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Elastic strip with a crack under periodic loading

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AN ELASTIC strip containing a semi-infinite crack is subject to periodic displacement along the edges. Using the complex Fourier transforms and the approximate Wiener-Hopf technique, the dynamic stress intensity factor, the crack opening and the stress field in front of the crack tip are determined. The state of stress is evaluated by means of the numerical inversion of transforms along the real axis and is expressed in terms of the excitation frequency ω.

Pasmo sprężyste, zawierające półnieskończoną szczelinę, poddane jest okresowym w czasie wymuszeniem kinematycznym na swych brzegach. Za pomocą zespolonej transformacji Fouriera oraz przybliżonej procedury faktoryzacyjnej Wienera-Hopfa wyznaczono współczynnik dynamicznej koncentracji naprężenia, rozwarcie szczeliny oraz stan naprężenia przed wierzchołkiem szczeliny. Stan naprężenia w pasmie wyznaczono odwracając otrzymane transformacje w sposób numeryczny wzdłuż osi rzeczywistej; jest on funkcją częstości ω wymuszonych przemieszczeń brzegów pasma.

Упругая полоса, содержавшая полубесконечную трещину, подвергнута периодическим во времени, кинематическим вынуждением на своих границах. С помощью комплексного преобразования Фурье и приближенной процедуры факторизации Винера-Хопфа определены коэффициент динамической концентрации напряжений, раскрытие трещины и напряженное состояние перед вершиной трещины. Напряженное состояние в полосе определено обращая полученные изображения численным образом вдоль действительной оси; оно является функцией частоты ω вынужденных перемещений границ полосы.

1. Introduction

KNOWLEDGE of the exact distribution of stresses and strains in the vicinity of a crack proves to be of primary importance for the general solution of the crack problem. In linear elastic bodies and brittle fracture problems it is sometimes sufficient to know the value of the stress intensity factor (SIF), i.e. the singular behaviour of stresses at the crack tip, in order to be able to estimate the behaviour of the crack. In dynamic crack problems, however, in which the influence of inertia forces is difficult to determine, it may prove necessary to know the state of stress and strain within the entire region of the strip. Therefore the present paper deals not only with the determination of the stress intensity factor but also presents the results concerning the crack opening, stress distribution in front of the crack and also the stresses at other points of the strip.

2. Formulation of the problem

Let us consider an infinite elastic strip of width $2h$ containing a semi-infinite crack; the edges of the strip are fixed in the longitudinal direction, $u(x, \pm h) = 0$ and are periodi-

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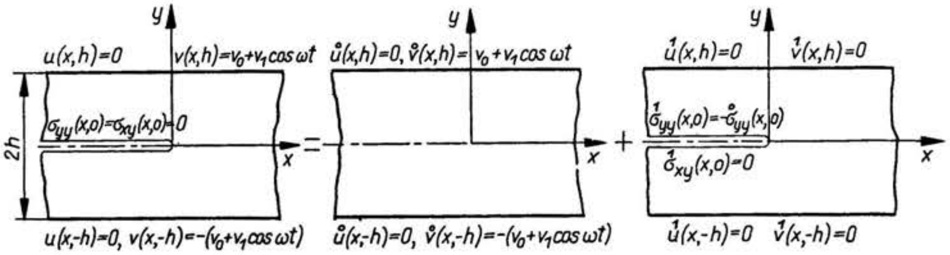
cally excited in the transversal direction by the displacements $v(x, \pm h) = \pm(v_0+v_1\cos\omega t)$. The plane state of strain is considered, and the body is assumed to be linearly elastic. The statical component of displacement $\pm v_0$ is so assumed that the crack edges do not touch each other in the process of vibration.

The solution is obtained by superposition of the following two partial problems shown in Fig. 1:



Problem (a) Problem (b)

FIG. 1.

(a) A strip without the crack, with the boundary conditions $\overset{0}{u}(x, \pm h) = 0$, $\overset{0}{v}(x, \pm h) = \pm(v_0+v_1\cos\omega t)$, and

(b) A strip with the crack, with the boundary conditions $\overset{1}{u}(x, \pm h) = 0$, $\overset{1}{v}(x, \pm h) = 0$, the crack being loaded by $\overset{1}{\sigma}_{yy}(x, 0) = -\overset{0}{\sigma}_{yy}(x, 0)$.

The static problem of $v^{(\text{stat})}(x, \pm h) = \pm v_0$ will not be considered in this paper since the corresponding solutions may be found, e.g. in the paper [1] by W. G. KNAUSS, or may be derived from the dynamic solution by the limiting procedure of $\omega \to 0$, v_1 being replaced with v_0.

Let us start with the Lamé equations written in Cartesian coordinates

$$\begin{aligned} \mu\nabla^2 u+(\lambda+\mu)\theta_{,x} &= \varrho\ddot{u}, \\ \mu\nabla^2 v+(\lambda+\mu)\theta_{,y} &= \varrho\ddot{v}. \end{aligned} \tag{2.1}$$

Here λ, μ are the Lamé material constants, ϱ — mass density, θ — dilatation, $\nabla^2 = \partial^2/\partial x^2+\partial^2/\partial y^2$, and $\partial/\partial x(\ ) = (\ )_{,x}$ $\partial/\partial t(\ ) = (\dot{\ })$. Using the Helmholtz representation

$$u(x, y, t) = \varphi_{,x}+\psi_{,y}, \quad v(x, y, t) = \varphi_{,y}-\psi_{,x} \tag{2.2}$$

it will be assumed that the state of harmonic vibrations allows for writing the corresponding solutions in the form of products,

$$\varphi(x, y, t) = \varphi^*(x, y)\cos\omega t, \quad \psi(x, y, t) = \psi^*(x, y)\cos\omega t. \tag{2.3}$$

On substituting the Eqs. (2.2) and (2.3) into (2.1) we obtain two partial differential equations for the amplitudes $\varphi^*(x, y)$ and $\psi^*(x, y)$,

$$\nabla^2\varphi^*+\varkappa_1^2\varphi^* = 0, \quad \nabla^2\psi^*+\varkappa_2^2\psi^* = 0 \tag{2.4}$$

with $\varkappa_1 = \omega/c_1$, $\varkappa_2 = \omega/c_2$, and $c_1^2 = (\lambda+2\mu)/\varrho$, $c_2^2 = \mu/\varrho$.

Let us, moreover, introduce the two-sided complex Fourier transform

$$F(\alpha, y) = F^-(\alpha, y)+F^+(\alpha, y) \tag{2.5}$$

with

$$F^-(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(x, y)e^{i\alpha x}dx \quad \text{reg. for } \operatorname{Im}\{\alpha\} < \tau_+,$$

$$F^+(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x, y)e^{i\alpha x}dx \quad \text{reg. for } \operatorname{Im}\{\alpha\} > \tau_-.$$

The corresponding inverse formula has the form

$$\begin{gathered} f(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau_0}^{\infty+i\tau_0} F(\alpha, y)e^{-i\alpha x}d\alpha, \\ \alpha = \sigma + i\tau \quad \text{and} \quad \tau_0 \in (\tau_-, \tau_+). \end{gathered} \tag{2.6}$$

The Fourier transforms applied to the Eqs. (2.4) yield the ordinary differential equations

$$\left[\frac{d^2}{dy^2} - (\alpha^2 - \varkappa_1^2)\right]\Phi^*(\alpha, y) = 0, \quad \left[\frac{d^2}{dy^2} - (\alpha^2 - \varkappa_2^2)\right]\Psi^*(\alpha, y) = 0 \tag{2.7}$$

with the general solutions

$$\begin{aligned} \Phi^*(\alpha, y) &= C_1(\alpha)\operatorname{sh}(y\sqrt{\alpha^2 - \varkappa_1^2}) + C_2(\alpha)\operatorname{ch}(y\sqrt{\alpha^2 - \varkappa_1^2}), \\ \Psi^*(\alpha, y) &= C_3(\alpha)\operatorname{sh}(y\sqrt{\alpha^2 - \varkappa_2^2}) + C_4(\alpha)\operatorname{ch}(y\sqrt{\alpha^2 - \varkappa_2^2}). \end{aligned} \tag{2.8}$$

In this manner we obtain the transformed displacement field and also, by means of Hooke's law, the transformed field of stresses,

$$\begin{aligned} U^*(\alpha, y) &= -i\{\alpha[C_1(\alpha)\operatorname{sh}(y\sqrt{\alpha^2 - \varkappa_1^2}) + C_2(\alpha)\operatorname{ch}(y\sqrt{\alpha^2 - \varkappa_1^2}) \\ &\qquad + i\sqrt{\alpha^2 - \varkappa_2^2}[C_3(\alpha)\operatorname{ch}(y\sqrt{\alpha^2 - \varkappa_2^2}) + C_4(\alpha)\operatorname{sh}(y\sqrt{\alpha^2 - \varkappa_2^2})]\}, \\ V^*(\alpha, y) &= \sqrt{\alpha^2 - \varkappa_1^2}\,[C_1(\alpha)\operatorname{ch}(y\sqrt{\alpha^2 - \varkappa_1^2}) + C_2(\alpha)\operatorname{sh}(y\sqrt{\alpha^2 - \varkappa_1^2})] \\ &\qquad + i\alpha[C_3(\alpha)\operatorname{sh}(y\sqrt{\alpha^2 - \varkappa_2^2}) + C_4(\alpha)\operatorname{ch}(y\sqrt{\alpha^2 - \varkappa_2^2})], \\ \Sigma^*_{xx}(\alpha, y) &= -\mu\{(2\alpha^2 + \varkappa_2^2 - 2\varkappa_1^2)[C_1(\alpha)\operatorname{sh}(y\sqrt{\alpha^2 - \varkappa_1^2}) + C_2(\alpha)\operatorname{ch}(y\sqrt{\alpha^2 - \varkappa_1^2})] \\ &\qquad + 2i\alpha\sqrt{\alpha^2 - \varkappa_2^2}[C_3(\alpha)\operatorname{ch}(y\sqrt{\alpha^2 - \varkappa_2^2}) + C_4(\alpha)\operatorname{sh}(y\sqrt{\alpha^2 - \varkappa_2^2})]\}, \\ \Sigma^*_{yy}(\alpha, y) &= \mu\{(2\alpha^2 - \varkappa_2^2)\,[C_1(\alpha)\operatorname{sh}(y\sqrt{\alpha^2 - \varkappa_1^2}) + C_2(\alpha)\operatorname{ch}(y\sqrt{\alpha^2 - \varkappa_1^2})] \\ &\qquad + 2i\alpha\sqrt{\alpha^2 - \varkappa_2^2}[C_3(\alpha)\operatorname{ch}(y\sqrt{\alpha^2 - \varkappa_2^2}) + C_4(\alpha)\operatorname{sh}(y\sqrt{\alpha^2 - \varkappa_2^2})]\}, \\ \Sigma^*_{xy}(\alpha, y) &= -i\mu\{2\alpha\sqrt{\alpha^2 - \varkappa_1^2}\,[C_1(\alpha)\operatorname{ch}(y\sqrt{\alpha^2 - \varkappa_1^2}) + C_2(\alpha)\operatorname{sh}(y\sqrt{\alpha^2 - \varkappa_1^2})] \\ &\qquad + i(2\alpha^2 - \varkappa_2^2)\,[C_3(\alpha)\operatorname{sh}(y\sqrt{\alpha^2 - \varkappa_2^2}) + C_4(\alpha)\operatorname{ch}(y\sqrt{\alpha^2 - \varkappa_2^2})]\}. \end{aligned} \tag{2.9}$$

The unknown integration functions $C_i(\alpha)$ are found from the corresponding boundary conditions.

3. Boundary conditions

Problem (a). (The strip without the crack). Transformed boundary conditions have the form $\overset{0}{U}{}^*(\alpha, \pm h) = 0$ and $\overset{0}{V}{}^*(\alpha, \pm h) = \pm v_1\sqrt{2\pi}\,\delta(\alpha)$. The Dirac delta-function occurring in that expression allows for a closed-form inverse transform

$$\overset{0}{u}{}^*(x, y) = 0, \quad \overset{0}{v}{}^*(x, y) = \frac{v_1 \sin(\varkappa_1 y)}{\sin(\varkappa_1 h)}, \quad \overset{0}{\sigma}{}^*_{xy}(x, y) = 0,$$

$$\text{(3.1)} \qquad \overset{0}{\sigma}{}^*_{xx}(x, y) = \frac{2\mu\nu}{1-2\nu} \frac{v_1 \varkappa_1 \cos(\varkappa_1 y)}{\sin(\varkappa_1 h)},$$

$$\overset{0}{\sigma}{}^*_{yy}(x, y) = \frac{2\mu(1-\nu)}{1-2\nu} \frac{v_1 \varkappa_1 \cos(\varkappa_1 y)}{\sin(\varkappa_1 h)}, \quad \varkappa_1 h \neq n\pi.$$

The static solution to be superposed over the dynamic one may be obtained from the Eq. (3.1) by assuming $\omega \to 0$ (or $\varkappa_1 \to 0$) and replacing v_1 with v_0.

Problem (b). (The strip with a loaded crack). Owing to the symmetry, only the upper half-strip is considered. The transformed boundary conditions are

$$\text{(3.2)} \qquad \begin{gathered} \overset{1}{U}{}^*(\alpha, h) = 0, \quad \overset{1}{V}{}^*(\alpha, h) = 0, \quad \overset{1}{\Sigma}{}^*_{xy}(\alpha, 0) = 0, \\ \overset{1}{V}{}^*(\alpha, 0) = V^-(\alpha) \quad \text{und} \quad \overset{1}{\Sigma}{}^*_{yy}(\alpha, 0) = \Sigma^-_{yy}(\alpha) + \Sigma^+_{yy}(\alpha) \end{gathered}$$

with

$$V^-(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \overset{1}{v}{}^*(x, 0) e_{i\alpha x} dx,$$

$$\Sigma^-_{yy}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \overset{1}{\sigma}{}^*_{yy}(x, 0) e^{i\alpha x} dx = -\frac{p_1}{\sqrt{2\pi}} \left[-\frac{i}{\alpha} + \pi\delta(\alpha) \right] \quad \text{reg. for } \operatorname{Im}\{\alpha\} \leqslant 0.$$

Here $p_1 = \overset{0}{\sigma}{}^*_{yy}(x, 0)$. In order to be able to evaluate the inverse transform along the real axis, the axis itself will be included to the strip of regularity — except the point $\alpha = 0$ (similar to the assumption of [3]).

By satisfying the boundary conditions $(3.2)_{1-4}$ we obtain the integration functions $C_i = C_i\{V^-(\alpha)\}$, and inserting them into the Eq. $(3.2)_5$ yields the Wiener-Hopf integral equation

$$\text{(3.3)} \qquad V^-(z) = -\frac{h}{\mu} H(z) [\Sigma^+_{yy}(z) - P_1(z)] \quad \text{reg. for } -\varepsilon < \operatorname{Im}\{z\} \leqslant 0.$$

The functions to be determined are $V^-(z)$ and $\Sigma^+_{yy}(z)$, and the kernel $H(z)$ is equal to

$$H(z) = f_1(z)/f_2(z), \quad H(z) = H(-z),$$

$$f_1(z) = \sigma_2^2 \sqrt{z^2 - \sigma_1^2} \Big[z^2 \operatorname{sh}\sqrt{z^2 - \sigma_2^2} \operatorname{ch}\sqrt{z^2 - \sigma_1^2}$$
$$- \sqrt{(z^2 - \sigma_1^2)(z^2 - \sigma_2^2)} \operatorname{sh}\sqrt{z^2 - \sigma_1^2} \operatorname{ch}\sqrt{z^2 - \sigma_2^2}$$

$$f_2(z) = [4z^4+(2z^2-\sigma_2^2)^2]\sqrt{(z^2-\sigma_1^2)(z^2-\sigma_2^2)}\,\mathrm{ch}\sqrt{z^2-\sigma_1^2}\,\mathrm{ch}\sqrt{z^2-\sigma_2^2}$$
$$-z^2[(2z^2-\sigma_2^2)^2+4(z^2-\sigma_1^2)(z^2-\sigma_2^2)]\,\mathrm{sh}\sqrt{z^2-\sigma_1^2}\,\mathrm{sh}\sqrt{z^2-\sigma_2^2}$$
$$-4z^2(2z^2-\sigma_2^2)\sqrt{(z^2-\sigma_1^2)(z^2-\sigma_2^2)}\,.$$

Here the following notations are introduced:

$$P_1(z) = \frac{hp_1}{\sqrt{2\pi}}\left[-\frac{i}{z}+\pi\delta(z)\right], \quad z = ah, \quad \sigma_1 = \varkappa_1 h \quad \text{and} \quad \sigma_2 = \varkappa_2 h.$$

An exact, analytical factorization of the Eq. (3.3) is not possible, and so we shall apply the approximate procedure proposed by W. T. KOITER [4] according to which the kernel is written in the form

$$H(z) = \overline{H}(z)H_1(z)\,. \tag{3.4}$$

The functions $\overline{H}(z)$ and $H_1(z)$ must satisfy the following conditions:

$\overline{H}(z)$ is the approximate kernel function with $\overline{H}(0) = H(0)$ and $\overline{H}(\infty) = H(\infty)$, it contains all the zeros and poles of $H(z)$ in the region $|\mathrm{Im}\{z\}| < \varepsilon$ for $0 < z < \infty$;

$H_1(z)$ is the residual function with the properties $H_1(0) = H_1(\infty) = 1$ and has no zeros and poles in the region $|\mathrm{Im}\{z\}| < \varepsilon$ for $0 < z < \infty$.

According to B. NOBLE [5], we have then

$$H_1(z) = H_1^+(z)/H_1^-(z) \tag{3.5}$$

with the following notations

$$\ln H_1^+(z) = \frac{1}{2\pi i}\int\limits_{-\infty+i\gamma_2}^{\infty+i\gamma_2}\frac{\ln H_1(\xi)}{\xi-z}\,d\xi,$$

$$\ln H_1^-(z) = \frac{1}{2\pi i}\int\limits_{-\infty+i\gamma_1}^{\infty+i\gamma_1}\frac{\ln H_1(\xi)}{\xi-z}\,d\xi,$$

and $-\varepsilon < \gamma_2 < \gamma_1 < \varepsilon$. From the symmetry properties it follows that also the functions H_1^+ and H^- satisfy the conditions $H_1^\pm(0) = H_1^\pm(\infty) = 1$.

Discussion of the kernel function $H(z)$ at real values of z in the interval $0 < z < \infty$ concerning the zeros and poles yields in the most interesting region of $\sigma_1 < \pi/2$ and $0 \leqslant \nu < 0.4$ (ν — Poisson's ratio) the results:

Case i: $\sigma_2 < \pi/2$, $f_1(z)$, $f_2(z)$ have no zeros;

Case ii: $\sigma_2 < \pi/2$, $f_1(z)$ has a single zero at $z = z_1$,
$f_2(z)$ has a single zero at $z = z_2$.

Using the above results let us assume for the approximate kernel function the following, relatively inaccurate representation:

$$\overline{H}(z) = \overline{\overline{H}}(z) = \frac{1-\nu}{\sqrt{z^2+A^2}}\,\frac{z^2-z_1^2}{z^2-z_2^2} \tag{3.6}$$

with the convention that $\begin{cases}\text{in Case i: } z_1 = z_2,\\ \text{in Case ii: } z_1 \neq z_2.\end{cases}$

Here $A = (1-\nu)z_1^2\sigma_2^2/(z_2^2\sigma_1\tan\sigma_1)$. In spite of its simplicity, the function satisfies both conditions required, $\bar{\bar{H}}(0) = H(0)$ and $\bar{\bar{H}}(\infty) = H(\infty)$, and also contains all the zeros and poles in the extended region of regularity. From [6] it follows that this simple assumption is sufficient in calculating the accurate value of the dynamic SIF. The representation (3.6) may be generalized and written in the form

$$\bar{H}(z) = \bar{\bar{H}}(z)\frac{M(z)}{L(z)} \quad \text{with} \quad \frac{M(z)}{L(z)} = \frac{z^4+2\vartheta z^2 z_0^2+z_0^4}{z^4+z_0^4} \tag{3.7}$$

with $\vartheta = H(z_0)/\bar{\bar{H}}(z_0)-1$. Then $\bar{H}(z_0) = H(z_0)$ makes the original function and its approximation coincide at an additional arbitrary point z_0 thus increasing the accuracy of the new representation. In this paper z_0 will be selected as the point at which $|H(z_0)-\bar{\bar{H}}(z_0)|$ reaches its maximum value. This makes it possible to reduce the relative error $[H(z)-\bar{H}(z)]/H(z)$ within the region of ω and ν considered to be less than 5%. The values of z_0 and ϑ are determined for prescribed ω and ν by numerical methods.

The approximate function $\bar{H}(z)$ obtained in this manner may easily be factorized to yield

$$\bar{H}(z) = (1-\nu)\frac{z^2-z_1^2}{z^2-z_2^2}K^+(z)K^-(z). \tag{3.8}$$

Here

$$K^\pm(z) = \frac{M^\pm(z)}{R^\pm(z)L^\pm(z)} \quad \text{and} \quad R^\pm(z) = \sqrt{z\pm iA}\,.$$

Decomposition of $M(z)$

$$\begin{aligned} M^-(z) &= (z-z_1^M)(z-z_2^M) \\ M^+(z) &= (z-\bar{z}_1^M)(z-\bar{z}_2^M) \end{aligned} \quad \text{with the roots} \quad \begin{aligned} z_1^M &= \frac{z_0}{\sqrt{2}}\sqrt{1-\vartheta}+i\sqrt{1+\vartheta}], \\ z_2^M &= \frac{z_0}{\sqrt{2}}[-\sqrt{1-\vartheta}+i\sqrt{1+\vartheta}], \end{aligned}$$

enables us to factorize $L(z)$ by the limiting procedure

$$L^\pm(z) = \lim_{\vartheta\to 0} M^\pm(z) \quad \text{and} \quad \{z_1^L, z_2^L\} = \lim_{\vartheta\to 0}\{z_1^M, z_2^M\}.$$

Dashed symbols denote complex conjugate roots (e.g. $\bar{z}_1^M$ or $\bar{z}_2^M$) which may be obtained by replacing the imaginary i with $-i$.

Substituting the Eqs. (3.8) and (3.5) into (3.3) according to the Eq. (3.4) we obtain after certain transformations

$$-\frac{\mu}{h(1-\nu)}\frac{z^2-z_2^2}{z^2-z_1^2}\frac{H_1^-(z)}{K^-(z)}V^-(z) = K^+(z)H_1^+(z)\Sigma_{yy}^+(z)-E(z). \tag{3.9}$$

Functions $E(z) = K^+(z)H_1^+(z)P_1(z)$ may be factorited by elementary methods to give

$$E(z) = E^+(z)-E^-(z) \tag{3.10}$$

with the notations

$$E^+(z) = [K^+(z)H_1^+(z) - K^+(0)]P_1(z),$$
$$E^-(z) = -K^+(0)P_1(z).$$

The Eqs. (3.9) and (3.10) and the Liouville theorem make it possible to determine the sought-for functions,

$$V^-(z) = -\frac{h(1-\nu)}{\mu}\frac{z^2-z_1^2}{z^2-z_2^2}\frac{K^-(z)E^-(z)}{H_1^-(z)},$$
(3.11)
$$\Sigma_{yy}^+(z) = \frac{E^+(z)}{K^+(z)H_1^+(z)}.$$

The inverse transform performed according to the Eq. (2.6) yields the crack opening function $\overset{1}{v}{}^*(x, 0)$ for $x < 0$ and the stress $\overset{1}{\sigma}{}^*_{yy}(x, 0)$ for $x > 0$.

4. Dynamic stress intensity factor

If our interest is confined to the asymptotic behaviour of the original functions in the vicinity of the crack tip then we may utilize the well-known theorems by ABEL (cf. e.g. [5]) which make it possible to determine the behaviour of transformed functions at $z \to \pm\infty$ once the asymptotic behaviour of their originals at $x \to \pm 0$ is known. The Eq. (3.11) determines *exactly* the asymptotic behaviour of $V^-(z)$ and $\Sigma_{yy}^+(z)$ at $z \to \infty$ (since $H_1^\pm(\infty) = 1$)

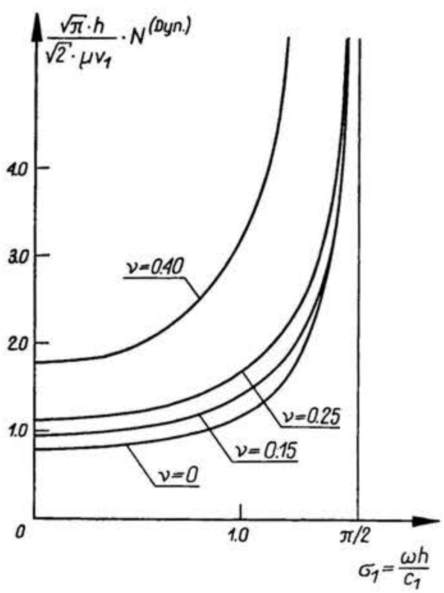


FIG. 2.

what enables us to write down the asymptotic behaviour of amplitudes $\overset{1}{v}^*(\xi, 0)$ and $\overset{1}{\sigma}_{yy}(\xi, 0)$ for $\xi = x/h \to 0$ in the original region

$$(4.1)\qquad \begin{aligned} \overset{1}{v}^*(\xi, 0) &= \frac{2(1-\nu)h}{\mu} N^{(\mathrm{Dyn})}\sqrt{-\xi} \quad &\text{for} \quad \xi \to -0,\\ \overset{1}{\sigma}{}^*_{yy}(\xi, 0) &= \frac{N^{(\mathrm{Dyn})}}{\sqrt{\xi}} \quad &\text{for} \quad \xi \to +0 \end{aligned}$$

with the dynamic SIF

$$(4.2)\qquad N^{(\mathrm{Dyn})} = \frac{2\mu v_1}{h\sqrt{2\pi(1-2\nu)}}\frac{z_2}{z_1}\sqrt{\frac{2\sigma_1}{\sin 2\sigma_1}} \quad \begin{cases}\text{Case i: } z_1 = z_2\\ \text{Case ii: } z_1 \neq z_2.\end{cases}$$

Fig. 2 represents the reduced value of the dynamic SIF $N^{(\mathrm{Dyn})} = \sqrt{\pi}hN^{(\mathrm{Dyn})}/(\sqrt{2}\mu v_1)$ as a function of $\sigma_1 = \omega h/c_1$ for several values of Poisson's ratio ν. Transition to the static case is determined by the limiting procedure

$$(4.3)\qquad \lim_{\omega\to 0} N^{(\mathrm{Dyn})} = N^{(\mathrm{Stat})} = \frac{2\mu v_1}{h\sqrt{2\pi(1-2\nu)}}.$$

The general solution may be found by superposing the static case due to the state of deformation $v^{(\mathrm{Stat})}(x, \pm h) = \pm v_0$.

5. Analytical determination of inverse transforms

Substituting in the Eq. (3.11) $H_1^+(z) = 1$ along the entire path of integration makes it possible to perform the required integration and determine the inverse transforms according to the Eq. (2.6),

$$(5.1)\qquad \begin{aligned} \overset{1}{v}^*(\xi, 0) &= -\frac{1-\nu}{\sqrt{2\pi}\mu}\int_{-\infty-i\varepsilon_1}^{\infty-i\varepsilon_1}\frac{z^2-z_1^2}{z^2-z_2^2}K^-(z)E^-(z)e^{-i\xi z}dz, \quad \xi < 0,\\ \overset{1}{\sigma}{}^*_{yy}(\xi, 0) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty-i\varepsilon_1}^{\infty-i\varepsilon_1}\frac{E^+(z)}{K^+(z)}e^{-i\xi z}dz, \quad \xi > 0,\ 0 < \varepsilon_1 < \varepsilon. \end{aligned}$$

The corresponding integrals are now evaluated by substituting the respective functions and by certain elementary transformations to yield the following results:

$$(5.2)\qquad \begin{aligned} \overset{1}{v}^*(\xi, 0) = -\frac{i(1-\nu)hp_1}{2\pi\mu\sqrt{iA}}\Big\langle \frac{z_1^2}{z_2^2}J(0, \xi) + ik_1\Big[\frac{z_1^2-(z_1^L)^2}{z_2^2-(z_1^L)^2}J(z_1^L, \xi)\\ -\frac{z_1^2-(z_2^L)^2}{z_2^2-(z_2^L)^2}J(z_2^L, \xi)\Big] + \frac{z_2^2-z_1^2}{2z_2}\Big\{\Big[\frac{1}{z_2} - \frac{ik_1(z_1^L-z_2^L)}{(z_2-z_1^L)(z_2-z_2^L)}\Big]J(z_2, \xi)\\ +\Big[\frac{1}{z_2} + \frac{ik_1(z_1^L-z_2^L)}{(z_2+z_1^L)(z_2+z_2^L)}\Big]J(-z_2, \xi)\Big\}\Big\rangle, \quad \xi < 0 \end{aligned}$$

with notations $k_1 = \sqrt{1+\vartheta}-1$ and

$$J(\beta, \xi) = \int_{-\infty-i\varepsilon_1}^{\infty-i\varepsilon_1} \frac{e^{-i\varepsilon_1}}{(z-\beta)\sqrt{z-iA}}\, dz$$

and, moreover, the stress

$$(5.3) \qquad \overset{1}{\sigma}{}^{*}_{yy}(\xi, 0) = \frac{ip_1}{2\pi\sqrt{iA}} \{Q(0, \xi)-ik_2[Q(\bar{z}_1^M, \xi)-Q(\bar{z}_2^M, \xi)]\}, \quad \xi > 0$$

with notations $k_2 = k_1/\sqrt{1-\vartheta}$ and

$$Q(\beta, \xi) = \int_{-\infty-i\varepsilon_1}^{\infty-i\varepsilon_1} \frac{\sqrt{z+iA}\, e^{-i\xi z}}{z-\beta}\, dz.$$

The partial integral $J(\beta, \xi)$ are determined by means of the Cauchy integral formula, the path of integration being selected according to Fig. 3a:

$$(5.4) \qquad J(\beta, \xi) = -2\pi i \operatorname{Res}(\beta) - \int_{C_1} \ldots - \int_{C_2} \ldots = -\frac{2\pi i e^{-i\xi\beta}}{\sqrt{-i}\sqrt{A+i\beta}} \operatorname{erf}\sqrt{-\xi(A+i\beta)}.$$

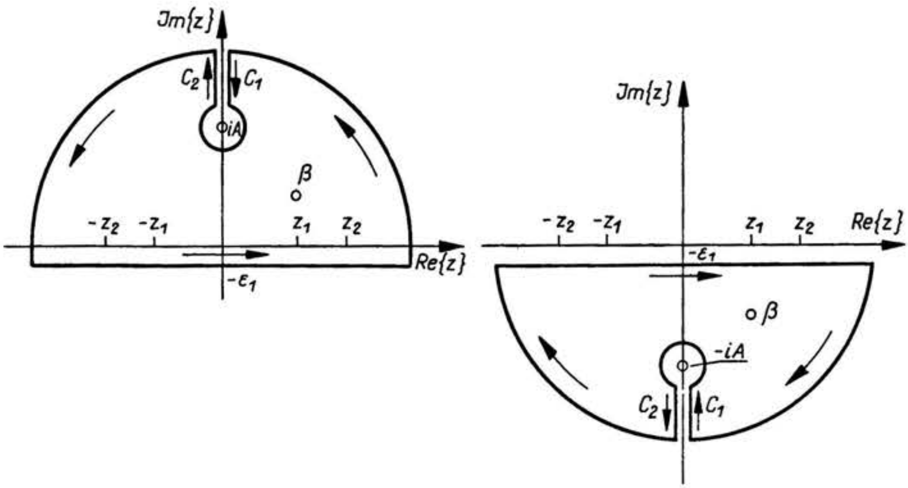


FIG. 3.

The other integral $Q(\beta, \xi)$ is found in a similar way for $\beta = \bar{z}_1^M$ or $\bar{z}_2^M$ and using the integration path shown in Fig. 3b,

$$(5.5) \qquad Q(\beta, \xi) = -2\pi i \operatorname{Res}(\beta) - \int_{C_1} \ldots - \int_{C_2} \ldots = 2\left[\sqrt{\frac{i\pi}{\xi}}\, e^{-A\xi} - i\sqrt{i}\,\pi\sqrt{A-i\beta}\, e^{-i\beta\xi} \operatorname{erf}\sqrt{\xi(A-i\beta)}\right].$$

In the case of $\beta = 0$ the pole lies outside the region of integration and so the corresponding result is

$$(5.6) \qquad Q(0, \xi) = -\int_{C_1} \ldots - \int_{C_2} = 2\left[\sqrt{\frac{-i\pi}{\xi}}\, e^{-A\xi} - \pi\sqrt{-iA}\, \operatorname{Erfc}\sqrt{A\xi}\right].$$

Evaluation of the contour integrals $\int\limits_{C_1} \ldots$ and $\int\limits_{C_2} \ldots$ in the Eqs. (5.4) and (5.5) is described in [7]. Substituting the Eqs. (5.4)–(5.6) into (5.2) or (5.3) yields

$$\text{(5.7)} \quad \overset{1}{v}{}^*(\xi, 0) = \frac{v_1}{\cos\sigma_1}\left\{\operatorname{erf}\sqrt{-A\xi} + \frac{ik_1\sqrt{A}z_2^2}{z_1^2}\left[\frac{z_1^2-(z_1^L)^2}{z_2^2-(z_1^L)^2}\,\frac{e^{-iz_1^L\xi}}{\sqrt{A+iz_1^L}} \times\right.\right.$$
$$\left. \times \operatorname{erf}\sqrt{-\xi(A+iz_1^L)} - \frac{z_1^2-(z_2^L)^2}{z_2^2-(z_2^L)^2}\,\frac{e^{-iz_2^L\xi}}{\sqrt{A+iz_2^L}}\operatorname{erf}\sqrt{-\xi(A+iz_2^L)}\right]$$
$$+\frac{z_2\sqrt{A}\,(z_2^2-z_1^2)}{2z_1^2}\left[\left(\frac{1}{z_2} - \frac{ik_1(z_1^L-z_2^L)}{(z_2-z_1^L)(z_2-z_2^L)}\right)\frac{e^{-iz_2\xi}}{\sqrt{A+iz_2}}\operatorname{erf}\sqrt{-\xi(A+iz_2)}\right.$$
$$\left.\left.+\left(\frac{1}{z_2} + \frac{ik_1(z_1^L-z_2^L)}{(z_2+z_1^L)(z_2+z_2^L)}\right)\frac{e^{iz_2\xi}}{\sqrt{A-iz_2}}\operatorname{erf}\sqrt{-\xi(A-iz_2)}\right]\right\}, \quad \xi < 0$$

and

$$\text{(5.8)} \quad \overset{1}{\sigma}{}^*_{yy}(\xi, 0) = \frac{\mu v_1}{\pi h\sqrt{A}}\,\frac{\sigma_2^2}{\sigma_1\sin\sigma_1}\left\{\sqrt{\frac{\pi}{\xi}}\,e^{-A\xi} - \pi\sqrt{A}\,\operatorname{Erfc}\sqrt{A\xi}\right.$$
$$-ik_2\pi\left[\sqrt{A-i\bar{z}_1^M}\,e^{-i\bar{z}_1^M\xi}\operatorname{erf}\sqrt{\xi(A-i\bar{z}_1^M)}\right.$$
$$\left.\left.-\sqrt{A-i\bar{z}_2^M}\,e^{-i\bar{z}_2^M\xi}\operatorname{erf}\sqrt{\xi(A-i\bar{z}_2^M)}\,\right]\right\}, \quad \xi > 0.$$

In Case i: $z_1 = z_2$ (coinciding zero points) the Eq. (5.7) is simplified to

$$\text{(5.9)} \quad \overset{1}{v}{}^*(\xi, 0) = \frac{v_1}{\cos\sigma_1}\left\{\operatorname{erf}\sqrt{-A\xi} + ik_1\sqrt{A}\left[\frac{e^{-iz_1^L\xi}}{\sqrt{A+iz_1^L}}\operatorname{erf}\sqrt{-\xi(A+iz_1^L)}\right.\right.$$
$$\left.\left.-\frac{e^{-iz_2^L\xi}}{\sqrt{A+iz_2^L}}\operatorname{erf}\sqrt{-\xi(A+iz_2^L)}\right]\right\}, \quad \xi < 0.$$

Figure 4 presents the form of the crack opening amplitude $\overset{1}{v}{}^*/v_1$ at various values of the excitation frequency ω. The amplitude of dynamic stresses $(h/\mu v_1)\overset{1}{\sigma}{}^*_{yy}(\xi, 0)$ at $\omega = 0.60\, c_1/h$ is shown in Fig. 6a.

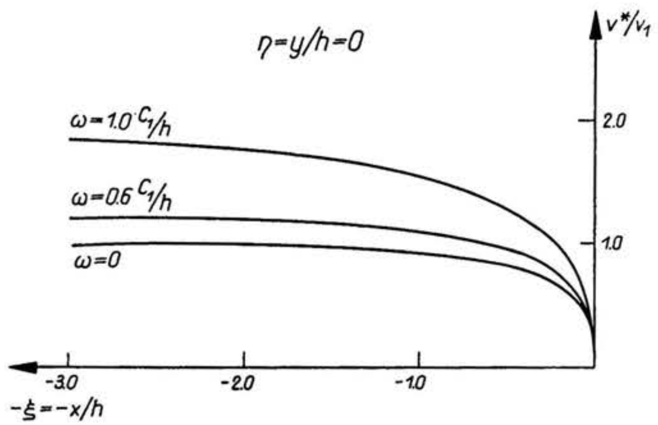


FIG. 4.

6. Numerical evaluation of inverse transforms

The inverse transforms at an arbitrary point of the strip must be computed numerically. Owing to the extension of the strip of regularity, the corresponding integrations may be performed along the real axis. In dealing with the residual functions $H_1(z)$, the identity $H_1^+(z) = 1$ must be assumed. The amplitudes of dynamic stresses and displacements at a point (ξ, η), with $\xi = x/h$, $\eta = y/h$, $\lambda = \mathrm{Re}\,\{z\}$, are determined by numerical evaluation of the integrals

$$\text{(6.1)}\qquad \begin{aligned} \overset{1}{\sigma}{}^*_{ij}(\xi, \eta) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Sigma_{ij}(\lambda, \eta) e^{-i\lambda\xi} d\lambda, \\ \overset{1}{u}{}^*_i(\xi, \eta) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_i(\lambda, \eta) e^{-i\lambda\xi} d\lambda. \end{aligned}$$

The integration functions $C_l = C_l\{V^-(\alpha)\}$ resulting from the boundary conditions $(3.2)_{1-4}$ are substituted in the Eq. (2.9) which makes it possible to represent the expressions along the real axis

$$\text{(6.2)}\qquad \begin{aligned} \Sigma_{ij}(\lambda, \eta) &= z_{ij}(\lambda, \eta) V^-(\lambda), \\ U_i(\lambda, \eta) &= z_i(\lambda, \eta) V^-(\lambda), \end{aligned}$$

the functions $Z_{ij}(\lambda, \eta)$ and $Z_i(\lambda, \eta)$ being either purely imaginary or real. On substituting the Eq. (6.2) into (6.1) and taking into account the decomposition of $V^-(\lambda)e^{i\lambda\xi}$ in the form

$$\text{(6.3)}\qquad V^-(\lambda)e^{i\lambda\xi} = \frac{hv_1}{\sqrt{2\pi}} \{\Phi_0(\lambda)e^{-i\lambda\xi}\delta(\lambda) + \Phi(\lambda) - i\Psi(\lambda)\},$$

with the notations

$$\begin{aligned} \Phi_0(0) &= \frac{\pi}{\cos\sigma_1}, \\ \Phi(\lambda) &= h(\lambda)\cos\lambda\xi + g(\lambda)\sin\lambda\xi, \\ \Psi(\lambda) &= h(\lambda)\sin\lambda\xi - g(\lambda)\cos\lambda\xi, \end{aligned}$$

and

$$\begin{aligned} h(\lambda) = &-\frac{\sqrt{A}}{2\varrho\cos\sigma_1(\lambda^4+\lambda_0^4)} \frac{1-(\lambda/\lambda_1)^2}{1-(\lambda/\lambda_2)^2} [T(\lambda)(\sqrt{\varrho+\lambda} - \sqrt{\varrho-\lambda}) \\ &+ S(\lambda)(\sqrt{\varrho+\lambda} + \sqrt{\varrho-\lambda})], \\ g(\lambda) = &-\frac{\sqrt{A}}{2\varrho\cos\sigma_1(\lambda^4+\lambda_0^4)} \cdot \frac{1-(\lambda/\lambda_1)^2}{1-(\lambda/\lambda_2)^2} [T(\lambda)(\sqrt{\varrho+\lambda} + \sqrt{\varrho-\lambda}) \\ &- S(\lambda)(\sqrt{\varrho+\lambda} - \sqrt{\varrho-\lambda})], \\ T(\lambda) = &\ (\lambda^2-\lambda_0^2)/\lambda + 2\sqrt{1+\vartheta}\,\lambda_0^2\lambda, \\ S(\lambda) = &\ \sqrt{2}\,\lambda_0(\lambda^2-\lambda_0^2)(\sqrt{1+\vartheta}-1), \end{aligned}$$

$$\varrho = \sqrt{A^2+\lambda^2} \quad \text{and} \quad \lambda_i = \mathrm{Re}\{z_i\} \quad \text{with} \quad i = 0, 1, 2$$

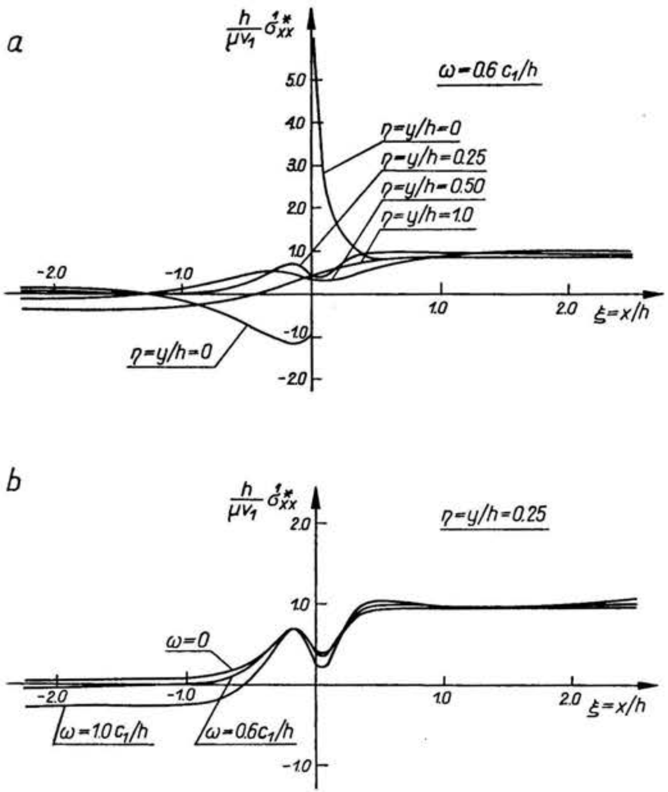

a
$\frac{h}{\mu v_1}\sigma^{1*}_{xx}$
$\omega = 0.6\ c_1/h$
$\eta = y/h = 0$
$\eta = y/h = 0.25$
$\eta = y/h = 0.50$
$\eta = y/h = 1.0$
$\xi = x/h$
$\eta = y/h = 0$
b
$\frac{h}{\mu v_1}\sigma^{1*}_{xx}$
$\eta = y/h = 0.25$
$\omega = 0$
$\xi = x/h$
$\omega = 1.0\ c_1/h$
$\omega = 0.6 c_1/h$

FIG. 5.

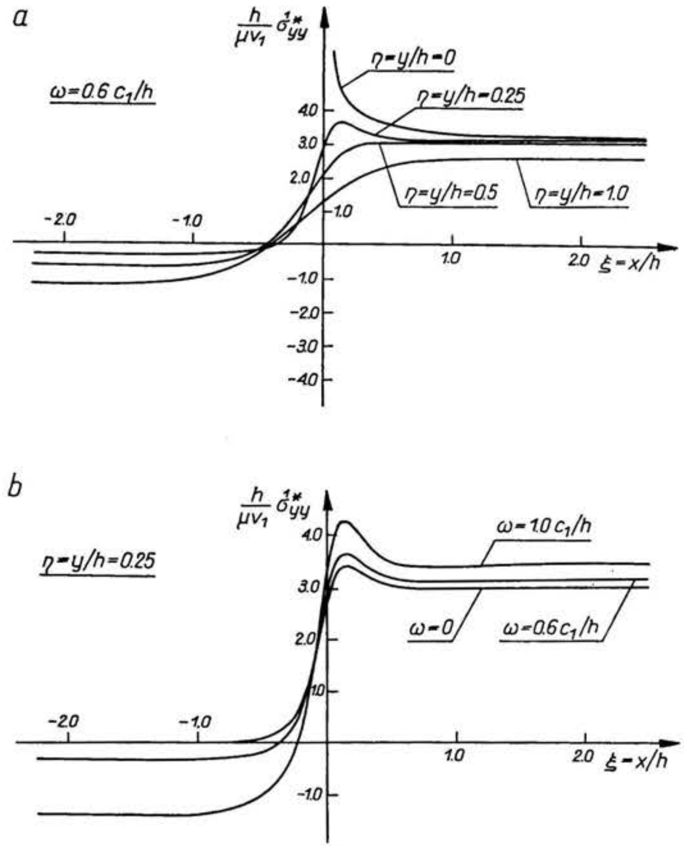

a
$\frac{h}{\mu v_1}\sigma^{1*}_{yy}$
$\omega = 0.6\ c_1/h$
$\eta = y/h = 0$
$\eta = y/h = 0.25$
$\eta = y/h = 0.5$
$\eta = y/h = 1.0$
$\xi = x/h$
b
$\frac{h}{\mu v_1}\sigma^{1*}_{yy}$
$\eta = y/h = 0.25$
$\omega = 1.0\ c_1/h$
$\omega = 0$
$\omega = 0.6 c_1/h$
$\xi = x/h$

FIG. 6.

we can represent the integral containing the Dirac delta-function in a closed form; using the properties of symmetry and antisymmetry of functions with respect to λ we finally obtain

$$
\overset{1}{\sigma}{}^{*}_{xx}(\xi,\eta) = \frac{\mu v_1}{h}\left[\frac{1}{2\cos\sigma_1}\lim_{\lambda\to 0} z_{xx}(\lambda,\eta) + \frac{1}{\pi}\int_0^\infty z_{xx}(\lambda,\eta)\Phi(\lambda)\,d\lambda\right],
$$

$$
\overset{1}{\sigma}{}^{*}_{yy}(\xi,\eta) = \frac{\mu v_1}{h}\left[\frac{1}{2\cos\sigma_1}\lim_{\lambda\to 0} z_{yy}(\lambda,\eta) + \frac{1}{\pi}\int_0^\infty z_{yy}(\lambda,\eta)\Phi(\lambda)\,d\lambda\right],
$$

$$
\overset{1}{\sigma}{}^{*}_{xy}(\xi,\eta) = \frac{\mu v_1}{h}\frac{1}{\pi}\int_0^\infty z_{xy}(\lambda,\eta)\Psi(\lambda)\,d\lambda, \tag{6.4}
$$

$$
\overset{1}{u}{}^{*}(\xi,\eta) = v_1\frac{1}{\pi}\int_0^\infty z_u(\lambda,\eta)\Psi(\lambda)\,d\lambda,
$$

$$
\overset{1}{v}{}^{*}(\xi,\eta) = v_1\left[\frac{1}{2\cos\sigma_1}\lim_{\lambda\to 0} z_v(\lambda,\eta) + \frac{1}{\pi}\int_0^\infty z_v(\lambda,\eta)\Phi(\lambda)\,d\lambda\right].
$$

Convergence of the improper integrals in the region $0 < \eta \leqslant 1$ is very good, and certain difficulties arise only in evaluating the stress $\overset{1}{\sigma}{}^{*}_{xx}(\xi, 0)$ at $\eta = 0$. The difficulties may easily

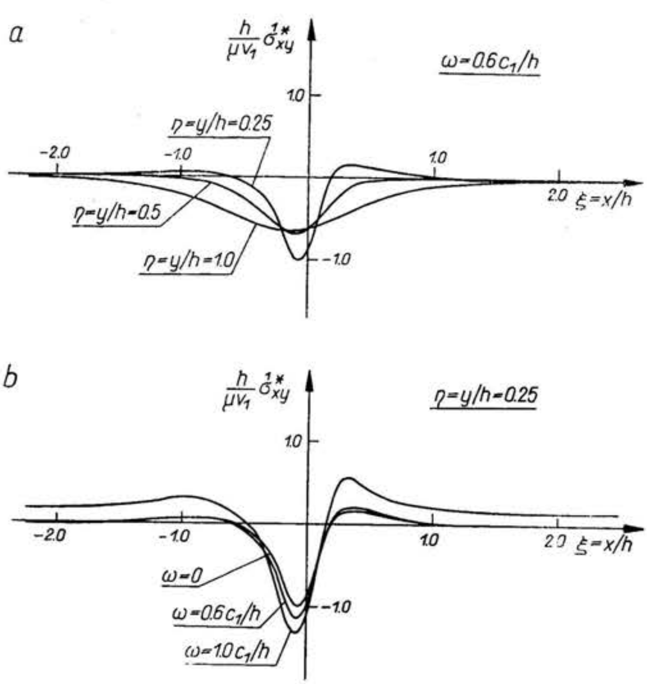


FIG. 7.

be removed by separating the numerically non-convergent term which occurs also in calculating the other stress $\overset{1}{\sigma}{}^*_{yy}(\xi, 0)$ and using the formula (5.8).

The statical part of stresses which is needed for constructing the general solution and which results from the forced displacements $\pm v_0$ may be obtained by the limiting procedure

$$\overset{1}{\sigma}{}^{(\mathrm{Stat})}_{ij} = \lim_{\substack{\omega \to 0 \\ v_1 \to v_0}} \overset{1}{\sigma}{}^*_{ij} \cos\omega t \quad \text{and} \quad \overset{1}{u}{}^{(\mathrm{Stat})}_{i} = \lim_{\substack{\omega \to 0 \\ v_1 \to v_0}} \overset{1}{u}{}^*_{i} \cos\omega t. \tag{6.5}$$

Superposition of all the partial problems yields the final form of the state of stress and displacement,

$$\begin{aligned} \sigma_{ij}(\xi, \eta) &= \overset{0}{\sigma}{}^{(\mathrm{Stat})}_{ij} + \overset{1}{\sigma}{}^{(\mathrm{Stat})}_{ij} + (\overset{0}{\sigma}{}^*_{ij} + \overset{1}{\sigma}{}^*_{ij})\cos\omega t, \\ u_i(\xi, \eta) &= \overset{0}{u}{}^{(\mathrm{Stat})}_{i} + \overset{1}{u}{}^{(\mathrm{Stat})}_{i} + (\overset{0}{u}{}^*_{i} + \overset{1}{u}{}^*_{i})\cos\omega t. \end{aligned} \tag{6.6}$$

In Figs. 5, 6 and 7 the dynamic components of stress amplitudes in the form of $(h/\mu v_1)\overset{1}{\sigma}{}^*_{ij}$ only along the axis $\xi = x/h$ are shown; Figs. 5a, 6a and 7a demonstrate the dependence on $\eta = y/h$ for $\omega = 0.6c_1/h$, while Figs. 5b, 6b and 7b — the dependence of stresses on the excitation frequency ω at $\eta = 0.25$. Poisson's ratio $\nu = 0.25$.

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