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Dual extremum principles and error bounds in the theory of plates with large deflections

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For the theory of plates with large deflections, the dual extremum principles are considered. Starting from the extremum principle of displacements, a strictly static complementary energy principle is given. With these two theorems, global error bounds can be calculated for plate problems with large deflections. This is shown on some numerical examples.

W pracy rozważane są dualne zasady ekstremalne w teorii płyt z dużymi odkształceniami. Wychodząc z zasady ekstremum przemieszczeń podano ściśle statyczną zasadę energii dopełniającej. Na podstawie tych dwóch zasad można obliczyć globalne oszacowania błędu dla zagadnień płyty z dużymi ugięciami. Zilustrowano to na kilku przykładach liczbowych.

В работе рассмотрены дуальные экстремальные принципы в теории плит с большими деформациями. Начиная из принципа экстремум перемещений дается точно статический принцип дополнительной энергии. На основе этих двух принципов можно вычислить глобальную оценку ошибки для задач плиты с большими прогибами. Это иллюстрируется на нескольких числовых примерах.

Notations

THE INDICES $\alpha, \beta, \gamma, \delta$ have the values 1, 2; the indices i, j, k, l have the values 1, 2, 3.

$\delta_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$ Kronecker symbol,

$\varepsilon_{ijk} = \begin{cases} +1 & i, j, k \text{ cyclic}, \\ -1 & i, j, k \text{ noncyclic}, \\ 0 & i, j, k \text{ otherwise}, \end{cases}$

$x_i (i = 1, 2, 3)$ rectangular coordinates of the undeformed plate,

$u_i^{(0)} = (u, v, w)$ components of the displacement of the plate,

$u_{,i}$ partial differentiation of the function u with respect to the coordinate direction x_i,

$e_{\alpha\beta}^{(m)} (m = 0, 1)$ Green strain tensor of the order m,

$S_{\alpha\beta}^{(m)} (m = 0, 1)$ Kirchhoff stress tensor of the order m,

$T_{\alpha i}^{(m)} (m = 0, 1)$ Piola stress tensor of the order m,

$N_{\alpha i}$ membrane forces,

$M_{\alpha\beta}$ moments,

W strain energy density per undeformed plate area,

W_c complementary energy density per undeformed plate area.

1. Introduction

In linear elasticity the very effective methods for the calculation of global and pointwise bounds are based on the two dual extremum principles: the theorem of potential energy and the theorem of complementary energy. In non-linear elasticity the principle of potential energy is well known, whereas the principle of complementary energy has only been considered for some special cases as one-dimensional problems or discrete problems [1–3] and the so-called semi-linear materials [4–5]. Recently the complementary energy principle was given for the von KÁRMÁN plate theory [6].

The linear KIRCHHOFF plate theory is only valued for those plate problems, in which the plate deflection is much smaller than the plate thickness. In all other cases, in which the deflection is of the same order as the plate thickness, the membrane forces can no longer be neglected. This leads to a geometric non-linear problem, described by non-linear equilibrium equations and a non-linear compatibility condition.

In this paper the dual extremum principles for the theory of plates with large deflections are considered. Starting with the extremum principle of displacements, a strict complementary energy principle can be deduced by a Legendre transformation, using the displacement gradient tensor and the PIOLA stress tensor as dual quantities. The variables of the complementary energy functional are the stress resultants and the stress moments. The necessary conditions are three linear equilibrium equations and linear static boundary conditions.

With the dual extremum principles, global error bounds can be calculated for plate problems including large deflections. Numerical results are given for two examples.

2. The non-linear plate theory according to von Kármán

Using the Lagrange description, the plate is referred to rectangular coordinates x_i $(i = 1, 2, 3)$ with x_1 and x_2 in the middle plane and the faces at $x_3 = \pm h/2$. Presuming the validity of Kirchhoffs's hypothesis, the components of displacement u_i $(i = 1, 2, 3)$ are given to [7]:

$$u_1 = u - x_3 w_{,1} \qquad u_2 = v - x_3 w_{,2} \qquad u_3 = w, \tag{2.1}$$

in which u, v, w are functions of x_1 and x_2 only. Here $w_{,\alpha}$ means partial differentiation of the function w with respect to the coordinate x_α. With the usual assumptions the non-linear GREEN strain tensor is defined as

$$e_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + u_{3,\alpha} u_{3,\beta}), \qquad \alpha, \beta = 1, 2. \tag{2.2}$$

Indices notation with summation convention is used.

If $N_{\alpha\beta}$ are the stress resultants and $M_{\alpha\beta}$ are the stress moments of the plate with respect to the original unloaded configuration, the stress-strain relations are:

$$\begin{aligned} N_{\alpha\beta} &= \frac{Eh}{1-\nu^2}[(1-\nu)e^0_{\alpha\beta} + \nu\delta_{\alpha\beta} e^0_{\gamma\gamma}], \\ M_{\alpha\beta} &= -\frac{Eh^3}{12(1-\nu^2)}[(1-\nu)w_{,\alpha\beta} + \nu\delta_{\alpha\beta} w_{,\gamma\gamma}]. \end{aligned} \tag{2.3}$$

If q is the transverse load per unit area of the undeformed plate, the equilibrium equations are defined as:

$$N_{\alpha\beta,\alpha} = 0, \quad M_{\alpha\beta,\alpha\beta} + N_{\alpha\beta} w_{,\alpha\beta} + q = 0. \tag{2.4}$$

With the skew-symmetric ε-tensor, the compatibility condition is

$$\varepsilon_{3\alpha\gamma}\varepsilon_{3\beta\delta}\left(N_{\alpha\beta,\gamma\delta} + \frac{Eh}{2} w_{,\alpha\beta} w_{,\gamma\delta}\right) = 0. \tag{2.5}$$

The equilibrium condition $(2.4)_2$ and the compatibility condition (2.5) are non-linear partial differential equations.

3. The extremum principle of displacements for the non-linear plate theory

To calculate the total potential energy of the plate, we use the components of displacements (2.1) in the following notation:

$$u_\alpha = u_\alpha^{(0)} + x_3 u_\alpha^{(1)}, \quad \alpha = 1, 2, \quad u_3 = u_3^{(0)}, \tag{3.1}$$

where $u_i^{(0)}$ $(i = 1, 2, 3)$ and $u_\alpha^{(1)}$ $(\alpha = 1, 2)$ are functions of x_1 and x_2 only. Kirchhoff's hypothesis is given as:

$$u_\alpha^{(1)} = -u_{3,\alpha}^{(0)} = -w_{,\alpha}. \tag{3.2}$$

With (3.1) the Green strain tensor (2.2) can be written in the following form:

$$e_{\alpha\beta} = e_{\alpha\beta}^{(0)} + x_3 e_{\alpha\beta}^{(1)} \tag{3.3}$$

with the components

$$e_{\alpha\beta}^{(0)} = \frac{1}{2}\left(u_{\alpha,\beta}^{(0)} + u_{\beta,\alpha}^{(0)} + u_{3,\alpha}^{(0)} u_{3,\beta}^{(0)}\right), \quad e_{\alpha\beta}^{(1)} = \frac{1}{2}\left(u_{\alpha,\beta}^{(1)} + u_{\beta,\alpha}^{(1)}\right). \tag{3.4}$$

Let V_0 be the plate volume, F_0 the area of the plate middle surface, S_{0p} the boundary surface with prescribed load distribution and C_{0p} the related boundary line of the middle surface, S_{0v} the boundary surface with prescribed displacements and C_{0v} the related boundary line of the middle surface. The index 0 refers to the undeformed plate configuration. The total potential energy is defined by the function:

$$I = \int_{V_0} W_{V_0}(u_{j,i})\, dV_0 - \int_{F_0} q u_3^{(0)} dF_0 - \int_{S_{0p}} P_i^* u_i\, dS_{0p}; \tag{3.5}$$

P_i^* are the surface forces given on the boundary S_{0p}. $W_{V_0}(u_{j,i})$ is the energy density, measured per unit volume of the undeformed plate:

$$W_{V_0}(u_{j,i}) = \frac{E}{2(1-\nu^2)}\left[(1-\nu) e_{\alpha\beta} e_{\alpha\beta} + \nu (e_{\alpha\alpha})^2\right]. \tag{3.6}$$

Integrated over the plate thickness, we have the energy density per undeformed plate area:

$$(3.7)\qquad W(u_{j,i}) = \int_{-h/2}^{h/2} W_{V_0}(u_{j,i})dx_3 = \frac{Eh}{2(1-\nu^2)}[(1-\nu)e^{(0)}_{\alpha\beta}e^{(0)}_{\alpha\beta}+\nu(e^{(0)}_{\alpha\alpha})^2] + \frac{Eh^3}{24(1-\nu^2)}[(1-\nu)e^{(1)}_{\alpha\beta}e^{(1)}_{\alpha\beta}+\nu(e^{(1)}_{\alpha\alpha})^2].$$

If we introduce the following given stress resultants and stress moments on the boundary S_{0p}:

$$(3.8)\qquad N^*_{n\alpha} = \int_{-h/2}^{h/2} P^*_\alpha dx_3,\quad Q^*_{n3} = \int_{-h/2}^{h/2} P^*_3 dx_3,\quad M^*_{n\alpha} = \int_{-h/2}^{h/2} P^*_\alpha x_3 dx_3,$$

the functional (3.5) is given as:

$$(3.9)\qquad I = \int_{F_0} [W(u_{j,i})-qu^{(0)}_3]dF_0 - \int_{C_{0p}} [N^*_{n\alpha}u^{(0)}_\alpha + M^*_{n\alpha}u^{(1)}_\alpha + Q^*_{n3}u^{(0)}_3]dC_0.$$

We introduce the symmetric non-linear Kirchhoff stress tensor of order zero and order one:

$$(3.10)\qquad S^{(m)}_{\alpha\beta} = \frac{\partial W}{\partial e^{(m)}_{\beta\alpha}},\quad m = 0, 1.$$

The differentiation of the strain energy (3.7) leads to the membrane forces and moments of the plate:

$$(3.11)\qquad S^{(0)}_{\alpha\beta} = N_{\alpha\beta},\quad S^{(1)}_{\alpha\beta} = M_{\alpha\beta}.$$

Correspondingly we introduce the unsymmetric non-linear Piola stress tensor of order zero and one:

$$(3.12)\qquad \begin{aligned} T^{(0)}_{\alpha\beta} &= \frac{\partial W}{\partial u^{(0)}_{\beta,\alpha}} = S^{(0)}_{\alpha\beta} = N_{\alpha\beta},\\ T^{(0)}_{\alpha 3} &= \frac{\partial W}{\partial u^{(0)}_{3,\alpha}} = S^{(0)}_{\alpha\beta}u^{(0)}_{3,\beta} = N_{\alpha 3},\\ T^{(1)}_{\alpha\beta} &= \frac{\partial W}{\partial u^{(1)}_{\beta,\alpha}} = S^{(1)}_{\alpha\beta} = M_{\alpha\beta}. \end{aligned}$$

We consider now a geometric-admissible variation of the functional (3.9) and calculate the stationary value of the functional I:

$$(3.13)\qquad \delta I = \int_{F_0} [\delta W(u_{j,i})-q\delta u^{(0)}_3]dF_0 - \int_{C_{0p}} [N^*_{n\alpha}\delta u^{(0)}_\alpha + M^*_{n\alpha}\delta u^{(1)}_\alpha + Q^*_{n3}\delta u^{(0)}_3]dC_{0p}.$$

With (3.12) the variation of the strain energy density $W(u_{j,i})$ leads to:

$$(3.14)\qquad \delta W(u_{j,i}) = N_{\alpha\beta}\delta u^{(0)}_{\beta,\alpha} + N_{\alpha 3}\delta u^{(0)}_{3,\alpha} + M_{\alpha\beta}\delta u^{(1)}_{\beta,\alpha}.$$

With the components n_α of the unit normal vector to the curve C_0 and with the geometric conditions

$$\frac{\partial}{\partial x_1} = n_1 \frac{\partial}{\partial n} - n_2 \frac{\partial}{\partial s}, \qquad \frac{\partial}{\partial x_2} = n_2 \frac{\partial}{\partial n} + n_1 \frac{\partial}{\partial s}, \tag{3.15}$$

in which $\partial/\partial n$ and $\partial/\partial s$ respectively mean partial differentiation with respect to normal and tangential directions of the boundary curve C_0, we introduce the following notations:

$$\begin{aligned} N_{ni} &= n_\alpha N_{\alpha i}, & M_{\beta n} &= n_\alpha M_{\alpha\beta}, \\ M_{nn} &= n_\alpha M_{\alpha n}, & M_{ns} &= \varepsilon_{3\alpha\beta} n_\alpha M_{\beta n}, \\ Q_{n3} &= M_{\alpha n,\alpha}. \end{aligned} \tag{3.16}$$

With (3.2), (3.14), (3.15), (3.16) and using the Gauss divergence theorem, the variational equation (3.13) leads to the stationary value:

$$\begin{aligned} \delta I = &- \int_{F_0} [N_{\alpha\beta,\alpha} \delta u_\beta^{(0)} + (M_{\alpha\beta,\alpha\beta} + N_{\alpha 3,\alpha} + q)\delta w] dF_0 \\ &+ \int_{C_{0p}} [(N_{n\alpha} - N_{n\alpha}^*) \delta u_\alpha^{(0)} + (Q_{n3} + M_{ns,s} + N_{n3} - Q_{n3}^* - M_{ns,s}^*)\delta w \\ &- (M_{nn} - M_{nn}^*) \delta w_{,n}] dC_{0p} - (M_{ns} - M_{ns}^*) \delta w | C_{0p} \\ &+ \int_{C_0 V} [N_{n\alpha} \delta u_\alpha^{(0)} + (Q_{n3} + M_{ns,s} + N_{n3}) \delta w - M_{nn} \delta w_{,n}] dC_{0V} - M_{ns} \delta w | C_{0v} = 0, \end{aligned} \tag{3.17}$$

where the notation $(\ )|C_{0p}$ indicates the difference in values at the ends of the boundary curve C_{0p}.

For a geometric-admissible variation of the displacements $u_i^{(0)}$, satisfying the geometric boundary conditions on C_{0V}, the stationary value of the functional I leads to the Euler equations in V_0:

$$N_{\alpha\beta,\alpha} = 0, \quad M_{\alpha\beta,\alpha\beta} + N_{\alpha 3,\alpha} + q = 0, \tag{3.18}$$

and to the static boundary conditions on C_{0p}:

$$\begin{aligned} &N_{n\alpha} - N_{n\alpha}^* = 0, \quad Q_{n3} + M_{ns,s} + N_{n3} - Q_{n3}^* - M_{ns,s}^* = 0, \\ &M_{nn} - M_{nn}^* = 0, \quad M_{ns} - M_{ns}^* \text{ continuous.} \end{aligned} \tag{3.19}$$

The Eqs. (3.18) are the equilibrium conditions in a linear and strictly static formulation. The Eqs. (3.19) are the static boundary conditions in a strictly static and linear formulation as well.

If $(3.12)_1$ and $(3.12)_2$ are introduced into the equilibrium equation $(3.18)_2$ and if $(3.18)_1$ is taken into consideration, the equation $(3.18)_2$ leads to the non-linear equilibrium equation $(2.4)_2$ according to von Kármán. Correspondingly the equations (3.19) with $(3.12)_2$ define the non-linear static boundary conditions of the non-linear plate theory.

In this section, it has been shown that the boundary value problem of the non-linear plate theory corresponds to the stationary value of the variational functional I. It can be shown now that, for a special class of functions $u_i^{(0)}$ the variational functional I not only has a stationary value for the exact solution, but this value is a minimum as well. To prove this, we have to show that the energy density $W(u_{j,i})$ is a convex function of the components of the asymmetric displacement gradient tensor $u_{j,i}$. This will be considered in detail in a forthcoming paper.

If $u_i^{(0)}$ is the displacement field of the exact solution, and if $u_i^{(0)\sim}$ is a geometric-admissible displacement field belonging to that class of functions, for which the functional I has a minimum, then:

$$I(u_i^{(0)}) \leqslant I(u_i^{(0)\sim}) \tag{3.20}$$

holds.

4. The extremum principle of complementary energy

Starting from the extremum principle of displacements from Sec. 3, we construct a complementary energy principle in such a way, that the Euler equation (3.18) and the natural boundary conditions (3.19) of the extremum principle of displacements are the necessary conditions of the complementary energy principle.

We use the Legendre transformation

$$W_c = \sum_{m=0}^{1} u_{\alpha,\beta}^{(m)} T_{\beta\alpha}^{(m)} + u_{3,\alpha}^{(0)} T_{\alpha 3}^{(0)} - W(u_{j,i}) = u_{\alpha,\beta}^{(0)} N_{\beta\alpha} + u_{\alpha,\beta}^{(1)} M_{\beta\alpha} + u_{3,\alpha}^{(0)} N_{\alpha 3} - W(u_{j,i}) \tag{4.1}$$

with the complementary energy density W_c.

In the transformation (4.1) we have to express the components of the unsymmetric displacement gradient tensor $(u_{\beta,\alpha}^{(m)}, u_{3,\alpha}^{(0)})$ by the components of the unsymmetric Piola stress tensor $(T_{\alpha\beta}^{(m)}, T_{\alpha 3}^{(0)})$. This leads to the equations:

$$\begin{aligned} \frac{1}{2}(u_{\alpha,\beta}^{(0)} + u_{\beta,\alpha}^{(0)} + u_{3,\alpha}^{(0)} u_{3,\beta}^{(0)}) &= \frac{1}{Eh}[(1+\nu) N_{\alpha\beta} - \nu\delta_{\alpha\beta} N_{\gamma\gamma}], \\ \frac{1}{2}(u_{\alpha,\beta}^{(1)} + u_{\beta,\alpha}^{(1)}) &= \frac{12}{Eh^3}[(1+\nu) M_{\alpha\beta} - \nu\delta_{\alpha\beta} M_{\gamma\gamma}], \end{aligned} \tag{4.2}$$

$$u_{3,\alpha}^{(0)} = \frac{\varepsilon_{3\alpha\gamma}\varepsilon_{3\alpha\delta}(N_{\alpha 3} N_{\gamma\delta} - N_{\gamma 3} N_{\alpha\delta})}{\det(N_{\alpha\beta})} = \varphi_\alpha, \tag{4.3}$$

in which the index α is not summed and $\det(N_{\alpha\beta})$ means the determinant of the tensor $N_{\alpha\beta}$. With (4.2) and (4.3), the complementary energy density W_c of (4.1) can be calculated as a function of the stress resultants $N_{\alpha i}$ $(\alpha = 1, 2;\ i = 1, 2, 3)$ and of the moment stresses $M_{\alpha\beta}$:

$$W_c = \frac{1}{2Eh}[(1+\nu) N_{\alpha\beta} N_{\alpha\beta} - \nu(N_{\alpha\alpha})^2] + \frac{12}{2Eh^3}[(1+\nu) M_{\alpha\beta} M_{\alpha\beta} - \nu(M_{\alpha\alpha})^2] + \frac{1}{2}\varphi_\alpha N_{3\alpha} \tag{4.4}$$

with the functions φ_α according to (4.3). In general, these functions are incompatible.

Introducing (4.1) into (3.9), the functional I can be written in the form:

$$(4.5)\qquad I = -\int\limits_{F_0} W_c dF_0 + \int\limits_{F_0} [u^{(0)}_{\alpha,\beta} N_{\beta\alpha} + u^{(1)}_{\alpha,\beta} M_{\beta\alpha} + u^{(0)}_{3,\alpha} N_{\alpha 3} - q u^{(0)}_3] dF_0 - \int\limits_{C_{0p}} [N^*_{n\alpha} u^{(0)}_\alpha + M^*_{n\alpha} u^{(1)}_\alpha + Q^*_{n3} u^{(0)}_3] dC_{0p}.$$

Integration by part and using the Gauss divergence theorem leads to

$$(4.6)\qquad I = -\int\limits_{F_0} W_c dF_0 + \int\limits_{C_{0v}} [N_{n\alpha} u^{(0)*}_\alpha + (Q_{n3} + M_{ns,s} + N_{n3}) w^* - M_{nn} w^*_{,n}]\, dC_{0v} - M_{ns}\, w^* | C_{0v}$$
$$- \int\limits_{F_0} [N_{\beta\alpha,\beta} u^{(0)}_\alpha + (M_{\beta\alpha,\beta\alpha} + N_{\alpha 3,\alpha} + q) w] dF_0 + \int\limits_{C_{0v}} [(N_{n\alpha} - N^*_{n\alpha}) u^{(0)}_\alpha$$
$$+ (Q_{n3} + M_{ns,s} + N_{n3} - Q^*_{n3} - M^*_{ns,s}) w - (M_{nn} - M^*_{nn}) w_{,n}] dC_{0p} - (M_{ns} - M^*_{ns}) w | C_{0p}$$

with given displacements u^*_i on the boundary C_{0v}.

If the functional (4.6) is varied in such a way, that the equilibrium conditions (3.18). and the static boundary conditions (3.19) are always satisfied, then the last two integral expressions of (4.6) vanish and this leads to the complementary energy functional I_c:

$$(4.7)\qquad I_c = -\int\limits_{F_0} W_c dF_0 + \int\limits_{C_{0v}} [N_{n\alpha} u^{(0)*}_\alpha + (Q_{n3} + M_{ns,s} + N_{n3}) w^* - M_{nn} w^*_{,n}]\, dC_{0v} - M_{ns} w^* | C_{0v}$$

with the complementary energy density W_c according to (4.4).

In the complementary energy functional I_c of (4.7) we have to vary the stress resultants $N_{\alpha i}$ and the moments $M_{\alpha\beta}$, satisfying the equilibrium equations (3.18) and the static boundary conditions (3.19).

Now, we have to show that for all static-admissible stress states, satisfying (3.18) and (3.19), the functional I_c assumes its stationary value for the solution of the boundary value problem of the non-linear plate theory. To prove this, we use the method of Lagrange multipliers. In the complementary energy functional I_c the Eqs. (3.18) and (3.19) are taken into account by the Lagrange multipliers $\lambda^{(0)}_i$:

$$(4.8)\qquad I_c = -\int\limits_{F_0} W_c dF_0 - \int\limits_{F_0} [N_{\alpha\beta,\beta} \lambda^{(0)}_\alpha + (M_{\alpha\beta,\beta\alpha} + N_{\alpha 3,\alpha} + q) \lambda^{(0)}_3] dF_0$$
$$+ \int\limits_{C_{0v}} [N_{n\alpha} u^{(0)*}_\alpha + (Q_{n3} + M_{ns,s} + N_{n3}) w^* - M_{nn} w^*_{,n}] dC_{0v} - M_{ns} w^* | C_{0v}$$
$$+ \int\limits_{C_{0p}} [(N_{n\alpha} - N^*_{n\alpha}) \lambda^{(0)}_\alpha + (Q_{n3} + M_{ns,s} + N_{n3} - Q^*_{n3} - M^*_{ns,s}) \lambda^{(0)}_3$$
$$- (M_{nn} - M^*_{nn}) \lambda^{(0)}_{3,n}] dC_{0p} - (M_{ns} - M^*_{ns}) \lambda^{(0)}_3 | C_{0p}.$$

To vary the complementary energy functional I_c, we have to consider the variation of the complementary energy density W_c:

$$\delta W_c = \frac{\partial W_c}{\partial N_{\alpha\beta}} \delta N_{\alpha\beta} + \frac{\partial W_c}{\partial N_{\alpha 3}} \delta N_{\alpha 3} + \frac{\partial W_c}{\partial M_{\alpha\beta}} \delta M_{\alpha\beta}. \tag{4.9}$$

With the differentiation of (4.4),

$$\begin{aligned} \frac{\partial W_c}{\partial N_{\alpha\beta}} &= \frac{1}{Eh}[(1+\nu)N_{\alpha\beta} - \nu\delta_{\alpha\beta}N_{\gamma\gamma}] - \frac{1}{2}\varphi_\alpha\varphi_\beta, \\ \frac{\partial W_c}{\partial N_{\alpha 3}} &= \varphi_\alpha, \\ \frac{\partial W_c}{\partial M_{\alpha\beta}} &= \frac{12}{Eh^3}[(1+\nu)M_{\alpha\beta} - \nu\delta_{\alpha\beta}M_{\gamma\gamma}] \end{aligned} \tag{4.10}$$

is obtained.

With (4.9) and (4.10) the variation of I_c leads to

$$\begin{aligned} \delta I_c = &- \int \left\{ \left[\frac{1}{Eh}((1+\nu)N_{\alpha\beta} - \nu\delta_{\alpha\beta}N_{\gamma\gamma}) - \frac{1}{2}\varphi_\alpha\varphi_\beta \right] \delta N_{\alpha\beta} \right. \\ &\left. + \frac{12}{Eh^3}((1+\nu)M_{\alpha\beta} - \nu\delta_{\alpha\beta}M_{\gamma\gamma})\delta M_{\alpha\beta} + \varphi_\alpha \delta N_{\alpha 3} \right\} dF_0 \\ &- \int_{F_0} [\delta N_{\beta\alpha,\beta}\lambda_\alpha^{(0)} + (\delta M_{\beta\alpha,\beta\alpha} + \delta N_{\alpha 3,\alpha})\lambda_3^{(0)}] dF_0 \\ &+ \int_{C_{0v}} [\delta N_{n\alpha}u_\alpha^{(0)*} + \delta(Q_{n3} + M_{ns,s} + N_{n3})w^* - \delta M_{nn}w^*_{,n}] dC_{0v} - \delta M_{ns}w^*|C_{0v} \\ &+ \int_{C_{0p}} [\delta N_{n\alpha}\lambda_\alpha^{(0)} + \delta(Q_{n3} + M_{ns,s} + N_{n3})\lambda_3^{(0)} - \delta M_{nn}\lambda_{3,n}^{(0)}] dC_{0p} - \delta M_{ns}\lambda_3^{(0)}|C_{0p}. \end{aligned} \tag{4.11}$$

After some partial integrations and after using the Gauss divergence theorem, the stationary value of the complementary energy functional I_c is obtained:

$$\begin{aligned} \delta I_c = &- \int_{F_0} \left\{ \left[\frac{1}{Eh}((1+\nu)N_{\alpha\beta} - \nu\delta_{\alpha\beta}N_{\gamma\gamma}) - \frac{1}{2}\varphi_\alpha\varphi_\beta - \frac{1}{2}(\lambda_{\alpha,\beta}^{(0)} + \lambda_{\beta,\alpha}^{(0)}) \right] \delta N_{\beta\alpha} \right. \\ &\left. + \left[\frac{12}{Eh^3}((1+\nu)M_{\alpha\beta} - \nu\delta_{\alpha\beta}M_{\gamma\gamma}) + \lambda_{3,\alpha\beta}^{(0)} \right] \delta M_{\alpha\beta} + [\varphi_\alpha - \lambda_{3,\alpha}^{(0)}]\delta N_{\alpha 3} \right\} dF_0 \\ &- \int_{C_{0v}} [(\lambda_\alpha^{(0)} - u_\alpha^{(0)*})\delta N_{n\alpha} + (\lambda_3^{(0)} - w^*)\delta(Q_{n3} + M_{ns,s} + N_{n3}) \\ &- (\lambda_{3,n}^{(0)} - w^*_{,n})\delta M_{nn}] dC_{0v} + (\lambda_3^{(0)} - w^*)\delta M_{ns}|C_{0v} = 0. \end{aligned} \tag{4.12}$$

The stationary condition (4.12) leads to the Euler equations in V_0:

$$\text{(4.13)}\quad \begin{aligned} &\frac{1}{Eh}[(1+\nu)N_{\alpha\beta}-\nu\delta_{\alpha\beta}N_{\gamma\gamma}]-\frac{1}{2}\varphi_\alpha\varphi_\beta=\frac{1}{2}(\lambda^{(0)}_{\alpha,\beta}+\lambda^{(0)}_{\beta,\alpha}),\\ &-\frac{12}{Eh^3}[(1+\nu)M_{\alpha\beta}-\nu\delta_{\alpha\beta}M_{\gamma\gamma}]=\lambda^{(0)}_{3,\alpha\beta},\end{aligned}$$

$$\text{(4.14)}\quad \varphi_\alpha=\lambda^{(0)}_{3,\alpha},$$

and to the natural boundary conditions on C_{0v}:

$$\text{(4.15)}\quad \lambda^{(0)}_\alpha-u^{(0)*}_\alpha=0,\quad \lambda^{(0)}_3-w^*=0,\quad \lambda^{(0)}_{3,n}-w^*_{,n}=0.$$

If we identify the Lagrange multipliers $\lambda^{(0)}_i$ with the displacements of the plate $u^{(0)}_i$, then (4.13) are the stress-strain relations, (4.14) are the compatibility conditions of the functions φ_α and (4.15) are the geometric boundary conditions of the non-linear plate problem. Eliminating the Lagrange multipliers in $(4.13)_1$ and introducing (4.14), the compatibility condition (2.5) according to von KÁRMÁN is obtained.

With (4.13), (4.14) and (4.15) we have proved that the solution of the boundary-value problem of the non-linear plate theory corresponds to the stationary value of the complementary energy functional I_c. It can be shown now that, for some class of elastic stress states $(N_{\alpha l}, M_{\alpha\beta})$, the complementary energy functional I has a maximum. In a forthcoming paper it will be proved that these elastic stress states belong to stable equilibrium.

If $(N^{\approx}_{\alpha i}, M^{\approx}_{\alpha\beta})$ is a stress state belonging to a stable equilibrium, and if $(N_{\alpha l}, M_{\alpha\beta})$ is the exact solution, then

$$\text{(4.16)}\quad I_c(N^{\approx}_{\alpha i}, M^{\approx}_{\alpha\beta})\leqslant I_c(N_{\alpha l}, M_{\alpha\beta})$$

holds.

With the upper bound of the inequality (3.20) and the lower bound of the inequality (4.16), the values of the functionals I and I_c for the exact solution can be bounded from above and below:

$$\text{(4.17)}\quad I_c(N^{\approx}_{\alpha i}, M^{\approx}_{\alpha\beta})\leqslant I_c(N_{\alpha l}, M_{\alpha\beta})=I(u^{(0)}_i)\leqslant I(u^{(0)\sim}_i).$$

Using geometric and static-admissible approximations with unknown coefficients, we are able to determine these coefficients by converging variational processes.

5. Numerical results

As a first example we consider a square plate with constant transverse load q, all sides simply supported with vanishing displacements (u, v, w) (Fig. 1):

Introducing dimensionless coordinates $x_1=\bar{x}_1/a$; $x_2=\bar{x}_2/a$, we have the geometric boundary conditions:

$$\text{(5.1)}\quad \begin{aligned} &u=v=w=0,\quad x_1=\pm1,\quad -1\leqslant x_2\leqslant+1,\\ &u=v=w=0,\quad -1\leqslant x_1\leqslant+1,\quad x_2=\pm1,\end{aligned}$$

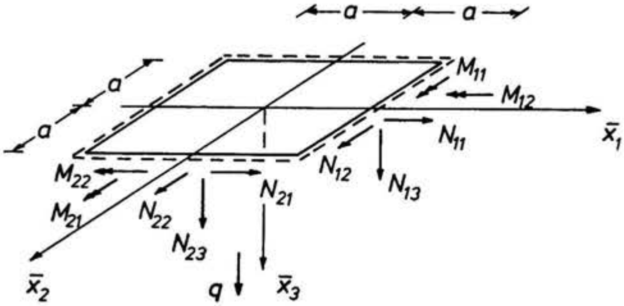

FIG. 1.

and the static boundary conditions:

$$
(5.2)\qquad \begin{aligned} M_{11} &= 0, \quad x_1 = \pm 1, \quad -1 \leqslant x_2 \leqslant +1,\\ M_{22} &= 0, \quad -1 \leqslant x_1 \leqslant +1, \quad x_2 = \pm 1. \end{aligned}
$$

For different values of q and $\nu = 0.3$, we have to approximate the unknown elastic states by variational processes.

As a geometric-admissible displacement field, we choose

$$
(5.3)\qquad \begin{aligned} u_1^{(0)} &= u = c_1(x_1 - x_1^3)(1 - x_2^2),\\ u_2^{(0)} &= v = c_1(1 - x_1^2)(x_2 - x_2^3),\\ u_3^{(0)} &= w = c_2(1 - x_1^2)(1 - x_2^2), \end{aligned}
$$

with unknown coefficients c_1 and c_2. These coefficients will be determined by minimizing the functional I according to (3.9).

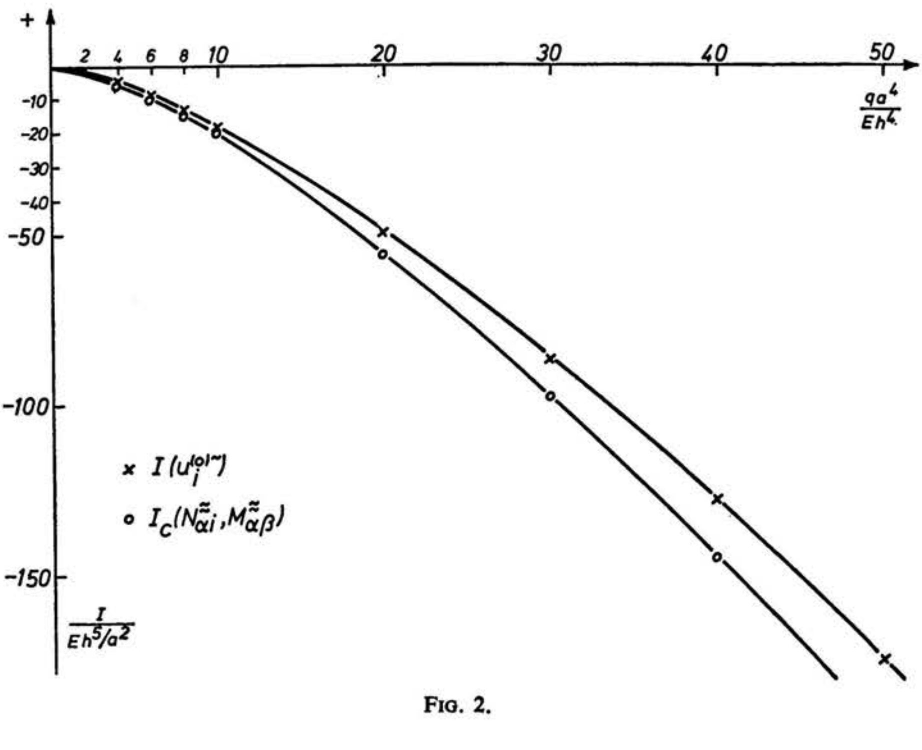

FIG. 2.

As a static-admissible stress state, we choose

$$
\begin{aligned}
&M_{11} = \frac{q_1}{4}(1-x_1^2)(1+\alpha x_2^2), \quad N_{11} = N_{22} = \beta, \quad N_{12} = 0,\\
(5.4)\qquad &M_{22} = \frac{q_1}{4}(1+\alpha x_1^2)(1-x_2^2), \quad N_{13} = (q-q_1)\left[-\frac{x_1}{2}+\gamma\left(x_1x_2^2-\frac{x_1^3}{3}\right)\right],\\
&M_{12} = \frac{q_1}{12}\alpha(x_1^3x_2+x_2^3x_1), \quad N_{23} = (q-q_1)\left[-\frac{x_2}{2}+\gamma\left(x_1^2x_2-\frac{x_2^3}{3}\right)\right].
\end{aligned}
$$

The unknown coefficients will be determined by maximizing the complementary functional I_c according to (4.7). With the inequality (4.17), upper and lower bounds for the exact values of the functional can be calculated. The results are shown in Fig. 2.

As a second example we consider a square plate with constant load q, with two sides simply supported and with vanishing displacements, and with two sides free. The geometric boundary conditions are given as

$$
(5.5)\qquad u = v = w = 0, \quad x_1 = \pm 1, \; -1 \leqslant x_2 \leqslant +1,
$$

and the static boundary conditions are

$$
(5.6)\qquad
\begin{aligned}
&M_{11} = 0, && x_1 = \pm 1,\; -1 \leqslant x_2 \leqslant +1,\\
&\left.\begin{aligned} &N_{22} = 0\\ &N_{21} = 0\\ &M_{22} = 0\\ &Q_{23}+M_{21,1}+N_{23} = 0 \end{aligned}\right\} && -1 \leqslant x_1 \leqslant +1,\; x_2 = \pm 1.
\end{aligned}
$$

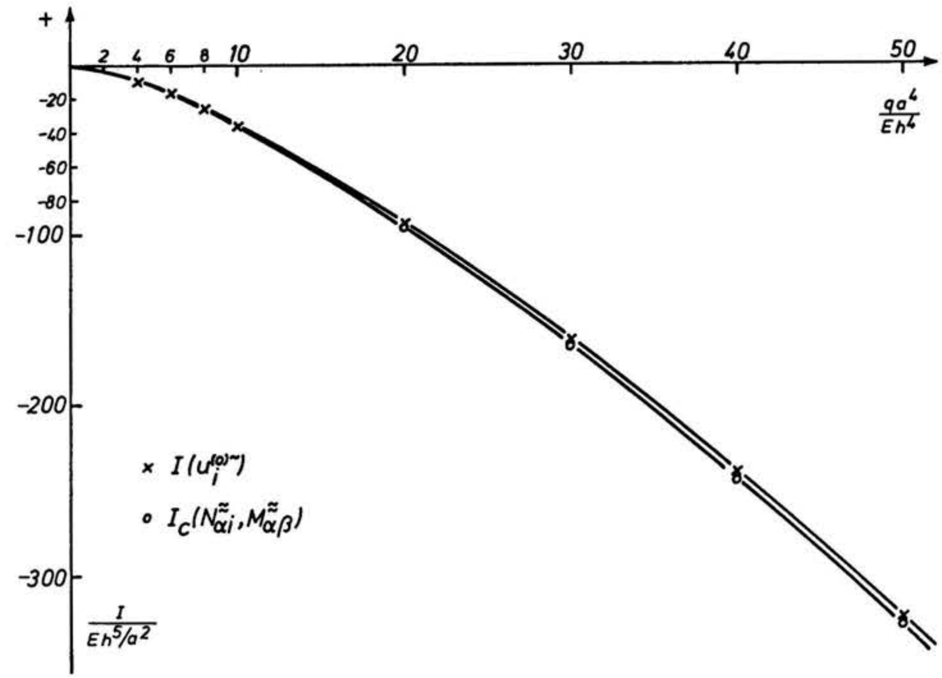

FIG. 3.

We choose a geometric admissible displacement field

$$u = c_1(x_1 - x_1^3), \quad v = c_2 x_2(1 - x_1^2), \quad w = c_3(1 - x_1^2), \tag{5.7}$$

and a static-admissible stress field

$$\begin{aligned} &M_{11} = \frac{q_1}{2}(1 - x_1^2), \quad N_{11} = \alpha, \quad N_{22} = N_{12} = 0, \\ &M_{22} = M_{12} = 0, \quad N_{13} = -(q - q_1)x_1, \quad N_{23} = 0. \end{aligned} \tag{5.8}$$

The unknown coefficients in (5.7) and (5.8) are determined by minimizing (3.9) and maximizing (4.7). With the approximations (5.7) and (5.8) the upper and lower bounds for the functional are calculated. The results are shown in Fig. 3.

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