow us to decompose the unit step response into : a linear ramp of slope $\frac{1}{T}$, a step of amplitude $-\frac{1}{2}$, and a saw-tooth wave that linearly decreases from $\frac{1}{2}$ to $-\frac{1}{2}$ in every period from kT to $(k+1)T$.

In both cases, the periodic component can easily be expressed as a series of harmonic terms (for $t > 0$). It is therefore natural to approximate the step response of the unity-feedback system by retaining the non-periodic component together with a suitable number of the first harmonics of the periodic component.

A rational approximation $W_a(s)$ of the transcendental transfer function $W(s)$ of the above-mentioned feedback system is obtained by dividing the Laplace transform of the approximate step response by the Laplace transform $\frac{1}{s}$ of the step input. The rational approximant $G_a(s)$ of the delayor transfer function is then determined as:

$$G_a(s) = \frac{W_a(s)}{1 \mp W_a(s)}, \tag{1.2}$$

where the minus sign applies to the case of negative feedback and the plus sign to that of positive feedback. It turns out [3] that $G_a(s)$ is a stable biproper rational function having the form of a Blaschke product; precisely, negative feedback supplies even-order approximants and positive feedback produces odd-order approximants.

Obviously, the same result could be achieved by referring to different inputs (even an impulse), but the choice of the unit step is particularly convenient. According to the terminology suggested in [4], the rationale of such a procedure consists in retaining the "input component" (and the "resonant component", if any) and in truncating the periodic "system component" of the response.

In [2] a feedback structure is used as well, but another approximation criterion is adopted, which leads to different models depending on the chosen input. In particular, the family of inputs considered in [2] is : $\{u(t) = t^m, m \in \mathbb{N}, t > 0\}$ and the procedure exploits several properties of Bernoulli numbers and polynomials.

In the following, the above approaches are presented in a unified form which allows us to point out their respective features and to derive the related approximants in an easier way. Finally, criteria are given to choose the approximation that is most suited to the application at hand, also taking into account the standard Padé approximation and a further approximation presented in [5].

2 Derivation of the approximant

For the sake of simplicity, we shall almost exclusively refer to the case of negative feedback; only a brief mention will be made of the case of positive feedback.

2.1 Negative feedback

The transfer function $W(s)$ of the negative feedback system with forward-path transfer function $G(s) = e^{-Ts}$ is:

$$W(s) = \frac{G(s)}{1+G(s)} = \frac{1}{e^{Ts}+1} \tag{2.1}$$

whose singularities (poles) are the roots of $e^{Ts} = -1$, i.e.:

$$s = \pm jp_k := \pm j(2k-1)\frac{\pi}{T},\ k \in \mathbb{Z}_+ .$$

$W(s)$ can also be interpreted as the Laplace transform of the sequence of positive and negative impulses forming the *derivative* of the step response described in the Introduction. Therefore, it is the sum of a constant equal to $\frac{1}{2}$ (corresponding to the step component in the just-mentioned step response) and a series of "harmonic" terms associated with the above poles:

$$W(s) = \frac{1}{2} + \sum_{k=1}^{\infty} \left[\frac{r_k}{s - jp_k} + \frac{\bar{r}_k}{s + jp_k} \right] ,$$

where the bar denotes conjugate and, using the standard formula for the residues:

$$r_k = \lim_{s \to jp_k} (s - jp_k) W(s) = -\frac{1}{T} .$$

It follows that:

$$W(s) = \frac{1}{2} - \frac{2}{T} \sum_{k=1}^{\infty} \frac{s}{s^2 + (2k-1)^2 \frac{\pi^2}{T^2}} . \tag{2.2}$$

In order to compare the results in [2] and [3], let us consider a canonical input of the form:

$$u_i(t) = \frac{t^{i-1}}{(i-1)!} ,\ t > 0 \tag{2.3}$$

whose Laplace transform is:

$$U_i(s) = \frac{1}{s^i}$$

(in [3] only the case of $i = 1$ is considered, whereas the inputs used in [2] differ from (2.3) by a scaling factor which is irrelevant for the following considerations).

On the basis of (2.2) the Laplace transform of the (forced) response to (2.3), i.e.:

$$Y_i(s) = \frac{1}{s^i} W(s) ,$$

can be rewritten as

$$Y_i(s) = \frac{1}{s^i} \sum_{h=0}^{i-1} c_h s^h + \sum_{k=1}^{\infty} \frac{\alpha_{ki} + \beta_{ki} s}{s^2 + (2k-1)^2 \frac{\pi^2}{T^2}} ,$$

where for i even:

$$\alpha_{ki} = 0 , \qquad \beta_{ki} = (-1)^{\frac{i}{2}} \frac{2}{T} \left[\frac{T}{(2k-1)\pi} \right]^i \tag{2.4i}$$

and for i odd:

$$\alpha_{ki} = (-1)^{\frac{i-1}{2}} \frac{2}{T} \left[\frac{T}{(2k-1)\pi} \right]^{(i-1)} , \qquad \beta_{ki} = 0 . \tag{2.4ii}$$

Therefore, $W(s)$ can also be presented in the alternative form:

$$W(s) = \frac{Y_i(s)}{U_i(s)} = \sum_{h=0}^{i-1} c_h s^h + \sum_{k=1}^{\infty} \frac{\alpha_{ki} s^i + \beta_{ki} s^{i+1}}{s^2 + (2k+1)^2 \frac{\pi^2}{T^2}} \,. \tag{2.5}$$

Each term of the series in (2.5) is given by the sum of a polynomial of degree $i-1$ (quotient of the division of its numerator by its denominator) and a strictly proper rational function (whose numerator is the remainder of the division). Therefore (2.5) becomes:

$$W(s) = \sum_{h=0}^{i-1} c_h s^h + \sum_{k=1}^{\infty} \left\{ \sum_{h=0}^{i-1} d_{ki,h} s^h + \frac{\gamma_{ki} + \delta_{ki} s}{s^2 + (2k-1)^2 \frac{\pi^2}{T^2}} \right\} \tag{2.6}$$

which can be rewritten as:

$$W(s) = \sum_{h=0}^{i-1} \left(c_h + \sum_{k=1}^{\infty} d_{ki,h} \right) s^h + \sum_{k=1}^{\infty} \frac{\gamma_{ki} + \delta_{ki} s}{s^2 + (2k-1)^2 \frac{\pi^2}{T^2}} \,. \tag{2.7}$$

By comparing (2.7) with (2.2), one finds that

$$c_0 + \sum_{k=1}^{\infty} d_{ki,0} = \frac{1}{2}\,, \tag{2.8}$$

$$c_h + \sum_{k=1}^{\infty} d_{ki,h} = 0\,, \quad h > 0 \tag{2.9}$$

$$\gamma_{ki} = 0\,, \quad \forall\, k, i$$

$$\delta_{ki} = -\frac{2}{T}\,, \quad \forall\, k, i\,.$$

The procedure suggested in [3] could alternatively be presented with reference to expression (2.7) where coefficients related to the specific input appear. Precisely, the approximant $W_a(s)$ is obtained in this case by adding to the exact value $\frac{1}{2}$ of the first sum (cf. (2.8) and (2.9)) the first K (harmonic) terms of the second summation:

$$W_a(s) = \frac{1}{2} - \frac{2}{T} \sum_{k=1}^{K} \frac{s}{s^2 + (2k-1)^2 \frac{\pi^2}{T^2}}$$

which is *independent* of the input $u_i(t)$.

The procedure suggested in [2] refers instead to expressions (2.5) or (2.6), and the approximation consists in truncating the summation over k, where each addendum is formed by a polynomial and a strictly proper harmonic term. Therefore the resulting $W_a(s)$ is:

$$W_a(s) = \sum_{h=0}^{i-1} c_h s^h + \sum_{k=1}^{K} \frac{\alpha_{ki} + \beta_{ki} s}{s^2 + (2k-1)^2 \frac{\pi^2}{T^2}}\,, \tag{2.10}$$

which does depend on i and it is not proper because the part added to the harmonic terms does not reduce to the constant $\frac{1}{2}$, as is instead the case in $W(s)$. Nevertheless, the approximant $G_a(s) = W_a(s)/(1 - W_a(s))$ of e^{-Ts} turns out to be biproper.

As concerns the computation of the above approximants, the suggested approach seems to be preferable to that adopted in [2] because:

(i) coefficients c_h, which correspond to the first i Maclaurin expansion coefficients of:

$$W(s) = \frac{1}{1 + \sum_{h=0}^{\infty} \frac{(Ts)^h}{h!}},$$

can be easily be evaluated using the classic Padé procedure, and

(ii) formulae (2.4i) and (2.4ii) immediately supply coefficients α_{ki}, β_{ki}.

2.2 Positive feedback

Considerations analogous with those of Section 2.1 lead to the following transfer function in the case of positive feedback:

$$W(s) = \frac{1}{Ts} - \frac{1}{2} + \frac{2}{T} \sum_{k=1}^{\infty} \frac{s}{s^2 + \left(\frac{2k\pi}{T}\right)^2}, \tag{2.11}$$

so that $Y_i(s) = W(s)U_i(s)$ can be separated into a (harmonic) series associated with the imaginary conjugate poles of $W(s)$ and a strictly proper fraction with denominator s^{i+1}. Using the terminology in [4], the mentioned series corresponds to the "system component" of the forced response and the fraction corresponds to its "interaction component" because the poles of the latter are common to $W(s)$ and $U_i(s)$ (no "input component" is present in this case since $U_i(s)$ does not exhibit poles different from those of $W(s)$).

As shown in [3], the truncation of the series in (2.2) results in even-order biproper approximants $G_a(s)$, whereas the truncation of the series in (2.11) results in odd-order biproper approximants $G_a(s)$.

Instead, as shown in [2], truncating the series in (2.5) leads to odd-order approximants, whereas truncating the analogous series corresponding to positive feedback leads to even-order approximants.

2.3 Stability and approximation error

It has been proved [3] that the even-order rational approximations $G_a(s)$ of e^{-Ts} obtained from (2.1), as well as the odd-order ones obtained by truncating (2.11), are stable. Instead, as explicitly stated in [2] for inputs t^m, $m > 2$ (i.e., using the previous notation, $u_i(t)$ with $i > 3$) the "alternating sign of the Bernoulli numbers makes the approximation in general unstable [...] Hence, from a practical point of view, any improvement with respect to the approximants obtained in [3] is to be found with $p = 1$", i.e., $i = 2, 3$.

The approximation accuracy can be evaluated by referring, e.g., to the "closed-loop error":

$$E(s) := W(s) - W_a(s).$$

From (2.1) we get:

$$E(s) = E_1(s) := -\frac{2}{T} \sum_{k=K+1}^{\infty} \frac{s}{s^2 + p_k^2},$$

whereas from (2.10) we have:

$$E(s) = E_2(s) := \sum_{h=0}^{i-1} \sum_{k=K+1}^{\infty} d_{ki,h} s^h + E_1(s).$$

Since $E(s)$ is a complex quantity, $|E_2(s)|$ may well be smaller than $|E_1(s)|$ for certain values of s (or $j\omega$).

3 Alternative approximants

As already pointed out, the procedure suggested in [2] leads to approximants that depend on the chosen canonical input. To improve the approximation within suitable frequency bands not centred at the origin, it is reasonable to resort to non-canonical inputs whose spectrum has larger amplitude there. A simple choice corresponds, e.g., to

$$U(s) = \frac{1}{1 + 2\xi\frac{s}{\omega_n} + \frac{s^2}{\omega_n^2}},$$

in which ω_n is at the centre of the band and ξ is suitably small.

The choice of the form of the input (as well as the order of the canonical input) is somewhat arbitrary and is influenced, in practice, by empiric considerations. Therefore, it makes sense to compare the results of the above procedures with those obtained in [5] using a heuristic procedure based on the direct approximation of the phase Bode diagram of $e^{-jT\omega}$ by means of a Blaschke product $B_n(j\omega)$ of order n. For n odd, the first factor of $B_n(s)$ has the form:

$$G_1(s) = \frac{1-\tau s}{1+\tau s}, \qquad \tau > 0$$

and the others have the form:

$$G_i(s) = \frac{1 - 2\xi_i\frac{s}{\omega_{ni}} + \frac{s^2}{\omega_{ni}^2}}{1 + 2\xi_i\frac{s}{\omega_{ni}} + \frac{s^2}{\omega_{ni}^2}}, \qquad 1 > \xi_i > 0, \qquad \omega_{ni} > 0, \tag{3.1}$$

whereas for n even all factors have form (3.1).

All the considered techniques produce unit-magnitude all-pass frequency responses so that the approximation they afford can be judged with reference to the phase deviation $\Delta(j\omega)$ from $-T\omega$ only. As $\omega \to \infty$, $\Delta(j\omega) \to \infty$ in all cases. Therefore, reasonable criteria for choosing the method most suited to the specific application are: (i) the bandwidth B_ϵ where $|\Delta(j\omega)|$ is less than a specified value ϵ, or (ii) the maximum Δ_B of $|\Delta(j\omega)|$ in a prescribed band B.

By way of example, Fig. 1 shows $\Delta(j\omega)$ vs ω for the 4-th order all-pass approximants of $e^{-j\omega}$, ($T = 1$) obtained according to (2.1) with $K = 2$ (curve a), to the procedure suggested in [2] for $u_3(t) = t^2$ (curve b), to the standard Padé procedure (curve c), and to the heuristic method in [5] (curve d). For instance, with reference to criterion (i) above, the Padé approximant is best for ϵ very small, the method suggested in [2] is optimal for $\epsilon \simeq 10°$, and the heuristic method and the method suggested in [3] are preferable for $\epsilon \geq 45°$.

Analogous results are obtainable for approximants of different order.

4 Conclusions

The approximation procedure presented in [2] and [3] have been embedded in a unified frame which points out well their respective features and allows us to determine the parameters of the approximants in an easier way. Criteria have been provided for choosing the approximation method that is most suited to the specific application.

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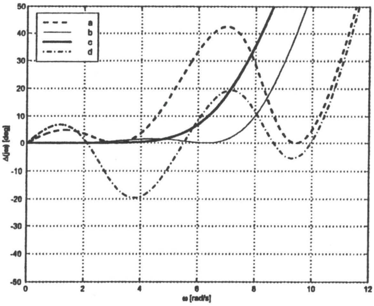

FIG. 1. Phase deviations $\Delta(j\omega)$ for the considered 4-th order approximants.

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