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**Group judgement.
Some applications to decision
problems in transport**

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Group judgement. Some application to decision problems in transport

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Abstract

Long term planning of transport development is a complex multicriteria decision problem, in which aspects of risk and indeterminacy are to be taken into account. Such problems cannot be solved well with the use of formalized deterministic or stochastic mathematical programming methods. It is due to the fact that beyond models built on the basis of such an approach there is a lot of factors that are very difficult to be formalized (e.g. landscape or social impacts), but have to be taken into consideration in the planning process.

Introduction

Transport development planning is a complex decision task to be solved under conditions of risk and indeterminacy. The first of factors mentioned is related to problems in which random variables to be considered have known or unknown probability distributions. In the former case methods of probability calculus are to be applied, in the later one has to use methods of mathematical statistics. The second factor makes both approaches meaningless. This is the case of events of unique and unitary character; such an assumption makes the probability concepts useless.

Transport investments (highways, high-speed railway systems, airports, bridges, etc.) are of this kind. Moreover, in cases under consideration it is difficult to formulate a fully formalized criterion taking into account interests of all the parties involved in planning and implementation of transport development projects. An exemplification of such a situation are difficulties with localization of the Warsaw segment of A-2 highway.

To solve such problems one has to make the use of expert judgements. Experts present opinion on the basis of their knowledge and experience. These opinions are then aggregated to form so called group judgements (Fig. 1).

Given a set of elements $\mathcal{O} = \{O_1, \dots, O_n\}$ (technologies, routes, projects, etc), a group of K experts, $\mathcal{E} = \{E_1, \dots, E_K\}$ and a criterion (set of criteria) $\mathcal{Q} = \{Q_1, \dots, Q_q\}$.

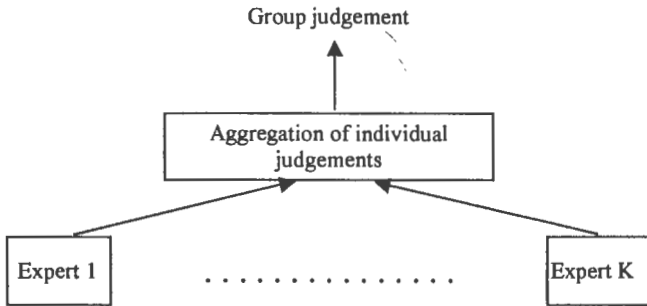


Fig. 1. Process of determining a group judgement

The process of determining a group judgement could be also described by means of input-output approach. Input data consists of the set of elements \mathcal{O} and the set of criteria \mathcal{Q} . The task to be accomplished by an expert is:

- (i) to order the set of elements \mathcal{O}
 - (ii) to classify the elements of this set, when the number of classes is given with respect to the chosen criterion \mathcal{Q} .
- If the number of the elements of the set is equal to the number of classes, both tasks (i) and (ii) are equivalent.

The very simple example of classification is the division of the set \mathcal{O} into two classes - the best elements - chosen due to an adopted criterion and others. The output data is the group judgement determined according to the accepted rule of aggregation (Fig. 1).

If the number of elements is too large, then the task of determining a preference order of elements of the set \mathcal{O} becomes very difficult. In such a case the problem to be solved by experts is simplified - they are asked to perform pairwise comparisons only. The order of elements can be given in the order as well as in the number scale. The use of the order scale means that the experts present their judgements, taking into account the criterion \mathcal{Q} , using the terms of comparisons: "better", "worse", "equivalent". The use of the number scale means that experts should qualify how much (or to what extent) a given element is better (or worse) than other ones. This distinction sometimes leads to a misunderstanding, because numbers are often used to define the rank of an element in a preference order. However, these numbers do not define any quantitative relation among elements.

In Fig. 2 the basic elements of the process of group judgement taken into account in the input-output schema are presented.

Input data		Task	Scale of ordering
Set of elements \mathcal{O}	$\mathcal{O} = \{O_1, \dots, O_n\}$ $O_i, O_j \in \mathcal{O}$	To determine preference order of a set of elements	The order scale
Set of criteria \mathcal{Q}	$\mathcal{Q} = \{Q_1, \dots, Q_q\}$ $Q_s, Q_t \in \mathcal{Q}$		The number scale
		To classify elements of a set	

Fig. 2. Basic elements of the process of group judgement

It should be noticed that in complex problems of group judgement relations between the set of elements and the set of criteria should be taken into consideration too. These problems are discussed in the paper of Saaty (1986).

The aim of the paper is to present problems of determining a group judgement in the case when expert opinions and a group judgement are given in the form of a preference order. Problems related to determining a group judgement in the case when the number scale is used are discussed e.g. in the paper of Wagner (1997).

Methods used to determine a group judgement in the last case mentioned can be divided into two groups. One of them is based on pairwise comparisons of elements (the origin is the method of Condorcet), in the second method information about the position of an element in a preference order is applied (the origin is the method of Borda). In the paper some problems of applying of the methods mentioned in practical situations are presented and advantages and disadvantages of both of them are discussed. The method of Kemeny's median, proposed in the papers of Kemeny (1959) and Kemeny, Snell (1960) which recently receives more and more attention, is discussed in details.

Problems to be analysed can be exemplified with the case of evaluation of the impact of construction of the Trans-Sumatra highway. This example shows the complexity of problems accompanying the solution of such tasks (Saaty, Takizawa 1986).

The Trans-Sumatra highway (TSH) was built in the late 1970's. It was originally planned that the highway would stretch from the extreme southern part to the northern one to ease the flow of goods and passengers both to and from areas within the island and between Sumatra and already developed island of Java.

After the completion of the TSH project some agencies were interested in analysing the overall impact of the highway according to the perception of the local societies. Such a study was carried out at the regional planning office of each region involved. In Fig. 3 and 4 positive and negative impacts taken into account in this analysis are shown.

Criteria	IMPACT	National		Regional
	IMPACT	Economic	Social	Others
		Time saving Resource and fund allocation Restrain in price increase Interregional trade Intraregional trade Employment Increase in government revenue	Local pride Communication Safety and reliability	Environment accessibility National security Comfort in travelling
Variants	Status quo policy	Push agriculture policy	Balanced growth policy	

Fig. 3 Positive impact of Trans-Sumatra highway

Criteria	IMPACT	National		Regional
	IMPACT	Economic	Social	Others
		Operating and maintenance cost Other mode of transportation cost	Change in life style and traditional values Forced movement of people Local employment replacement Envy of other regions	Pollution Distortion in ecosystem
Variants	Status quo policy	Push agriculture policy	Balanced growth policy	

Fig. 4 Negative impact of Trans-Sumatra highway

1. Basic notions and definitions

1.1. Pairwise comparisons

Given two elements $O_i, O_j \in \mathcal{O}$. The expert is asked to express his opinion on

- (i) the alternative O_i is better (according to a given criterion \mathcal{O}) than O_j ; this condition is written as $O_i \succ_{\mathcal{O}} O_j$
- (ii) the alternative O_i is worse (according to a given criterion \mathcal{O}) than O_j ; this condition is written as $O_i \prec_{\mathcal{O}} O_j$
- (iii) the alternative O_i is equivalent (according to a given criterion \mathcal{O}) to O_j ; this condition is written as $O_i \approx_{\mathcal{O}} O_j$

The disadvantage of this approach is the large number of comparisons to be performed $\left(\frac{n}{2}(n-1)\right)$ as well as the possibility that nontransitive judgements occur, i.e. it may happen that in an expert judgement we have $O_i \succ O_j, O_j \succ O_1$ but $O_i \prec O_1$. There are many methods that allow to determine group judgement in the case when nontransitivity of expert judgements occur (e.g. Bury, Petriczek, Wagner, 2000).

1.2. Preference orders

Given set of elements \mathcal{O} , the criterion \mathcal{O} and K experts. Each of experts is asked to determine the preference order of elements of this set. Two cases are to be distinguished:

- (i) for $O_i, O_j \in \mathcal{O}$ expert judgements may have the form $O_i \succ O_j$ or $O_j \succ O_i$ only,
- (ii) some elements may be considered as equivalent, i.e. judgements given in the form $O_i \approx O_j$ are accepted.

For the case (i) the preference order of element is as follows O_{i_1}, \dots, O_{i_k} and an element placed at the first position is regarded as the best one and that placed at the last position is regarded as the worst one, in other words the element placed at the position i_1 is better (according to the given criterion \mathcal{C}) than that placed on the position i_{t+1} , i.e. $O_{i_1} > O_{i_{t+1}}$.

For the case (ii) the preference order of elements is

$$\underbrace{O_{i_{11}}, \dots, O_{i_{1k}}}_{\text{the first position}}, \underbrace{O_{i_{21}}, \dots, O_{i_{2k}}}_{\text{the second position}}, \dots, \underbrace{O_{i_{t1}}, \dots, O_{i_{tk}}}_{\text{the } t\text{-th position}},$$

where the number of elements placed at the j -th position is equal to l_j and $\sum_{j=1}^t l_j = n$, $t \leq n$.

In the extreme case $t=1$ and all the elements of the set \mathcal{O} are considered to be equivalent.

In practice, it is very important to consider both cases discussed. Several methods of determining group judgement are known and described in literature in detail, but some of them may be applied directly (i.e. without any modification of definitions used) only for the first case mentioned. The most known are the methods of Condorcet and Borda, devised still in XVIII century. The method of Condorcet is intuitively based on the assumption that the best (according to a given group of experts) element is the element that is ranked higher than any other alternative by a majority of group members (Michaud, 1988, Nurmi, 1987, Saari, 1995, Young, Levenglick, 1978). This element is called the Condorcet winner.

In the case (i) for every set of preference orders given by experts one can form a matrix $S=[s_{ij}]$. Each element s_{ij} of this matrix is equal to the number of experts who consider the element O_i to be better than O_j with respect to the criterion \mathcal{C} . The matrix S is called the outranking matrix (Nurmi, 2000). According to the definition presented the element O_i is the Condorcet winner if $s_{ij} > K/2$ for each $i \neq j$. To avoid ambiguity it is assumed that the number of experts is odd. It is evident from the given definition of the Condorcet winner that it is based on pairwise comparisons. On the other side in the method of Borda information on the position of an element in the preference order is directly used. Namely, the element at the first position is given rank $(n-1)$, the element at the second position is given rank $(n-2)$ and so on. The element at the last position has the rank 0. Let \mathcal{G}_t^i ($i=1, \dots, n$; $t=1, \dots, n$) denotes the number of experts in whose judgement the element O_i takes the t -th position. Hence the rank of element O_i - denoted by β_B^i - is, due to Borda rule,

$$\beta_B^i = \sum_{t=1}^n \mathcal{G}_t^i, \quad 0 \leq \beta_B^i \leq K. \quad (1)$$

It is easy to prove that $K(n-1) \geq \beta_B^i \geq 0$, $i=1, \dots, n$.

Even though the rank of an element in the Borda method is determined subject to its position in the preference order, one can prove that this rank can be directly computed making use of the outranking matrix S .

Given the set of experts' preference orders $P^{(k)} = \{P^1, \dots, P^K\}$. For this set one can construct the S matrix:

$$S = \begin{array}{c|cccc} & O_1 & O_2 & \dots & O_n \\ \hline O_1 & - & s_{12} & \dots & s_{1n} \\ O_2 & s_{21} & - & \dots & s_{2n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ O_n & s_{n1} & s_{n2} & \dots & s_{nn} \end{array} \quad (2)$$

For the case considered it is clearly $s_{ij} + s_{ji} = K$.

It should be pointed out that there exists the relation between the sum of elements in a row of S matrix and the \mathcal{G}_i^t coefficients ($i=1, \dots, n; t=1, \dots, n$)

$$\sum_{j=1, j \neq i}^n s_{ij} = (n-1)\mathcal{G}_1^i + (n-2)\mathcal{G}_2^i + \dots + [n - (n-1)]\mathcal{G}_{n-1}^i + 0 \cdot \mathcal{G}_n^i = \beta_i^i \quad (3)$$

The relation (3) can be proved easily. If \mathcal{G}_1^i is the number of experts who put element O_i at the first position, then \mathcal{G}_1^i will be the component of $(n-1)$ elements s_{ij} . If \mathcal{G}_2^i is the number of experts who put element O_i at the second position, then \mathcal{G}_2^i will be the component of $(n-2)$ elements s_{ij} . As a result of this reasoning, the relation (3) is obtained

Already in the XVIII century it was known that the Condorcet winner may differ from that of Borda (Nurmi, 2000, Saari, 1995). An interesting example illustrating this observation was given by Fishburn (1973).

Example 1.

Given five elements $\mathcal{O} = \{O_1, O_2, O_3, O_4, O_5\}$. Preference orders given by five experts are as follows:

$$P^1 = \{O_4, O_5, O_1, O_2, O_3\}$$

$$P^2 = \{O_5, O_1, O_3, O_2, O_4\}$$

$$P^3 = \{O_3, O_4, O_5, O_1, O_2\}$$

$$P^4 = \{O_4, O_5, O_2, O_3, O_1\}$$

$$P^5 = \{O_5, O_2, O_1, O_4, O_3\}$$

(4)

The outranking matrix for this example is as follows

	O_1	O_2	O_3	O_4	O_5	$\sum s_{ij}$
O_1	–	3	3	2	0	8
O_2	2	–	3	2	0	7
O_3	2	2	–	2	1	7
O_4	3	3	3	–	3	12
O_5	5	5	4	2	–	16

$$\text{majority of voices} = 3 > \frac{5}{2} \quad (5)$$

It follows from the matrix S that the Condorcet winner is the element O_4 and the Borda winner is element O_5 .

Making use of this example H.Nurmi (2000) showed that the Borda method does not determine as a winner an element that is chosen as winner by all the experts except one, the only condition is that the number of elements is greater than that of experts. To prove this conclusion let us assume that $(K-1)$ experts put element O_1 at the first position and one expert put element O_2 at this position. We also assume that in judgements of $(K-1)$ experts the element O_2 is placed at the second position and the element O_1 is placed at the lowest position by the remaining one. If the element O_2 , instead of O_1 , is to be the Borda winner, then the following condition is to be satisfied

$$(K-1)(n-1)+1 \cdot 0 < (K-1)(n-2)+1 \cdot (n-1).$$

This condition holds true for $K < n$.

In the case of Condorcet method one has to take into account the possibility that so called Condorcet paradox can occur. The very simple example of this situation is as follows.

Example 2.

Given preference order:

$$P^1 = \{O_1, O_2, O_3\}$$

$$P^2 = \{O_2, O_3, O_1\}$$

$$P^3 = \{O_3, O_1, O_2\}$$

(6)

In this example in distinct preference orders a given element is located at distinct positions.

The outranking matrix has the form

$$S = \begin{array}{c|ccc|c} & O_1 & O_2 & O_3 & \\ \hline O_1 & - & 2 & 1 & \\ \hline O_2 & 1 & - & 2 & \\ \hline O_3 & 2 & 1 & - & \\ \hline \end{array}, \text{ majority of experts} = 2 > \frac{3}{2}$$

(7)

Hence the Condorcet winner does not exist for this case.

Problems encountered when applying the methods of Condorcet and Borda were the reason that new methods of group judgement determining have been developed. Among others are those of plurality, Copeland, Haare, Coombs, Nanson, Pareto, French election, Banks' chains, Black. An excellent review of these methods is given in the book by H.Nurmi (1987).

Due to the definition of Condorcet paradox presented above one can suppose that if the experts' judgements creating the paradox were removed or added, the group judgement determined with the use of Condorcet method should not change. H.Nurmi (2000) showed that this conclusion is not true.

Example 3.

Given following judgements of eleven experts

$$P^1, \dots, P^7 = \{O_1, O_2, O_3\}$$

$$P^8, \dots, P^{11} = \{O_2, O_3, O_1\}$$

(8)

The outranking matrix is as follows

$$S = \begin{array}{c|ccc|c} & O_1 & O_2 & O_3 & \sum s_{ij} \\ \hline O_1 & - & 7 & 7 & 14 \\ \hline O_2 & 4 & - & 11 & 15 \\ \hline O_3 & 4 & 0 & - & 4 \\ \hline \end{array} \text{ majority of experts} = 6 > \frac{11}{2}$$

(9)

Hence the Condorcet winner is element O_1 ; the Borda winner is element O_2 .

Assume now that 12 new persons join the group of experts and that their judgements are as follows:

$$P^{12}, \dots, P^{15} = \{O_1, O_3, O_2\}$$

$$P^{16}, \dots, P^{19} = \{O_2, O_1, O_3\}$$

$$P^{20}, \dots, P^{23} = \{O_3, O_2, O_1\}$$

(10)

These judgements form the Condorcet paradox. After taking them into account the outranking matrix is as follows

$$S = \begin{array}{c|ccc|c} & O_1 & O_2 & O_3 & \sum s_{ij} \\ \hline O_1 & - & 11 & 15 & 26 \\ \hline O_2 & 12 & - & 15 & 27 \\ \hline O_3 & 8 & 8 & - & 16 \\ \hline \end{array} \text{ majority of experts} = 12 > \frac{23}{2}$$

(11)

Now, the Condorcet winner is the element O_2 , which remains also the Borda winner.

The conclusion that group judgement derived with Borda method does not change after adding or removing opinions that create the Condorcet paradox, seems to be obvious if one takes into account how the ranks of the elements are determined in this method.

It also seems to be rational a suggestion that adding or removing the same number of opposed preferences should not affect group judgement.

Given a preference order P^1

$$P^1 = \{O_{i_1}, O_{i_2}, \dots, O_{i_{n-1}}, O_{i_n}\} \quad (12)$$

the following preference order is regarded as opposed one

$$\bar{P}^1 = \{O_{i_n}, O_{i_{n-1}}, \dots, O_{i_2}, O_{i_1}\} \quad (13)$$

D.G.Saari (1995) proposed the following. When analysing experts' judgements in order to determine the group judgement one can decompose them into three components: the first related to the Condorcet paradox, the second representing a reversal part and the third called basic. Nurmi (2000) pointed out that the sequence in which the decomposition is performed affects the group judgement. Nevertheless such decomposition provides many information on problems associated with determining a group judgement. Saari (1995) proved this conclusion distinctly in his book.

Let $2K_1$ denotes the number of experts having opposed opinions. The outranking matrix is as follows

$$S = \begin{array}{c|cccc|c} & O_{i_1} & O_{i_2} & \dots & O_{i_{n-1}} & O_{i_n} & \sum s_{ij} \\ \hline O_{i_1} & - & K_1 & \dots & K_1 & K_1 & (n-1)(K_1-1) \\ \hline O_{i_2} & K_1 & - & \dots & K_1 & K_1 & (n-1)(K_1-1) \\ \hline \vdots & \vdots & \vdots & - & \vdots & \vdots & \vdots \\ \hline O_{i_{n-1}} & K_1 & K_1 & \dots & - & K_1 & (n-1)(K_1-1) \\ \hline O_{i_n} & K_1 & K_1 & \dots & K_1 & - & (n-1)(K_1-1) \end{array} \quad (14)$$

It follows from this example that adding or removing opposed opinions does not affect the rank of an element determined with the Borda method. Saari (1995) proved that this is the case only for group judgement determined on the basis of the position of an element in preference orders given by experts.

Nurmi (2000) approved this conclusion applying the plurality method to the Example 3. For the plurality of experts the winner is element O_1 because it is located at the first position by 7 experts. If now we add judgements of 4 experts - $P^{12}, \dots, P^{15} = \{O_2, O_1, O_3\}$ as well as opposed judgements - $P^{16}, \dots, P^{19} = \{O_3, O_1, O_2\}$ of 4 experts too, then the plurality winner is element O_2 placed at the first position by 8 experts; element O_1 is considered as the first by 7 experts only.

The Condorcet method, based on pairwise comparisons, was till now regarded as the model method of determining group judgement. However, it can be applied only when the Condorcet winner exists. Recent papers of Nurmi (2000), Saari (1995) and Saari, Merlin (2000) question this conclusion.

As mentioned in Introduction a method to which more and more attention is paid is Kemeny's median. The initial interest was seriously diminished when computing problems of applying Kemeny's median to determine group judgement were encountered. However, the properties of the median were the reason of renewed interest in this method (Saari, Merlin, 2000) and (Young, Levenglick, 1978). In Bury, Petriczek, Wagner (1999) it was shown that numerical difficulties related to determining of the median could be overcome when applying some heuristic algorithms.

2. Kemeny's median

The classical definition of Kemeny's median is related to pairwise comparisons. Litvak (1982) proposed a new definition that refers to the notion of preference vectors. Both approaches will be presented here.

2.1. Kemeny's median defined with the use of pairwise comparisons

Given a preference order of n alternatives presented by the k -th expert ($k=1, \dots, K$) $O_{i_1}, O_{i_2}, \dots, O_{i_n}$.

For this preference order the following matrix of pairwise comparisons can be constructed

$$A^k = \begin{bmatrix} a_{11}^k & \dots & a_{1n}^k \\ \vdots & \ddots & \vdots \\ a_{n1}^k & \dots & a_{nn}^k \end{bmatrix}, \text{ where } a_{ij}^k = \begin{cases} 1 & \text{for } O_i > O_j \\ 0 & \text{for } O_i \approx O_j \\ -1 & \text{for } O_i < O_j \end{cases} \quad (15)$$

Definition 1. (Litvak, 1982) Assume that two preference orders P^1 and P^2 are given. The distance between these two preference orders can be expressed as follows

$$d(P^1, P^2) = \sum_{i=1}^n \sum_{j=1}^n |a_{ij}^1 - a_{ij}^2| \quad (16)$$

It can be proved that the distance defined in such a way satisfies all the axioms determining the measure of "closeness" in a unique way (Litvak, 1982).

Given a set of preference orders $\{P^k\} = (P^1, \dots, P^K)$. The distance of some preference order P from this set is defined as follows (Litvak, 1982)

$$d(P, P^{(k)}) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^K d_{ij}(P, P^{(k)}) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^K d_{ij}^k = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^K |a_{ij}^k - a_{ij}^P| \quad (17)$$

Taking into account that according to (15)

$$|a_{ij}^k - a_{ij}^P| + |a_{ji}^k - a_{ji}^P| = 2|a_{ij}^k - a_{ij}^P| = 2|a_{ji}^k - a_{ji}^P| \quad (18)$$

the expression (17) can be rewritten as follows (Bury, Petriczek, Wagner, 1999)

$$d = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^K |a_{ij}^k - a_{ij}^P| = \sum_{\substack{i, j \\ (i,j) \in I_p^{(1)}}} \sum_{k=1}^K |a_{ij}^k - a_{ij}^P| + \sum_{\substack{i, j \\ (i,j) \in I_p^{(2)}}} \sum_{k=1}^K |a_{ij}^k - a_{ij}^P| \quad (19)$$

where

$I_p^{(1)}$ - the set of indices (i,j) for which $O_i > O_j$ in the preference order P or - in other words - the set of indices for which $a_{ij}^P = 1$.

$I_p^{(2)}$ - the set of indices (i,j) for which $O_i \approx O_j$ in the preference order P or - in other words - the set of indices for which $a_{ij}^P = 0$.

Assume that in the preference order P $O_i > O_j$, i.e. $a_{ij}^P = 1$. In order to determine the distance of this preference order from a given set $\{P^k\}$, one can make use of coefficients defined as follows (Litvak, 1982)

$$r_{ij} = \sum_{k=1}^K d_{ij}(P, P^{(k)}) = \sum_{k=1}^K |a_{ij}^k - a_{ij}^p| = \sum_{k=1}^K |a_{ij}^k - 1| \quad (20)$$

They are called the loss coefficients and the matrix $R=[r_{ij}]$ is called the loss matrix. It is assumed that $r_{ij}=0$ for all $i=j$.

It should be noted that elements of the matrix R depend upon the form of preference orders P^k ($k=1, \dots, K$) only.

Making use of the coefficients r_{ij} ($i, j = 1, \dots, n$) the formulae (19) can be rewritten in the following form Bury, Petriczek, Wagner (2000):

$$d = \frac{1}{2} \sum_i \sum_{\substack{j \\ (i,j) \in I_p^{(1)}}} r_{ij} + \sum_i \sum_{\substack{j \\ (i,j) \in I_p^{(2)}}} \sum_{k=1}^K |a_{ij}^k| \quad (21)$$

Definition 2. (Litvak, 1982) A preference order P^M such that

$$M(P^1, \dots, P^K) = \arg \min_P d(P, P^{(k)}) \quad (22)$$

is called the Kemeny median of a set (P^1, \dots, P^K) .

In other words it is a preference order that in sense of the distance (17) is the "closest" one to all the preference orders of the set $\{P^k\}$.

The introduced definition of the Kemeny median brings about the necessity of considering two cases:

- i) there are no equivalent alternatives in the median
- ii) equivalent alternatives can occur in the median.

In the first case $I_p^{(2)} = \{i, i\}$, $i=1, \dots, n$. Hence it follows from (19) that

$$d = \sum_{(i,j) \in I_p^{(1)}} r_{ij} \quad (23)$$

This result was obtained by Litvak (1982).

This relation can be used to determine the lower bound H of the distance (19). Litvak has shown that the following theorem is satisfied.

Theorem 1 (Litvak, 1982).

$$H = \sum_{i=1}^n \sum_{j=1}^n \min(r_{ij}, r_{ji}) \quad (24)$$

It is easy to prove (Bury, Petriczek, Wagner, 1999) that

$$r_{ji} = 2K - r_{ij}, \quad i < j \quad (25)$$

Hence, if follows from the expressions (24) and (25) that the lower bound H is equal to the sum of loss coefficients less or equal K located under or above the matrix diagonal. If $r_{ij}=r_{ji}$, then in the sum H a given element occurs only once.

Litvak (1982) has proved theorems that can be of some help when determining the Kemeny median in the first of cases considered.

Theorem 2 (Litvak, 1982). If the loss matrix corresponding to a given set $\{P^k\}$ is consistent, i.e. $r_{ij} \leq r_{ji}$, $r_{jl} \leq r_{lj} \Leftrightarrow r_{il} \leq r_{li}$, then a preference order $(O_{i_1}, \dots, O_{i_n})$ is the Kemeny median if and only if

$$r_{i_1, i_2} \leq r_{i_2, i_3}, \quad \forall i \in \{1, \dots, n-1\} \quad (26)$$

Definition 3 (Litvak, 1982). A set $\{P^k\}$ of preference orders has the Condorcet property if and only if there exists the Condorcet winner for any subset of this set.

Theorem 3 (Litvak, 1982). If the loss matrix corresponding to a given set $\{P^k\}$ is consistent, then this set has the Condorcet property. The converse of this theorem also holds true.

Theorem 4 (Litvak, 1982). If the loss matrix corresponding to a given set $\{P^k\}$ is consistent then a preference order $(O_{i_1}, \dots, O_{i_n})$ is the Kemeny median if and only if

$$r_{i_v, i_{v+1}} \leq r_{i_{v+1}, i_v}, \quad v \in \{1, \dots, n-1\} \quad (27)$$

Theorem 5 (Litvak, 1982). If a given set of preference orders $\{P^k\}$ has the Condorcet property, then the Kemeny median of this set is formed by the Condorcet winners determined for subsequent subsets of $\{P^k\}$ and the distance (19) is equal to its lower bound.

2.1.1 Heuristic algorithms for determining Kemeny's median

2.1.1.1. Litvak's algorithm

Steps of the algorithm are as follows. First, one has to find such alternative, say O_{i_1} , for

which $\sum_{j=1}^n r_{ij}$ is minimal. This alternative takes the first position. Then in the matrix R the row

and column corresponding to this alternative are deleted. The iteration process is repeated until an element taking the last position is determined. In order to verify whether obtained preference order is the Kemeny median one has to apply Theorem 2. The verification procedure should start from an alternative taking the last position. It should be emphasized that the algorithm described is based on the assumption that a matrix R is consistent.

2.1.1.2 Modified algorithm

A proposed algorithm consists of the following steps:

Step 1. For given expert judgements P^k ($k=1, \dots, K$) the pairwise comparisons matrices A^k are

determined, the expressions $\sum_{k=1}^K |a_{ij}^k|$ $k=1, \dots, K$, $k=1, \dots, K$, $i < j \leq n$, $i=1, \dots, n-1$ or $j=i+s$, $s=1, \dots, n-i$ are derived and the matrix R of r_{ij} coefficients (20) is computed.

Step 2. For each of two pairs (i, j) and (j, i) the coefficients $r_{ij}^* = \min(r_{ij}, r_{ji})$ are determined.

The pair of elements (i, j) for which $r_{ij} = r_{ij}^*$ is denoted as $(i, j)^*$.

Step 3. A matrix $T=[t_{ij}]$ of elements $t_{ij} = \begin{cases} 1 & \text{such that } r_{ij} = r_{ij}^* \\ 0 & \text{otherwise} \end{cases}$

and a vector $t_i=[t_{i1}, \dots, t_{in}]$ of elements $t_i = \sum_{j=1}^n t_{ij}$ is determined.

The value of $t^* = \max(t_1, \dots, t_n)$ is also computed.

Step 4. The numbers of elements for which $t_i = t^*$ are determined. The number of such alternatives is denoted as l_1 . These alternatives are then placed according to their growing numbers on the positions from 1 to l_1 .

Step 5. The numbers of alternatives for which $t_i = t^* - \tau_j$; $\tau_j = 1, \dots, t^*$ are determined. The number of such alternatives is denoted as l_j . These alternatives are then ordered

according to their growing numbers on the positions from $\sum_{w=1}^{j-1} l_w$ to $\sum_{w=1}^j l_w$. This procedure is repeated until the positions of all the alternatives are determined.

Step 6. For alternatives placed on positions from $\sum_{w=1}^{j-1} l_w$ to $\sum_{w=1}^j l_w$, they are denoted as $O_{i_{j1}}, \dots, O_{i_{jj}}$, the submatrix of matrix T of these alternatives is determined and the elements $t_{i_{jz}} = \sum_{v=1}^{l_j} t_{i_{jz}i_{jv}}$, $z=1, \dots, l_j$ are determined.

Step 7. The value of $t_{ij}^* = \max(t_{ij_1}, \dots, t_{ij_{l_j}})$ is determined.

Step 8. For alternatives $O_{i_{j1}}, \dots, O_{i_{j_{l_j}}}$ the steps 4°, 5° are repeated until the preference order of all the elements is determined. This procedure terminates when the order of only two alternatives is to be determined. If for these alternatives $r_{i_{jz}i_{jv}} = r_{i_{jv}i_{jz}}$ then the sequence $O_{i_{jz}}, O_{i_{jv}}$ as well as the sequence $O_{i_{jv}}, O_{i_{jz}}$ may occur in the median.

Step 9. The step 8° is repeated for all the groups of alternatives for which $l_j > 1$, $j=s < n$.

During each of the steps 4° to 9° it is verified whether for given order of alternatives O_{i_1}, \dots, O_{i_n} the condition $r_{i_t i_y} = r_{i_y i_t}^*$, $n > y > t$, $t=1, \dots, n-1$, holds true. If for alternatives O_{i_t} and O_{i_y} it is not the case, then the whole sequence of alternatives $(O_{i_1}, \dots, O_{i_n})$ is subject to further analysis.

Having the median computed one has to check whether introduction of equivalent alternatives accomplished according to the relation (19) would decrease the value of the median.

Example 4.

Given the set of 5 elements O_1, O_2, O_3, O_4, O_5 and the set of 11 preference orders determined for those alternatives:

	O_1	O_2	O_3	O_4	O_5
P^1	4	1	3	3	2
P^2	3	2	3	3	1
P^3	3	2	3	1	2
P^4	2	3	1	2	2
P^5	1	2	3	2	3
P^6	2	3	2	1	2
P^7	3	2	1	3	4
P^8	4	2	1	3	4
P^9	2	1	4	2	3
P^{10}	1	3	2	2	1
P^{11}	1	1	2	3	1

The matrix of loss coefficients R is

	O_1	O_2	O_3	O_4	O_5
O_1	0	13	11	12	11
O_2	9	0	10	9	10
O_3	11	12	0	11	14
O_4	10	13	11	0	9
O_5	11	12	8	13	0

T matrix is as follows

	O_1	O_2	O_3	O_4	O_5	t_i
O_1	0	0	0	0	0	0
O_2	1	0	1	1	1	4
O_3	1	0	0	0	0	1
O_4	1	0	1	0	1	3
O_5	1	0	1	0	0	2

The values of r_{ij}^* coefficients are as follows

(i, j)	r_{ij}	r_{ji}	r_{ij}^*	$(i,j)^*$	$\min r_{ij} - \sum_{k=1}^K a_{ij}^k $
1, 2	13	9	9	2, 1	-1
1, 3	11	11	11	3, 1	3
1, 4	12	10	10	4, 1	3
1, 5	11	11	11	5, 1	5
2, 3	10	12	10	2, 3	-1
2, 4	9	13	9	2, 4	-1
2, 5	10	12	10	2, 5	1
3, 4	11	11	11	4, 3	3
3, 5	14	8	8	5, 3	-1
4, 5	9	13	9	4, 5	-1

H=98

One can then rewrite the R matrix

	O_2	O_4	O_5	O_1	O_3	$\sum_{j=1}^n r_{ij}$
O_2	0	9	10	9	10	38
O_4	13	0	9	10	11	30
O_5	12	13	0	11	8	19
O_1	13	12	11	0	11	11
O_3	12	11	14	11	0	

$\Sigma=98$

The resulting order of elements is O_2, O_4, O_5, O_3, O_1 and the distance is 98.

If equivalent elements can occur in the median the distance can be improved e.g. if the order of elements is $O_2, O_4, (O_1, O_3, O_5)$, then the distance is 91.

2.2 Kemeny's median defined with the use of preference vector notion

Definition 4 (Litvak, 1982). The preference vector related to a preference order P^k is defined as follows

$$\pi^k = \{\pi_1^k, \dots, \pi_n^k\}, \quad k=1, \dots, K \quad (28)$$

where π_i^k is equal to the number of alternatives that precede the i -th one in the preference order under consideration.

Definition 5 (Litvak, 1982). Given two preference vectors π^{k_1} and π^{k_2} . A measure of the distance between these two vectors is defined as follows

$$d(\pi^{k_1}, \pi^{k_2}) = \sum_{i=1}^n |\pi_i^{k_1} - \pi_i^{k_2}| \quad (29)$$

It can be proved that so defined measure satisfies all axioms necessary to determine in a unique way the measure of preference vector closeness (Litvak, 1982).

Definition 6 (Litvak, 1982). Given a set of preference orders $\{P^k\}$. The distance of some preference order P from this set is given by

$$d(\pi, \pi^{(k)}) = \sum_{i=1}^n \sum_{k=1}^K |\pi_i^P - \pi_i^k| \quad (30)$$

Definition 7 (Litvak, 1982). A preference order P^M such that

$$M(P^1, \dots, P^K) = \arg \min_{\pi} \sum_{k=1}^K d(\pi, \pi^{(k)}) \quad (31)$$

is called the Kemeny median of a set $\{P^1, \dots, P^K\}$.

In the papers Bury, Petriczek, Wagner (1999) and Bury, Wagner (2000) results presented by Litvak (1982) were generalized to facilitate the computation of the distance (30).

Let us introduce the following notation

$$\rho_i^{k(j)} = |\pi_i^P - \pi_i^k| = |\pi_i^P - \pi_{i(j)}^k| \quad i=1, \dots, n; \quad k=1, \dots, K \quad (32)$$

where π_i^k is the number of preceding alternatives in the case when the i -th alternative takes the j -th position in the preference order P .

Summing coefficients $\rho_i^{k(j)}$ over k ($k=1, \dots, K$) we obtain $\rho_i^{(j)}$

$$\rho_i^{(j)} = \sum_{k=1}^K \rho_i^{k(j)} \quad (33)$$

This coefficient represents the aggregated difference among the position of i -th alternative in the preference order P and its positions in P^k ($k=1, \dots, K$).

Using this notation one can rewrite the expression (30) as follows

$$d = \sum_{i=1}^n \sum_{k=1}^K |\pi_i^P - \pi_i^k| = \sum_{i=1}^n \sum_{j=1}^s \rho_i^{(j)} x_{ij}, \quad (34)$$

where

$$x_{ij} = \begin{cases} 1 & \text{if the } i\text{-th alternative takes the } j\text{-th position in a preference order } P \\ 0 & \text{otherwise} \end{cases}$$

and s - the number of positions in the preference order under consideration.

Two cases are to be considered

- i) there are no equivalent alternatives in the median. In this case the numbers of position and alternatives are equal, i.e. $s=n$.
- ii) equivalent alternatives can occur in the median. In this case it is assumed that the number of alternatives located at the j -th position is equal l_j . Hence $\sum_{j=1}^s l_j = n$.

In the first case considered we have (Bury, Petriczek, Wagner, 2000)

$$d = \sum_{i=1}^n \sum_{j=1}^n \rho_i^{(j)} x_{ij} \quad (35)$$

The lower bound of this distance is as follows (Bury, Petriczek, Wagner, 2000)

$$G = \sum_{i=1}^n \rho_{i \min}, \quad \text{where } \rho_{i \min} = \min_j [\rho_i^{(1)}, \dots, \rho_i^{(n)}] \quad (36)$$

In the case under consideration the problem of determining the Kemeny median can be defined as the following zero-one integer programming problem (Litvak, 1982)

$$\sum_{i=1}^n \sum_{j=1}^n \rho_i^{(j)} x_{ij} \rightarrow \min, \quad \text{subject to} \quad (37)$$

$$\sum_{j=1}^n x_{ij} = 1, \quad i = 1, \dots, n, \quad \sum_{i=1}^n x_{ij} = 1, \quad j = 1, \dots, n \quad (38)$$

Having computed $\rho_i^{(j)}$, we can make use of them also in the second case discussed. In this case we have to introduce the following modification:

$$\min_{\substack{x_{ij} \in \{0,1\} \\ i=1, \dots, n \\ j=1, \dots, l_j}} \sum_{i=1}^n \sum_{t=1}^{l_j} \rho_i^{(t)} x_{it}, \quad (39)$$

subject to

$$\text{for each } i, i=1, \dots, n \sum_{t=1}^s x_{it} = 1 \text{ (i-th alternative can be located only at one position)} \quad (40)$$

$$\text{for each } i, t=1, \dots, s \sum_{i=1}^n x_{it} = l_t \text{ (} l_t \text{ alternatives are located at the t-th position)} \quad (41)$$

It can be shown that $\rho_i^{(1)}$ can be computed in the following way

$$\rho_i^{(1)} = \rho_i^{(j_t)}, \quad \text{where } j_t = 1 + \sum_{w=1}^{t-1} l_w \quad (42)$$

It follows from the reasoning presented that the optimisation problem (39) is a parametric one, because to solve it one has to determine the values of parameters s and l_1, \dots, l_s .

It should be noted that the Kemeny median (derived using both definitions) is not unique. The following example justifies this conclusion.

Example 5.

Let us assume that the set $\{P^k\}$ consists of four preference orders and they are as follows

P^1 : O_2, O_4, O_1, O_3 ;

P^2 : $O_1, (O_3, O_4), O_2$;

P^3 : $(O_2, O_3), O_4, O_1$;

P^4 : $O_3, O_2, (O_1, O_4)$;

The notation (O_i, O_j) means that elements in brackets are equivalent.

Let us consider two preference orders

P^{M_1} : O_3, O_2, O_4, O_1 ;

P^{M_2} : O_2, O_3, O_4, O_1 ;

$$d(P^{M_1}, P^{(k)}) = 4 + 4 + 2 + 5 = 15$$

$$d(P^{M_2}, P^{(k)}) = 4 + 4 + 2 + 5 = 15$$

It can be shown that the minimal value of the distance (31) is equal 15. So both the preference orders P^{M_1} and P^{M_2} are the Kemeny median in the sense of Definition 6. Hence, it is not unique.

2.2.1 Heuristic algorithm for determining Kemeny's median (case (i))

Given the matrix $[\rho]$ with elements $\rho_i^{(j)}$, $j=1, \dots, n$; $i=1, \dots, n$.

Let $\rho_{\min}^{(j)}$ denotes the minimal value of $\rho_i^{(j)}$ in the j -th column

$$\rho_{\min}^{(j)} = \min_i \rho_i^{(j)}, \quad j=1, \dots, n. \quad (43)$$

$$\text{A new variable } \delta_i^{(j)} \text{ is defined as follows} \quad \delta_i^{(j)} = \rho_i^{(j)} - \rho_{\min}^{(j)}, \quad j=1, \dots, n; \quad i=1, \dots, n. \quad (44)$$

Using this variable the problem of determining the median (34) can be formulated in the following way

$$\min_{x_{ij}} \sum_{i=1}^n \sum_{j=1}^n x_{ij} \rho_i^{(j)} = \min_{x_{ij}} \sum_{i=1}^n \sum_{j=1}^n x_{ij} (\rho_{\min}^{(j)} + \delta_i^{(j)}) = \min_{x_{ij}} \sum_{i=1}^n \sum_{j=1}^n x_{ij} \rho_{\min}^{(j)} + \min_{x_{ij}} \sum_{i=1}^n \sum_{j=1}^n x_{ij} \delta_i^{(j)} \quad (45)$$

Taking into account the constraint $\sum_{i=1}^n x_{ij} = 1$ we have

$$\min_{x_{ij}} \sum_{i=1}^n \sum_{j=1}^n x_{ij} \rho_i^{(j)} = \sum_{j=1}^n \rho_{\min}^{(j)} + \min_{x_{ij}} \sum_{i=1}^n \sum_{j=1}^n x_{ij} \delta_i^{(j)} \quad (46)$$

Since $\sum_{j=1}^n \rho_{\min}^{(j)}$ is fixed for a given matrix $[\rho]$, then the optimisation problem (46) reduces to determining the values of x_{ij} ($j=1, \dots, n; i=1, \dots, n$).

A heuristic algorithm described below can be used to solve this problem.

The matrix $[\delta_i^{(j)}]$ is constructed according to (44). A matrix $\Theta_0 = [\tau_a^{(j)}]$ is formed. The rows of this matrix correspond to natural numbers $a = 0, 1, 2, \dots$ and columns correspond to columns of the matrix $[\rho]$, $j = 1, \dots, n$. Elements $\tau_a^{(j)}$ are defined as follows

$$\tau_a^{(j)} = \begin{cases} O_i : \delta_i^{(j)} = a & i=1, \dots, n \\ \phi & : \delta_i^{(j)} \neq a \end{cases} \quad (47)$$

Step 1° A set \mathcal{A}_0 is formed; it consists of such elements O_i that appear only once in the first row of the Θ_0 matrix ($a=0$).

Step 2° A set \mathcal{A}_1 is formed; it consists of such elements O_i that appear two or more times in the first row of the Θ_0 matrix ($a=0$).

Step 3° Alternatives $O_i \in \mathcal{A}_0$ are located at positions corresponding to the columns in which they appear in the matrix Θ_0 .

Step 4° To each element of the set \mathcal{A}_1 a set of columns in which it appears is assigned.

$O_i \in \mathcal{A}_1 \Rightarrow J_b = \{j_{b_1}, \dots, j_{b_{l_b}}\}$, where l_b is the number of columns in which the alternative O_i appears.

Step 5° Alternatives $O_i \in \mathcal{A}_0 \cup \mathcal{A}_1$ are eliminated from all the rows of the matrix Θ_0 excluding the first one, i.e. that of $a=0$. As a result a new matrix Θ_1 is formed.

Step 6° Among alternatives $O_i \in \mathcal{A}_1$ that for which l_b is maximal is chosen. It is denoted by $O_{i_{b_2}}$. Let A_1 denotes the set of alternatives which do not appear in the first row of the matrix Θ_1 ($a=0$), $A_1 = \mathcal{O} - (\mathcal{A}_0 \cup \mathcal{A}_1)$. The column $j_{i_{b_2}}$ in which the alternative $O_{i_{b_2}}$ is to be placed is determined in the following way (columns determined in Step 3° are eliminated from the sets J_b)

$$\delta_{i_{b_2}}^{(j_{b_2})} = \max_{j \in J_{b_2}} \min_{O_i \in A_1} \delta_i^{(j)}. \quad (48)$$

In the remaining columns ($J_{b_2} - j_{b_2}$) the alternatives are located according to the rule

$$\min_{O_i \in A_1} \delta_i^{j_{b_2}}. \quad (49)$$

The alternatives located in the columns $j_{i_{b_2}} \in J_{b_2}$ are eliminated from the matrix Θ_1 as well as from the set A_1 . As a result a new matrix Θ_2 and a new set A_2 are obtained.

If all the operations mentioned in Step 6° can be accomplished in a unique way, then this procedure is repeated for the matrix Θ_2 . It is terminated when in a Θ_f matrix only one alternative appears in rows $a \geq 0$.

Step 7° Having all the alternatives placed in chosen columns, one has to calculate the distance d according to the formulae (34).

Step 8° Positions of two subsequent alternatives are interchanged and the value of d is calculated. If the distance is decreased, then the two positions of these alternatives are interchanged in the preference order determined in Step 6°. This procedure starts from the last alternative in the preference order determined in Step 6°.

Step 9° If any of the operations of Step 6° is not unique, then Steps 7° and 8° are to be repeated for all the preference orders under considerations.

Example 6.

Given the set of nine elements $\{O_1, O_2, O_3, O_4, O_5, O_6, O_7, O_8, O_9\}$ and judgements of eleven experts. On the basis of these judgements the matrix $[\rho_i^{(j)}]$ is constructed.

The matrix $[\rho_i^{(j)}]$ is as follows:

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
O_1	44	33	28	27	26	29	34	39	44
O_2	28	23	22	23	24	27	38	49	60
O_3	26	21	20	21	24	29	40	51	62
O_4	47	40	35	30	27	26	27	32	41
O_5	26	17	14	13	20	29	40	51	62
O_6	43	34	29	24	19	18	23	34	45
O_7	45	34	31	28	27	28	31	34	43
O_8	51	42	35	32	29	26	25	30	37
O_9	46	37	30	25	24	23	26	31	42

The smallest elements in each column are shaded. Hence, the matrix $[\delta_i^{(j)}]$ is the following:

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
O_1	18	16	14	14	7	11	11	9	7
O_2	2	6	8	10	5	9	15	19	23
O_3	0	4	6	8	5	11	17	21	25
O_4	21	23	21	17	8	8	4	2	4
O_5	0	0	0	0	1	11	17	21	25
O_6	17	17	15	11	0	0	0	4	8
O_7	19	17	17	15	8	10	8	4	6
O_8	25	25	21	19	10	8	2	0	0
O_9	20	20	16	12	5	5	3	1	5

$d_{\min}=197$

Having the matrix $[\delta_i^{(j)}]$ computed one can construct the matrix Θ_0 and determine the sets \mathcal{A}_6 and \mathcal{A}_7 . They are as follows: $\mathcal{A}_6 = \{O_3\}$, $\mathcal{A}_7 = \{O_5, O_6, O_8\}$.

Hence sets J_b are the following: $O_5: J_5 = \{1, 2, 3, 4\}, l_5 = 4$; $O_6: J_6 = \{5, 6, 7\}, l_6 = 3$;

$O_8: J_8 = \{8, 9\}, l_8 = 2$.

We have $l_{\max} = l_5$, therefore elements to be placed at positions $j \in J_5$ are to be determined first. To fix the position of the element O_5 one has to determine the number of column for which $\max_{j \in J_5} \min_{O_i \neq O_5} \delta_i^{(j)}$.

It follows from the analysis of the matrix Θ_0 that O_5 is to be placed at the fourth position. Elements to be located at the positions 2 and 3 are determined by the relation (49). Hence we have that O_3 is to be placed at the position 1, O_2 at the position 2, O_1 at the position 3. Deleting from the matrix Θ_0 the elements located at the positions 1 - 4 (except the row $a=0$) we obtain the matrix Θ_1 .

Now the position of the element O_6 is to be established, because $l_6 = 3$ and $l_8 = 2$. However $\max_{j \in J_6} \min_{O_i \neq O_6} \delta_i^{(j)}$ does not result in the unique solution (for the 5th and 6th position O_9 is located in the same row). Therefore we shall analyze the set $J_8 = \{8, 9\}$. From $\max_{j \in J_8} \min_{O_i \neq O_8} \delta_i^{(j)}$ it follows that

O_8 is to be placed at the position 9 and taking into account (49) we place O_9 at the position 8. Deleting elements O_8 and O_9 from the matrix Θ_1 we obtain the matrix Θ_2 .

Now we have to determine the position of the element O_6 . From $\max_{j \in J_6} \min_{O_i \neq O_6} \delta_i^{(j)}$ it follows that O_6 is to be placed at the position 6. Making use of (49) one can show that O_4 is to be placed at the position 7 and O_7 at the position 5.

Hence the obtained preference order is as follows: $O_3, O_2, O_1, O_5, O_7, O_6, O_4, O_9, O_8$.

From the matrix Θ_0 we have that $\delta_3^{(1)} = \delta_5^{(4)} = \delta_6^{(6)} = \delta_8^{(9)} = 0$ and $\delta_2^{(2)} = 6$; $\delta_1^{(3)} = 14$; $\delta_7^{(5)} = 8$; $\delta_4^{(7)} = 4$; $\delta_9^{(8)} = 1$. Making use of (46) we have $d = 197 + (6 + 14 + 8 + 4 + 1) = 230$.

The matrix Θ_0 is as follows:

a	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
0	O_3, O_5	O_5	O_5	O_5	O_6	O_6	O_6	O_8	O_8
1					O_5				O_9
2	O_2						O_8	O_4	
3							O_9		
4		O_3					O_4	O_6, O_7	O_4
5						O_2, O_3, O_9			O_5
6		O_2	O_3						O_7
7					O_1				O_1
8			O_2	O_3	O_4, O_7	O_4, O_8	O_7		O_6
9						O_2		O_1	
10				O_2	O_8	O_7			
11				O_6		O_1, O_3, O_5	O_1		
12				O_9					
13									
14			O_1	O_1					

a	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
15			O ₆	O ₇			O ₂		
16		O ₁	O ₉						
17	O ₆	O ₆ , O ₇	O ₇	O ₄			O ₃ , O ₅		
18	O ₁								
19	O ₇			O ₈				O ₂	
20	O ₉	O ₉							
21	O ₄		O ₄ , O ₈					O ₃ , O ₅	
22									
23		O ₄							O ₂
24									
25	O ₈	O ₈							O ₃ , O ₅

Using the integer programming approach (37, 38) implemented with the use of LINDO software the following preference order was obtained:

O₃, O₂, O₅, O₁, O₇, O₆, O₉, O₄, O₈.

Taking into account that $\delta_3^{(1)} = \delta_3^{(3)} = \delta_6^{(6)} = \delta_8^{(9)} = 0$ and $\delta_2^{(2)} = 6$; $\delta_1^{(4)} = 14$; $\delta_7^{(5)} = 8$; $\delta_9^{(7)} = 3$;

$\delta_4^{(8)} = 2$ we have $d=197+(6+14+8+3+2)=230$

Therefore both preference orders can be considered as the median.

It should be emphasized that in the case when the minimal value of elements in each column corresponds to distinct alternatives O_i, then the Kemeny's median is determined by positions of these elements and the distance (34) is equal to its lower bound.

Concluding remarks

Taking into account distinguishing features of planning problems related to long-term transport development one has to conclude that classical mathematical programming methods cannot be used to solve such problems and other methods have to be applied. In the paper basic aspects, both theoretical and practical, of applying expert judgements are discussed. It is pointed out that this approach can be used to support solving transport development problems under consideration.

Results of works carried in Poland in the area of long-term transport development planning point out that Polish planners do not have at their disposal adequate methodological tools making it possible to solve this problems under conditions of market economy. As a result, contracts for the diagnosis of Polish transport as well as planning of its development are transferred to foreign firms. Because of lack of knowledge of circumstances under which these tasks are to be implemented, related to political, economical, social, cultural and environmental aspects, solutions suggested by these firms cannot be fully accepted.

In authors' opinion methods presented in the paper can be of some help for Polish planners of transport development. In particular, they can be used to determine localization of Warsaw segment of A-2 highway.

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