Archives of Mechanics • Archiwum Mechaniki Stosowanej • 27, 3, pp. 473–484, Warszawa 1975

On material isomorphism in description of dynamic plasticity

P. PERZYNA (WARSZAWA)

PARTICULAR analysis of two material structures namely a structure of rate type and structure with internal state variables has been given. The main objective of this paper is to investigate the conditions under which both these material structures are isomoprhic, i.e. describe the same material. The second objective is to apply the results obtained to the description of dynamic plasticity. An essential feature of both structures considered is the form of the equation describing the evolution of internal states. This equation determines the evolution function. For every material structure the internal state is determined in a different way. The internal state and its evolution are substantial of the comparison of the material structures considered. Based on material isomorphism of the structures investigated the equivalence of the rate type theory of viscoplastic flow with the theory of viscoplasticity with internal state variables has been proved. All considerations have local character and concern finite elastic-plastic deformations.

Podano szczegółową analizę dwóch struktur materialnych, mianowicie struktury materialnej typu prędkościowego oraz struktury materialnej z parametrami wewnętrznymi. Celem tej analizy jest zbadanie warunków, przy których te dwie struktury są izomorficzne, tzn. opisują ten sam materiał. Celem następnym jest zastosowanie uzyskanych rezultatów do opisu dynamicznej plastyczności. Istotną cechą obydwu analizowanych struktur jest postać równania określającego ewolucję stanów wewnętrznych, które determinuje postać funkcji ewolucji. Dla każdej struktury stan wewnętrzny jest określony inaczej. Kluczem do porównania struktur jest stan wewnętrzny i jego zmiana w czasie, a więc ewolucja, dlatego funkcja ewolucji odgrywa w tym porównaniu rolę podstawową. Wykorzystując izomorfizm badanych struktur wykazano równoważność teorii lepkoplastycznego płynięcia typu prędkościowego z teorią lepkoplastyczności zbudowaną w ramach struktury z parametrami wewnętrznymi. Wszystkie rozważania mają charakter lokalny i dotyczą skończonych deformacji sprężysto-plastycznych.

Дается подробный анализ двух материальных структур, именно материальной структуры скоростного типа и материальной структуры с внутренними параметрами. Целью этого анализа является исследование условий, при которых эти две структуры изоморфичны, т. зн. описывают тот же самый материал. Очередной целью является применение полученных результатов для описания динамической пластичности. Существенным свойством обоих анализированных структур является вид уравнения определяющего эволюцию внутренних состояний, которые определяют вид функции эволюции. Для каждой структуры внутреннее состояние определено иначе. Ключом для сравнения структур является внутреннее состояние и его изменение во времени, т. е. эволюция. Поэтому функция эволюции играет в этом сравнении основную роль. Используя изоморфизм исследованных структур доказана эквивалентность теории вязкопластического течения скоростного типа с теорией вязкопластичности, построенной в рамках структуры с внутренними параметрами. Все рассуждения имеют локальный характер и касаются конечных упруго-пластических деформаций.

1. Introduction

THE OBJECTIVE of this paper is to give a particular analysis of two descriptions of dynamic plasticity, namely a theory with internal state variables and a theory of the rate type.

In the first part, an outline of the unique material structure for a dissipative material is given. The method of preparation space and the intrinsic state space are essential no-

http://rcin.org.pl

tions introduced for a mathematical description of a dissipative material. By different realizations of the method of preparation space we have obtained the material structure with internal state variables and the material structure of the rate type. An investigation of conditions under which these both material structures are isomorphic is given. An isomorphism of two unique material structures is understood as a similarity of these structures in the sense that these both structures describe the same material in a particle X of a body $\mathscr{B}$.

In the second part, an application of the results obtained to description of dynamic plasticity phenomena is presented. Two equivalent dynamic theories of plastic flow are constructed. Two particular examples of the description of an elastic-viscoplastic material are considered. The first is valid for medium strain rates and describes elastic-perfectly viscoplastic behaviour of a material and the second is valid for the entire range of deformation rate and concerns elastic-work-hardening viscoplastic behaviour.

It is important to note that all quantities and parameters introduced in the theories considered can easily be interpreted in terms of experience based on the analysis of dissipative mechanisms and on the experimental results for time-dependent plastic flow (cf. Refs. [4, 5, 7]).

2. General material structure ([1])

Let us consider a body $\mathscr{B}$ with particles X and assume that this body can deform inelastically. A motion of a body $\mathscr{B}$ is a single parameter family of configurations, i.e.

$$x = \chi(\mathrm{X}, t), \tag{2.1}$$

where x is the spatial position occupied by the material point X at time t. It is often convenient to identify the material point X with its position X in a fixed reference configuration $\varkappa$, and to write

$$x = \chi(X, t). \tag{2.2}$$

The gradient $F(t)$ of $\chi(X, t)$ with respect to X, i.e.

$$F(X, t) = \nabla\chi(X, t), \tag{2.3}$$

is the deformation gradient at X. The deformation gradient $F(t)$ describes all local properties of deformation at X.

In what follows, we shall use the right Cauchy-Green tensor $C(t) = F(t)^T F(t)$ as the fundamental local measure of deformation.

Let a continuous stress tensor $T_c(t)$ denote the Cauchy stress tensor of a particle X at time t. We shall use the second Piola-Kirchhoff stress tensor defined by the expression

$$T(t) = \big(\det F(t)\big)F(t)^{-1}T_c(t)\big(F(t)^{-1}\big)^T. \tag{2.4}$$

Let us introduce the notations as follows: $g = C(t)$ is the local configuration of X at time t, and $s = T(t)$ denotes the local response of a particle X at time t.

([1]) Cf. Ref. [6] in which a mathematical frame-free theory of dissipative materials is developed.

http://rcin.org.pl

A set of all possible configurations of a particle X will be denoted by $\mathcal{G}$ and will be called the configuration space (the deformation space).

A set of all possible responses of a particle X will be denoted by $\mathcal{S}$ and will be called the response space (the stress space).

We shall consider processes (2) in the configuration space $\mathcal{G}$ and processes in the response space $\mathcal{S}$.

A process

$$\text{(2.5)} \qquad \mathrm{P} \equiv \boldsymbol{C}\colon\ [0, d_{\mathrm{P}}] \to \mathcal{G}$$

will determine the change of the configuration of a particle X in the interval of time $[0, d_{\mathrm{P}}]$. A number d_{P} will be called the duration of the process P, and $\mathrm{P}^i = \mathrm{P}(0)$ and $\mathrm{P}^f = \mathrm{P}(d_{\mathrm{P}})$, the initial and final values of the process P, respectively.

A process

$$\text{(2.6)} \qquad \mathrm{Z} = \boldsymbol{T}\colon\ [0, d_{\mathrm{z}}] \to \mathcal{S}$$

will determine the change of the stress of a particle X in the interval of time $[0, d_{\mathrm{z}}]$.

DEFINITION 1. *Every pair* $(\mathrm{P}, \mathrm{Z}) \in \Pi \times \mathcal{Z}$ *such that* $\mathrm{Dom\,P} = \mathrm{Dom\,Z}$, where

$$\text{(2.7)} \qquad \begin{aligned} \Pi &\equiv \{\mathrm{P} | \mathrm{P}\colon\ [0, d_{\mathrm{P}}] \to \mathcal{G}\}, \\ \mathcal{Z} &\equiv \{\mathrm{Z} | \mathrm{Z}\colon\ [0, d_{\mathrm{z}}] \to \mathcal{S}\}, \end{aligned}$$

denote a set of the deformation processes and a set of the stress processes, respectively, is called a mechanical process for a particle X.

Let us introduce a space $\mathcal{K}$ connected with the configuration space $\mathcal{G}$ in such a way that elements of the space $\mathcal{K}$, which will be denoted by $k \in \mathcal{K}$, are the methods of preparation of the corresponding configurations from $\mathcal{G}$. The space $\mathcal{K}$ will be called the method of preparation space (3).

DEFINITION 2. *A non-empty set* $\mathcal{K}$ *will be called the method of preparation space for a particle* X *if*

$$\text{(2.8)} \qquad \bigvee_{\Sigma \subset \mathcal{G} \times \mathcal{K}} \ \bigvee_{\mathfrak{R} : (\Sigma \times \Pi)^* \to \mathcal{Z}} \ \bigwedge_{g \in \mathcal{G}} \ \bigwedge_{\mathrm{P} \in \Pi_g} \ \bigvee_{\mathcal{K}_g \subset \mathcal{K}} \mathfrak{R}(g, \cdot, \mathrm{P}) : \mathcal{K}_g \to \mathcal{Z}_{\mathrm{P}} \ \textit{is bijection},$$

where

$$\text{(2.9)} \qquad (\Sigma \times \Pi)^* = \{(\sigma, \mathrm{P}) \in \Sigma \times \Pi | \bigvee_{\mathcal{K}_{\mathrm{P}^i} \subset \mathcal{K}} \sigma \in \{\mathrm{P}^i\} \times \mathcal{K}_{\mathrm{P}^i}\},$$

$$\text{(2.10)} \qquad \Pi_g \equiv \{\mathrm{P} \in \Pi | \mathrm{P}^i = g\}.$$

DEFINITION 3. *A set*

$$\text{(2.11)} \qquad \Sigma \equiv \bigcup_{g \in \mathcal{G}} \{g\} \times \mathcal{K}_g, \quad \mathcal{K}_g \subset \mathcal{K}$$

is called the intrinsic state space (4) *of a particle* X.

(2) For a thorough discussion of properties of processes see NOLL [2].

(3) For a notion of the method of preparation see Refs. [3–5]. The precise definition of the method of preparation space was first introduced in Ref. [6]. We follow here the presentation from Ref. [6].

(4) The intrinsic state space Σ is due to Ref. [6]. It plays a similar role in the theory as the state space postulated by NOLL [2].

http://rcin.org.pl

The element $\sigma \in \Sigma$ is a pair of the configuration and the method of preparation, i.e.

$$\sigma \equiv (\mathrm{P}(t), \mathrm{A}(t)) = (g, k), \quad g \in \mathscr{G}, \quad k \in \mathscr{K}_g, \tag{2.12}$$

where by A we denote a process in the method of preparation space $\mathscr{K}$, i.e. A: $[0, d_{\mathrm{P}}] \to \mathscr{K}$.

We define two mappings as follows (5):

$$\hat{\mathrm{G}} \equiv \mathrm{pr}_{\mathscr{G}} : \Sigma \to \mathscr{G}, \qquad \hat{\mathrm{K}} \equiv \mathrm{pr}_{\mathscr{K}} : \Sigma \to \mathscr{K}, \tag{2.13}$$

which determine the projections from the intrinsic state space Σ on the configuration space $\mathscr{G}$ and on the method of preparation space $\mathscr{K}$, respectively.

DEFINITION 4. *A system* $(\mathscr{G}, \Pi, \Sigma, \mathfrak{R})$ *will be called the unique material structure defined in a particle* X.

The mapping

$$\mathfrak{R} : (\Sigma \times \Pi)^* \to \mathscr{Z} \tag{2.14}$$

introduced in the Definition 2, which will be called *the constitutive mapping*, expresses the general principle of determinism.

DEFINITION 5. *It is said that the mapping*

$$\dot{\mathrm{e}} : (\Sigma \times \Pi)^* \to \Sigma \tag{2.15}$$

is the evolution function (6), *if for every pair* $(\sigma_0, \mathrm{P}) \in (\Sigma \times \Pi)^*$

$$\mathfrak{R}(\dot{\mathrm{e}}(\sigma_0, \mathrm{P}), \mathrm{P}^f_{(0)}) = [\mathfrak{R}(\sigma_0, \mathrm{P})]^f, \tag{2.16}$$

where $[\mathfrak{R}(\sigma_0, \mathrm{P})]^f$ *denotes the final value of the stress process* $Z = \mathfrak{R}(\sigma_0, \mathrm{P})$ *and* $\mathrm{P}^f_{(0)}$ *is the process of duration zero.*

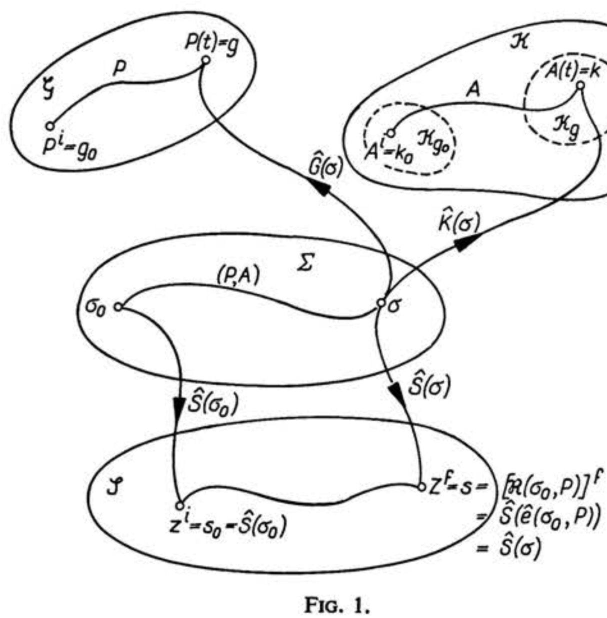

FIG. 1.

(5) For the mapping $\hat{\mathrm{G}}$ see NOLL [2] and for the mapping $\hat{\mathrm{K}}$ see Ref. [6].

(6) The evolution function $\hat{\mathrm{e}}$ is similar to that introduced by NOLL [2].

http://rcin.org.pl

If we introduce a new mapping (7)

(2.17) $$\hat{S}: \Sigma \to \mathcal{S}$$

by the expression

(2.18) $$\hat{S}(\sigma) \equiv \mathfrak{R}(\sigma, \hat{G}(\sigma)_{(0)}),$$

then the principle of determinism can be expressed by the relation

(2.19) $$Z(t) = \hat{S}(e(\sigma_0, P))$$

for every $(\sigma_0, P) \in (\Sigma \times \Pi)^*$, and for $d_P = t$, see Fig. 1.

The mapping $\hat{S}$ will be called the response function.

The system $(\mathcal{G}, \Pi, \Sigma, \hat{S}, \hat{e})$ is also the unique material structure in a particle X.

3. Internal state variable material structure (8)

Let use assume that a class of deformation processes contains only continuous processes.

PROPOSITION 1. *The method of preparation space is a finite dimensional vector space* $\mathcal{W}$, *i.e.*

(3.1) $$\mathcal{K} \equiv \mathcal{W},$$

the intrinsic state space Σ *is the set*

(3.2) $$\Sigma \equiv \{(g, k) | g \in \mathcal{G}, k \in \mathcal{W}_g\}$$

and

(3.3) $$(\Sigma \times \Pi)^* \equiv \{(\sigma, P) \in \Sigma \times \Pi | \hat{G}(\sigma) = P^i\}.$$

PROPOSITION 2. (i) *There exists a mapping*

(3.4) $$\hat{\alpha}: \Sigma \to \mathcal{W}$$

such that for every $P \in \Pi$ *and* $k_0 \in \mathcal{W}_{P^i}$ *the initial value problem*

(3.5) $$\frac{d}{d\tau} A(\tau) = \hat{\alpha}(P(\tau), A(\tau)), \quad A(0) = k_0$$

has a unique solution $A: [0, d_P] \to \mathcal{K} \equiv \mathcal{W}$.

(ii) *The constitutive mapping* $\mathfrak{R}; [\Sigma \times \Pi]^* \to \mathcal{Z}$ *satisfying for every pair* $(\sigma_0, P) \in (\Sigma \times \Pi)^*$ *the condition*

(3.6) $$\mathfrak{R}(\hat{G}(\sigma_0), \cdot, P): \mathcal{W}_{P^i} \to \mathcal{Z}_P \text{ must be bijection}$$

is such that the evolution function $\hat{e}$ *is of the form*

(3.7) $$e(\sigma_0, P) = \left(P^f, \mathfrak{F}(P, \hat{K}(\sigma_0))\right),$$

where $\mathfrak{F}$ *denotes the solution functional of the initial value problem* (3.5).

(7) Cf. NOLL [2]. It is worth to note that both mappings $\hat{e}$ and $\hat{S}$ are similar to those introduced by NOLL [2] but in this theory the mappings $\hat{e}$ and $\hat{S}$ are generated by the constitutive mapping $\mathfrak{R}$, cf. Ref. [6].

(8) We follow here the presentation given in Ref. [1].

http://rcin.org.pl

DEFINITION 6. *The unique material structure* $(\mathscr{G}, \Pi, \Sigma, \Re)$ *satisfying the Propositions 1 and 2 is called the material structure with internal state variables.*

The principle of determinism for the material structure with internal state variables is expressed by the constitutive equation

$$Z(t) = \hat{S}(P(t), A(t)). \tag{3.8}$$

4. Rate type material structure (9)

We assume that deformation processes $P \in \Pi$ are piecewise continuously differentiable, i.e. for every ti ... $t \in [0, d_P]$ exist left and right derivatives $\frac{d}{d\tau} P(\tau)\Big|_{\tau=t} = \dot{P}(t)$ which are equal for all ut a finite number of $t \in [0, d_P]$.

For su a class of processes Π we define the set

$$\mathscr{V}_g \equiv \left\{ \frac{d}{d\tau} P(\tau) \,\middle|\, P \in \Pi,\ \tau \in [0, d_P], P(\tau) = g \in \mathscr{G}, \right\}, \tag{4.1}$$

i.e. the class of all possible derivatives of deformation processes at a time at which they have the value g.

PROPOSITION 3. (i) *The method of preparation space is the stress space* $\mathscr{S}$, *i.e.*

$$\mathscr{K} \equiv \mathscr{S}, \tag{4.2}$$

the intrinsic state space Σ *is the set*

$$\Sigma \equiv \{(g, s) | g \in \mathscr{G}, s \in \mathscr{S}_g \equiv \mathscr{K}_g\} \tag{4.3}$$

and

$$(\Sigma \times \Pi)^* = \{(\sigma, P) \in \Sigma \times \Pi, \hat{G}(\sigma) = P^i\}. \tag{4.4}$$

(ii) *There exists a mapping*

$$\hat{\beta} : \mathscr{D} \to \mathscr{S}, \mathscr{D} \equiv \{(g, \dot{g}, s) | g \in \mathscr{G}, \dot{g} \in \mathscr{V}_g, s \in \mathscr{S}_g\}, \tag{4.5}$$

such that for every $P \in \Pi$ *and* $s_0 \in \mathscr{S}_{P^i}$ *the initial value problem*

$$\frac{d}{d\tau} Z(\tau) = \hat{\beta}(P(\tau), \dot{P}(\tau), Z(\tau)), \quad Z(0) = s_0 \tag{4.6}$$

has a unique solution Z: $[0, d_P] \to \mathscr{S}$.

(iii) *The constitutive mapping* $\Re$: $(\Sigma \times \Pi)^* \to \mathscr{Z}$ *satisfying for every pair* $(\sigma_0, P) \in (\Sigma \times \Pi)^*$ *the condition*

$$\Re(\hat{G}(\sigma_0), \cdot, P) : \mathscr{S}_{P^i} \to \mathscr{Z}_P \text{ must be bijection} \tag{4.7}$$

is such that the evolution function e *is of the form*

$$\hat{e}(\sigma_0, P) = (P^f, \mathfrak{Z}(P, s_0)), \tag{4.8}$$

where $\mathfrak{Z}(P, s_0)$ *denotes the solution functional of the initial value problem* (4.6).

(9) The rate type material structure was first developed by NOLL [2].

http://rcin.org.pl

(iv) *The constitutive mapping satisfies also the condition*

$$Z(t) = [\mathfrak{R}(\sigma_0, P)]^f = \mathfrak{Z}(P, s_0) = \hat{S}(\sigma) = \hat{K}(\sigma) = s. \tag{4.9}$$

DEFINITION 7. *The unique material structure* ($\mathcal{G}, \Pi, \Sigma, \mathfrak{R}$) *satisfying the Proposition* 3 *is called the material structure of the rate type.*

It is worth to note that the principle of determinism for the rate type material structure is expressed by the relation (4.9).

5. Isomorphic material structures

Suppose that we have defined, in one and the same particle X, two material structures ($\mathcal{G}, \Pi, \Sigma_1, \hat{S}_1, \hat{e}_1$) and ($\mathcal{G}, \Pi, \Sigma_2, \hat{S}_2, e_2$).

DEFINITION 8. *Two material structures* ($\mathcal{G}, \Pi, \Sigma_1, \hat{S}_1, \hat{e}_1$) *and* ($\mathcal{G}, \Pi, \Sigma_2, \hat{S}_2, \hat{e}_2$) *are materially isomorphic if there exists a bijection* $\iota\colon \Sigma_1 \to \Sigma_2$ *with the properties*

1) $\hat{S}_2(\iota(\sigma_1)) = \hat{S}_1(\sigma_1)$,

2) $\hat{G}_2(\iota(\sigma_1)) = \hat{G}_1(\sigma_1)$,

3) $\hat{e}_2(\iota(\sigma_1), P) = \iota(\hat{e}_1(\sigma_1, P))$.

This definition is a special case of the definition proposed by W. NOLL [2].

If two material structures defined in the same particle X are materially isomorphic, we also say that they describe at X the same material. So, a material is an equivalence class of material structures, the equivalence being material isomorphy (given by the Definition 8).

An isomorphism is thus a similarity. More important property of an isomorphism is that, if the unique material structure ($\mathcal{G}, \Pi, \Sigma_1, \hat{S}_1, e_1$) is isomorphic with the unique material structure ($\mathcal{G}, \Pi, \Sigma_2, \hat{S}_2, \hat{e}_2$) and if some features expressed by terms $\Sigma_1, \hat{S}_1, e_1$ are valid for the structure ($\mathcal{G}, \Pi, \Sigma_1, \hat{S}_1, e_1$), then the same features expressed by terms $\Sigma_2, \hat{S}_2, \hat{e}_2$ are preserved for the structure ($\mathcal{G}, \Pi, \Sigma_2, \hat{S}_2, \hat{e}_2$).

An isomorphism of two unique material structures is thus a similarity of these structures, under which the actions $\hat{S}_1$ and $\hat{e}_1$ defined on the set Σ_1 correspond to the actions $\hat{S}_2$ and $\hat{e}_2$ defined on the set Σ_2.

This is the main reason for which the study of isomorphism for different material structures is of great practical importance.

6. Discussion of a material isomorphism between internal state variable and rate type material structures

Let us write the main equations for the internal state variable material structure in the form

$$\sigma_1 = (P(t), A(t)) = (g, k) \in \Sigma_1, \quad g \in \mathcal{G}, k \in \mathcal{W}_g, \tag{6.1}$$

$$Z(t) = \hat{S}_1(P(t), A(t)), \tag{6.2}$$

$$\frac{d}{d\tau} A(\tau) = \alpha(P(\tau), A(\tau)), \quad A(0) = \hat{K}(\sigma_{10}) = k_0. \tag{6.3}$$

http://rcin.org.pl

We assume additionally that the Eq. (6.2) is such that $A(t)$ can be expressed as a function of $P(t)$ and $Z(t)$ for any time $t \in [0, d_P]$, i.e.

$$A(t) = \hat{N}(P(t), Z(t)). \tag{6.4}$$

The pair $(P(t), Z(t))$ determines the intrinsic state σ_2 at time t, i.e.

$$\sigma_2 = (P(t), Z(t)) = (g, s) \in \Sigma_2, \quad g \in \mathcal{G}, s \in \mathcal{S}_g. \tag{6.5}$$

The bijection $\iota: \Sigma_1 \to \Sigma_2$ is determined by the mapping $\hat{N}: \Sigma_2 \to \mathcal{K}_1 \equiv \mathcal{W}$.

Let us assume that deformation processes $P \in \Pi$ are continuous and piecewise differentiable with respect to time in the interval $[0, d_P]$ and that the constitutive function $\hat{S}_1$ is differentiable with respect to both variables $P(\tau)$ and $A(\tau)$, i.e. that the gradients $\partial_{P(\tau)}\hat{S}_1$ and $\partial_{A(\tau)}\hat{S}_1$ exist.

By differentiating the constitutive equation (6.2) with respect to time, we obtain the following evolution equation for the rate type structure

$$\frac{d}{d\tau} Z(\tau) = \hat{\beta}_0(P(\tau), Z(\tau)) + \hat{\beta}_1(P(\tau), Z(\tau))[\dot{P}(\tau)], \tag{6.6}$$

where

$$\begin{aligned} \hat{\beta}_0 &= \partial_{A(\tau)}\hat{S}_1\big(P(\tau), \hat{N}(P(\tau), Z(\tau))\big)\hat{\alpha}\big(P(\tau), \hat{N}(P(\tau), Z(\tau))\big), \\ \hat{\beta}_1 &= \partial_{P(\tau)}\hat{S}_1\big(P(\tau), \hat{N}(P(\tau), Z(\tau))\big), \end{aligned} \tag{6.7}$$

and the initial value $Z(0)$ is determined by the relation

$$Z(0) = \hat{S}_1(P(0), A(0)) = s_0. \tag{6.8}$$

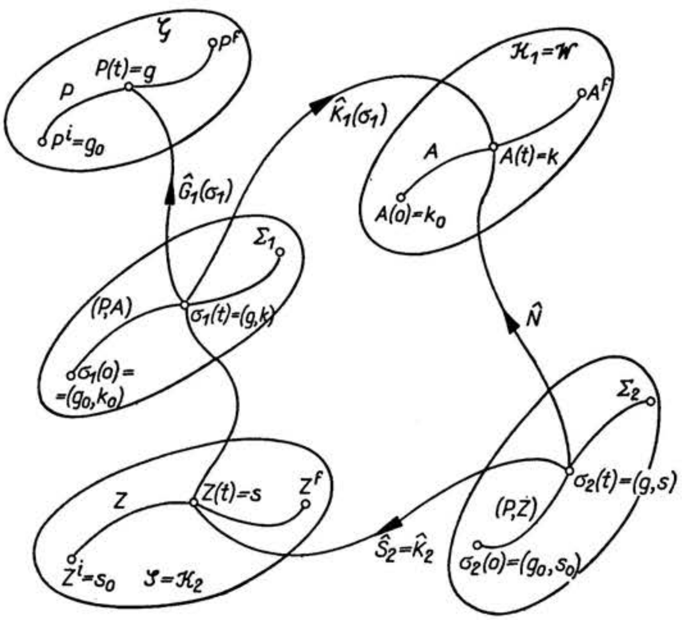

FIG. 2.

http://rcin.org.pl

If the functions $\hat{\beta}_0$ and $\hat{\beta}_1$ are such that the unique solution of the initial value problem (6.6)–(6.8) exists, then we can write

$$Z(t) = \mathfrak{Z}(P, s_0) = \hat{S}_2(\sigma_2) = \hat{K}_2(\sigma_2) = s. \tag{6.9}$$

The connections between different mappings introduced are explained in Fig. 2.

Under the conditions introduced, there is simple isomorphism between the internal state variable material structure and the rate type material structure.

Under above conditions both these material structures describe the same material at a particle X of a body $\mathscr{B}$.

7. Example 1. Elastic-viscoplastic material (medium strain rates)

Let us postulate that

$$A(t) = \boldsymbol{P}(t), \tag{7.1}$$

i.e. the inelastic deformation tensor $\boldsymbol{P}(t)$ is assumed as the internal state variable. So, the intrinsic state is determined by a pair

$$\sigma_1 = (\boldsymbol{C}(t), \boldsymbol{P}(t)) \in \Sigma_1. \tag{7.2}$$

The constitutive equation has the form

$$\boldsymbol{T}(t) = \hat{S}_1(\boldsymbol{C}(t), \boldsymbol{P}(t)). \tag{7.3}$$

The evolution equation is assumed as follows

$$\frac{d}{d\tau} \boldsymbol{P}(\tau) = \gamma \langle \Phi(\mathscr{F}(\tau)) \rangle \, \partial_{\boldsymbol{T}(t)} f \tag{7.4}$$

with the initial value

$$\boldsymbol{P}(0) = \boldsymbol{P}_0, \tag{7.5}$$

where γ is a viscosity constant, $\mathscr{F}(t)$ is the static yield condition and is assumed in the form

$$\mathscr{F}(t) = \frac{f(\boldsymbol{T}(t))}{\varkappa_0} - 1, \tag{7.6}$$

where $\varkappa_0$ is a yield constant, the dimensionless function $\Phi(\mathscr{F}(t))$ may be chosen to represent results of tests on the dynamic behaviour of materials and the symbol $\langle \Phi(\mathscr{F}(t)) \rangle$ is defined as follows

$$\langle \Phi(\mathscr{F}(t)) \rangle = \begin{cases} 0 & \text{for} \quad \mathscr{F}(t) \leqslant 0, \\ \Phi(\mathscr{F}(t)) & \text{for} \quad \mathscr{F}(t) > 0. \end{cases} \tag{7.7}$$

A material described by the relations (7.1)–(7.6) is called an elastic-perfectly viscoplastic material. The relations (7.1)–(7.6) defined the internal state variable structure of an elastic-perfectly viscoplastic material ([10]).

([10]) A thorough discussion of the internal state variable description of dynamic plasticity is given in Ref. [4].

http://rcin.org.pl

If the constitutive equation (7.3) can be written in the form

$$P(t) = \hat{N}(C(t), T(t)), \tag{7.8}$$

then the internal state σ_2 is determined by the expression

$$\sigma_2 = (C(t), T(t)) \in \Sigma_2. \tag{7.9}$$

Let us assume that all conditions explained in the previous section are satisfied for an elastic-perfectly viscoplastic material.

Differentiating the constitutive equation (7.3) with respect to time gives the evolution equation for the stress $T(t)$ in the form

$$\frac{d}{d\tau} T(\tau) = \hat{\boldsymbol{\beta}}_0(C(\tau), T(\tau)) + \hat{\boldsymbol{\beta}}_1(C(\tau), T(\tau))[\dot{C}(\tau)] \tag{7.10}$$

with the initial condition

$$T(0) = \hat{S}_1(C(0), P(0)), \tag{7.11}$$

where

$$\begin{aligned} \hat{\boldsymbol{\beta}}_0 &= \gamma \langle \Phi(\mathscr{F}(\tau)) \rangle \, \partial_{P(\tau)} \hat{S}_1 [\partial_{T(\tau)} f], \\ \hat{\boldsymbol{\beta}}_1 &= \partial_{C(\tau)} \hat{S}_1. \end{aligned} \tag{7.12}$$

The equations (7.8)–(7.12) define the rate type structure for an elastic-perfectly plastic material.

8. Example 2. Elastic-viscoplastic material (description in the entire range of strain rate)

Let us now postulate

$$A(t) = (\varkappa(t), \gamma(t), P(t)), \tag{8.1}$$

where $\varkappa(t)$ is the work-hardening scalar parameter, $\gamma(t)$ is the viscosity scalar parameter and $P(t)$ is the inelastic deformation tensor. The intrinsic state is determined by

$$\sigma_1 = (C(t), \varkappa(t), \gamma(t), P(t)) \in \Sigma_1. \tag{8.2}$$

We assume the constitutive equation in the form

$$T(t) = \hat{S}_1(C(t), \varkappa(t), \gamma(t), P(t)), \tag{8.3}$$

and postulate the static yield condition

$$\mathscr{F}(t) = \frac{f(T(t), P(t))}{\varkappa(t)} - 1. \tag{8.4}$$

The evolution equations for the internal state variables $\varkappa(t)$, $\gamma(t)$, $P(t)$ are postulated in the form

$$\begin{aligned} \dot{P}(t) &= \gamma(\tau) \langle \Phi(\mathscr{F}(\tau)) \rangle \, \partial_{T(\tau)} f, \\ \dot{\gamma}(\tau) &= \gamma(\tau) \langle \Phi(\mathscr{F}(\tau)) \rangle \, \mathrm{tr}\, [\hat{\boldsymbol{\Gamma}}_0(\sigma_1) \partial_{T(\tau)} f], \\ \dot{\varkappa}(\tau) &= \gamma(\tau) \langle \Phi(\mathscr{F}(\tau)) \rangle \, \mathrm{tr}\, [\hat{\mathbf{K}}_0(\sigma_1) \partial_{T(\tau)} f], \end{aligned} \tag{8.5}$$

http://rcin.org.pl

with the initial conditions

$$P(0) = P_0, \quad \gamma(0) = \gamma_0, \quad \varkappa(0) = \varkappa_0. \tag{8.6}$$

It is easy to show that for this case there is no simple transition to rate type structure of an elastic-viscoplastic material. But we shall prove that the internal state variable structure of an elastic-viscoplastic material determined by the Eqs. (8.1)–(8.6) is isomorphic with some mixed material structure.

To show this let us assume that the constitutive equation (8.3) can be written in the form

$$P(t) = \hat{\hat{\mathbf{N}}}\big(C(t), \varkappa(t), \gamma(t), T(t)\big). \tag{8.7}$$

Let us postulate that the intrinsic state σ_2 is now determined by the expression

$$\sigma_2 = \big(C(t), \varkappa(t), \gamma(t), T(t)\big) \in \Sigma_2 \tag{8.8}$$

We have assumed that the method of preparation space $\mathscr{K}$ is the Cartesian product of the two-dimensional vector space V_2 and the stress space $\mathscr{S}$, i.e.

$$\mathscr{K} \equiv V_2 \times \mathscr{S}. \tag{8.9}$$

In a similar way as in the previous section we can obtain the evolution equation for the stress tensor $T(t)$ in the form

$$\frac{d}{d\tau} T(\tau) = \hat{\hat{\boldsymbol{\beta}}}_0(\sigma_2) + \hat{\hat{\boldsymbol{\beta}}}_1(\sigma_2)\,[\dot{C}(\tau)], \tag{8.10}$$

where

$$\begin{aligned} \hat{\hat{\boldsymbol{\beta}}}_0 &= \gamma(\tau)\langle\Phi\big(\mathscr{F}(\tau)\big)\rangle\{\partial_{\varkappa(\tau)}\mathbf{S}_1 \operatorname{tr}[\hat{\mathbf{K}}_0\,\partial_{T(\tau)}f] \\ &\qquad + \partial_{\gamma(\tau)}\hat{\mathbf{S}}_1 \operatorname{tr}[\hat{\boldsymbol{\Gamma}}_0\,\partial_{T(\tau)}f] + \partial_{P(\tau)}\hat{\mathbf{S}}_1[\partial_{T(\tau)}f]\}, \\ \hat{\hat{\boldsymbol{\beta}}}_1 &= \partial_{C(\tau)}\hat{\mathbf{S}}_1, \end{aligned} \tag{8.11}$$

with the initial value

$$T(0) = \hat{\mathbf{S}}_1\big(C(0), \varkappa(0), \gamma(0), P(0)\big). \tag{8.12}$$

The evolution equations for $\varkappa(t)$ and $\gamma(t)$ have the same form as postulated by (8.5), i.e.

$$\begin{aligned} \dot{\gamma}(\tau) &= \gamma(\tau)\langle\Phi\big(\mathscr{F}(\tau)\big)\rangle \operatorname{tr}[\hat{\boldsymbol{\Gamma}}_0(\sigma_2)\partial_{T(\tau)}f], \\ \dot{\varkappa}(\tau) &= \gamma(\tau)\langle\Phi\big(\mathscr{F}(\tau)\big)\rangle \operatorname{tr}[\hat{\mathbf{K}}_0(\sigma_2)\partial_{T(\tau)}f], \end{aligned} \tag{8.13}$$

with

$$\gamma(0) = \gamma_0, \quad \varkappa(0) = \varkappa_0. \tag{8.14}$$

The equations (8.7)–(8.14) defined the mixed material structure which is isomorphic with the internal state variable structure of an elastic-viscoplastic material (8.1)–(8.6).

References

1. W. KOSIŃSKI and P. PERZYNA, *The unique material structures*, Bull. Acad. Polon. Sci., Série Sci. Techn., **21**, 655–662, 1973.

2. W. NOLL, *A new mathematical theory of simple materials*, Arch. Rat. Mech. Anal., **48**, 1–50, 1972.

http://rcin.org.pl

3. P. PERZYNA, *A gradient theory of rheological materials with internal structural changes*, Archives Mechanics, **23**, 845–850, 1971.
4. P. PERZYNA, *Internal variable description of plasticity*, in "Problems of Plasticity", Proc. Inter. Symp. on Foundations of Plasticity, Warsaw, August 30–September 2, 1972, ed. A. SAWCZUK, Noordhoff, pp. 145–170, Leyden 1974.
5. P. PERZYNA, *Physical theory of viscoplasticity*, Bull. Acad. Polon. Sci., Série Sci. Techn., **21**, 123–139, 1973.
6. P. PERZYNA and W. KOSIŃSKI, *A mathematical theory of materials*, Bull. Acad. Polon. Sci., Série Sci. Techn., **21**, 647–654, 1973.
7. P. PERZYNA, *The constitutive equations describing thermomechanical behaviour of materials at high rates of strain*, International Symposium, Oxford, April 2–4, 1974 Institute of Physics Conf. Ser. No. 21, pp. 138–153.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

http://rcin.org.pl