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One-dimensional shock waves in solids with internal state variables

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THE PROPAGATION of one-dimensional shock waves in solidlike materials with internal state variables is analyzed. The system of quasi-linear hyperbolic equations which governs the problem of the wave propagation in such materials is considered. It is proved by means of Rankine-Hugoniot relations that the internal state variables are continuous functions across the shock waves, but their time derivatives suffer jump discontinuity. The jump in the strain is assumed to be the amplitude of the wave. After the derivation of a differential equation which governs the changes of the amplitude of the wave in general case, the particular assumptions concerning the region ahead of the wave are formulated. It is assumed that this region is either in a homogeneous non-equilibrium state or in an equilibrium one. In both cases the governing equations are derived and the existence of a critical strain gradient is noticed. The conditions under which the amplitude grows or decays are specified. The discussion of the formation of shock waves is carried out. After JEFFREY [16] the notions of the critical time and the critical distance are introduced. The general investigations are examplified by an expansive shock wave propagating into a material of grade two. Using results contained in the previous papers of the author on acceleration waves, the explicit expressions for the critical time t_c and the critical distance X_c are given.

Analizowano rozprzestrzenianie się jednowymiarowych fal uderzeniowych w ciałach stałych z parametrami wewnętrznymi (wewnętrznymi zmiennymi stanu). Rozpatrzono układ quasi-liniowych równań hiperbolicznych, rządzący problemem propagacji fal w takich ciałach. Za pomocą związków Rankina-Hugoniota udowodniono, że parametry wewnętrzne są ciągłymi funkcjami na fali uderzeniowej, natomiast ich czasowa pochodna doznaje skokowej nieciągłości. Za amplitudę fali przyjęto skok odkształcenia. Po wyprowadzeniu równania różniczkowego, rządzącego zmianami amplitudy fali w ogólnym przypadku, sformułowano szczególne założenia odnośnie obszaru przed falą. Przyjęto, że obszar ten jest albo w jednorodnym stanie nierównowagi, albo w stanie równowagi. Dla obu przypadków wyprowadzono równania różniczkowe amplitud oraz stwierdzono istnienie krytycznego gradientu odkształcenia. Przeprowadzono dyskusję nad formowaniem się fal uderzeniowych. Za Jeffrey'em [16] wprowadzono pojęcie krytycznego czasu i krytycznej odległości. Ogólne badania zilustrowano przykładem rozciągającej fali uderzeniowej w materiale rzędu drugiego. Wykorzystując wyniki zawarte w poprzednich pracach autora, dotyczących fal przyspieszenia, podano jawne wyrażenie na krytyczny czas t_c oraz krytyczną odległość X_c.

Анализируется распространение одномерных ударных волн в твердых телах с внутренними параметрами (внутренними переменными состояния). Рассмотрена система квазилинейных гиперболических уравнений описывающая задачу распространения волн в таких телах. При помощи отношений Ренкина-Гюгонио доказано, что внутренние параметры являются непрерывными функциями на ударной волне, их же временная производная испытывает скачкообразный разрыв. За амплитуду волны принят скачок деформации. После вывода дифференциального уравнения описывающего изменения амплитуды волны в общем случае, сформулированы частные предположения касающиеся области перед волной. Принято, что эта область находится или в однородном состоянии неравновесия, или в состоянии равновесия. Для обоих случаев выведены дифференциальные уравнения амплитуд и констатировано существование критического градиента деформации. Проведено обсуждение формирования ударных волн. По Джефир [16] выведено понятие критического времени и критического расстояния. Общие исследования иллюстрированы примером растягивающей ударной волны в материале второго порядка. Используя результаты содержавшиеся в предыдущих работах автора, касающихся волн ускорения, дается явное выражение для критического времени t_c и критического расстояния X_c.

1. Introduction

IN the original theory of simple materials formulated by NOLL in 1958 the actual stress state is determined by the infinite past history of deformation. In 1972, NOLL formulated the new mathematical theory of simple materials which is free from this and the other defects and in which only deformation processes of finite duration, not infinite histories, ocurr in the description of tbe response of a material [1].

In the internal state variable description of a material take also place deformation processes of finite duration. This approach may be a good example of the material element defined by NOLL in [1]. On the other hand, it is also an example of the unique material structure defined by PERZYNA and KOSIŃSKI in [2, 3].

Since the internal variable approach has no defects mentioned above and has connections with the original and new theories of simple materials (see lecture cited in Sec. 2), this approach may safely be used in the description of various effects in dissipative materials, like plastic and viscoplastic ones([1]).

Particularly, the material with internal state variable is a very convenient model for carrying out investigations of the wave propagation phenomena in dissipative materials. Here, it should be emphasized that the initial boundary-value problem for materials with internal variables is governed by a system of hyperbolic equations.

In the present paper, shock waves in a solidlike material with internal variables will be investigated. The propagation of shock waves in this material was studied, in the case of fluid, by CHEN and GURTIN [4] and by KOSIŃSKI [5–7] and SZMIT [8] for solids.

In Sec. 2 the fundamental relations and theorem for a discussion of shock waves in the material under consideration is proved. By use of the Rankine-Hugoniot relation the problem of the jump discontinuity in the internal state variable function is solved.

In Sec. 3 the governing differential equation for an amplitude of a shock wave propagating through the material, which has been in homogeneous equilibrium and non-equilibrium states, is derived. The existence of a critical strain gradient λ is noticed. It is reamarked in the discussion of the behaviour of the amplitude in time, that the jump in the strain $[\![E]\!]$ will increase, decrease or remain constant according as the strain gradient $\partial_X E^-$ is greater than, smaller than or equal to λ.

In Sec. 4 the problem of the formation of shock waves in systems of hyperbolic non-linear equations is discussed. It is shown that the existence of the critical time t_c of the formation of shock wave was established in the previous investigations of the author on the acceleration waves.

Finally, the example of the expansive shock wave in the material of grade two is considered.

([1]) To examine that the internal state variable approach gives an adequete conceptual framework for the mathematical description of plasticity and yield, see works by DILLON, KRATOCHVÍL, PERZYNA, TEODOSIU, VALANIS and the others on plasticity and viscoplasticity.

2. The fundamental relations

In the present paper we consider one-dimensional motions in solidlike materials. *The motion* of a material point (particle) is given by the function χ of two variables such that its value $x = \chi(X, t)$ indicates the position at time t of the particle labelled X in the homogeneous reference configuration with mass density ϱ. *The displacement* of a particle is then defined by the function u of two variables such that

$$u(X, t) = \chi(X, t) - X \quad \text{for each } (X, t). \tag{2.1}$$

We denote, if they exist, the derivatives of the displacement function u [which according to (2.1) may be expressed by the derivatives of the motion function χ] as follows:

$$\begin{aligned} v(X, t) &= \frac{\partial}{\partial t} u(X, t), \quad & E(X, t) &= \frac{\partial}{\partial X} u(X, t), \\ \ddot{u}(X, t) &= \frac{\partial^2}{\partial t^2} u(X, t), \quad & \partial_X E &= \frac{\partial^2}{\partial X^2} u(X, t). \end{aligned} \tag{2.2}$$

We call them respectively *the velocity*, *the strain*, *the acceleration*, *the strain gradient* of a particle X at time t.

In the framework of the mechanical theories of continuous media, materials are characterized by one response function which determines the stress.

In the general theory of simple materials with memory the stress is determined at a material point X whenever the strain and its past history are known at X. In that approach the past history of the strain is an additional quantity (variable) which must be given in order to describe the response of a material. There is another approach in which the stress is determined when the actual values of strain and of additional parameters as well are prescribed. These parameters, called the *internal state variables*, are introduced by solution of an initial value problem. The solution is obtained in terms of the strain history (rather finite) and an initial value of the parameters (²).

Thus we consider a class of homogeneous materials with internal state variables described by *the constitutive equation*

$$T(X, t) = \mathcal{T}\big(E(X, t), \boldsymbol{\alpha}(X, t)\big), \quad t \in [0, \infty) \tag{2.3}$$

for the stress in particle X at time t, which is supplemented by *the evolution equation* (initial value problem) for the internal variable vector

$$\dot{\boldsymbol{\alpha}}(X, \tau) = \mathbf{a}\big(E(X, \tau), \boldsymbol{\alpha}(X, \tau)\big), \quad \boldsymbol{\alpha}(X, 0) = \boldsymbol{\alpha}_0(X), \quad \tau \in [0, \infty). \tag{2.4}$$

Here $\boldsymbol{\alpha}$ represents n-vector of internal variables (parameters), which was introduced to define uniquely the state of the particle of dissipative material (i.e. material with in-

(²) The problems of the similarity and the equivalence of these two approaches are very important in the theory of materials. There are some papers in the literature treating this subject. For instance WOJNO and KOSIŃSKI [9] formulated the conditions under which both approaches give the same stress for the same history of deformation. MAZILU and KOSIŃSKI examined in [10] the mathematical conditions under which the memory of a simple material can be parametrized and showed that materials with internal variables are a special case of materials with parametrical memory.

elastic, added to elastic, properties) ([3]). Internal variables may have different physical interpretation, for example, the work-hardening parameters, the inelastic or anelastic strains.

Since the pair $(E, \boldsymbol{\alpha})$ has to define uniquely the state of the particle, we must be sure that the evolution equation (2.4) has a unique solution. In [9] the theorem of existence and uniqueness of the solution of the evolution equation for internal variables was formulated. That theorem is true for the continuous strain (deformation) function. The present paper deals with discontinuous strain functions, which appear in the analysis of shock waves. Hence we must formulate the appropriate theorem for the case under consideration.

The basic system of equations for a one-dimensional material with internal state variables comprises: the law of motion

$$\varrho\frac{\partial v}{\partial t}-\frac{\partial T}{\partial X}-\varrho b=0, \tag{2.5}$$

the geometrical compatibility condition

$$\frac{\partial E}{\partial t}-\frac{\partial v}{\partial X}=0, \tag{2.6}$$

the evolution equation for internal state variables

$$\frac{\partial \boldsymbol{\alpha}}{\partial t}-\mathbf{a}(E, \boldsymbol{\alpha})=\mathbf{0} \tag{2.7}$$

and the constitutive equation

$$T=\mathscr{T}(E, \boldsymbol{\alpha}), \tag{2.8}$$

where b is a prescribed body force.

We can see that here we have to deal with the system of equations with two independent variables t and X and $n+2$ dependent variables v, E and $\boldsymbol{\alpha}$. Let $\mathbf{U}$ be the column vector with components $U_1 = v$, $U_2 = E$ and $U_3 = \boldsymbol{\alpha}$. By $\mathbf{F} = \mathbf{F}(\mathbf{U})$ we denote the column vector with components $F_1 = \varrho^{-1}\mathscr{T}(U_2, U_3)$, $F_2 = -U_1$, $F_3 = \mathbf{0}$. The letter $\mathbf{B}(\mathbf{U})$ denotes a vector with the components $B_1 = b$, $B_2 = 0$ and $B_3 = \mathbf{a}(U_2, U_3)$.

The above notations anable to rewrite the basic system of equations in the following divergence form (of generalized system of conservation laws, say):

$$\partial_t\mathbf{U}+\partial_X\mathbf{F}(\mathbf{U})+\mathbf{B}(\mathbf{U})=\mathbf{0}. \tag{2.9}$$

If we denote the gradient operator with respect to $\mathbf{U}$ by $\nabla_{\mathbf{U}}$, then

$$\partial_X\mathbf{F}=(\nabla_{\mathbf{U}}\mathbf{F})\partial_X\mathbf{U}, \tag{2.10}$$

or

$$\partial_X\mathbf{F}=\mathbf{A}\partial_X\mathbf{U},$$

where

$$\mathbf{A}(\mathbf{U})=\begin{bmatrix} 0 & -\varrho^{-1}\partial_E\mathscr{T}(U_2, U_3), & -\varrho^{-1}\partial_\alpha\mathscr{T}(U_2, U_3) \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

([3]) In [2, 3] PERZYNA and KOSIŃSKI defined the unique parametrical material structure in the mathematical framework for the description of dissipative materials. This structure contains materials with internal variables.

Using this result we can see that the system of Eqs. (2.9) [or (2.5)–(2.8)] transforms into the *quasi-linear system*

$$\partial_t \mathbf{U} + \mathbf{A}\partial_X \mathbf{U} + \mathbf{B} = \mathbf{0} \quad \text{with} \quad \mathbf{A} = \nabla_{\mathbf{U}} \mathbf{F}, \tag{2.11}$$

which is *hyperbolic* when the eigenvalues of **A** are real and distinct. In fact, its eigenvalues

$$\lambda_1 = -\sqrt{\varrho^{-1}\partial_E \mathscr{T}(U_2, U_3)}, \quad \lambda_k = 0, \quad \lambda_{2+n} = \sqrt{\varrho^{-1}\partial_E \mathscr{T}(U_2, U_3)}$$

are real provided that $\partial_E \mathscr{T}(U_2, U_3)$ is positive.

Since our system (2.11) is hyperbolic, the solutions **U** of it may have some singularities (e.g. the discontinuities of the first derivatives of **U** across the characteristics). Furthermore, the solutions themselves may suffer a jump discontinuity. The last case denotes that a shock wave takes place. In linear hyperbolic systems the discontinuity surface of the solution coincides with a characteristic manifold but in a quasi-linear system does not. This fact is the essential difference between linear and quasi-linear hyperblic system.

In a discussion of discontinuous solutions of hyperbolic equations the so-called Rankine-Hugoniot *relation* is often used.

Let $\sum \equiv \{(t, X)\colon t \in [0, \infty),\ X = Y(t)\}$ be a curve across which **U** has jump discontinuity. Then the vector

$$\mathbf{n} = \left(\frac{d}{dt} Y(t), -1\right) \tag{2.12}$$

is normal to $\sum$ at $(t, Y(t))$.

We use the well-known notation $[\![G]\!]$ for the jump $G^- - G^+$ in the function $G(X, t)$ across the discontinuity curve $\sum$, i.e.

$$G^- \equiv \lim_{X \to Y(t)^-} G(X, t), \quad G^+ \equiv \lim_{X \to Y(t)^+} G(X, t). \tag{2.13}$$

The Rankine-Hugoniot relation expresses ([4]) the continuity of the normal component (in two-dimensional $t-X$ space) of the vector field $(\mathbf{U}, \mathbf{F})$ across the discontinuity line, i.e.

$$\mathbf{n} \cdot ([\![\mathbf{U}]\!], [\![\mathbf{F}]\!]) = 0 \quad \text{or} \quad [\![(\mathbf{U}, \mathbf{F}) \cdot \mathbf{n}]\!] = 0. \tag{2.14}$$

In order to prove (2.14), one needs to assume that the vector **B** is a continuous function of **U** and to integrate the Eq. (2.9) over a domain D in t-X space. Using the Gaussian divergence theorem, in the limit as D shrinks to zero, we obtain (2.14).

If we denote the derivatives $\frac{d}{dt} Y(t)$ by $V(t)$ and call it *the intrinsic velocity* of the shock wave, then the Rankine-Hugoniot relation (2.14) can be written in the following, more convenient for further considerations, form

$$V[\![\mathbf{U}]\!] = [\![\mathbf{F}]\!]. \tag{2.15}$$

THEOREM 1. *In the motion with shock waves of the material described by the constitutive equation* (2.3) *and the evolution one* (2.4) *the internal state variable vector function* $\boldsymbol{\alpha}$ *has no jump discontinuity* ([5]) *across* $\sum$.

([4]) Cf. for example JEFFREY and TANIUTI [11].

([5]) In [5–7] other proofs of this fact are given.

P r o o f. The Rankine-Hugoniot relation applied to our case (2.9) gives (2.15). Since

$$\mathbf{U} = \begin{bmatrix} v \\ E \\ \boldsymbol{\alpha} \end{bmatrix} \quad \text{and} \quad \mathbf{F} = \begin{bmatrix} -\varrho^{-1}\mathscr{T}(E, \boldsymbol{\alpha}) \\ -v \\ \mathbf{0} \end{bmatrix}, \tag{2.16}$$

hence

$$\begin{aligned} \varrho V[\![v]\!] &= -[\![T]\!], \\ V[\![E]\!] &= -[\![v]\!], \\ [\![\boldsymbol{\alpha}]\!] &= \mathbf{0}, \end{aligned} \tag{2.17}$$

where (2.8) was used. The last equation in (2.17) gives the proof of the theorem.

The proved fact is fundamental for an investigation of shock waves in a material with internal state variables. It solves the question concerning the continuity of the internal variable function.

Let us notice that $(2.17)_1$ expresses the law of motion on the discontinuity curve $\sum$. The relation $(2.17)_2$ is often called *the kinematical condition of compatibility*. Here we should remark that the general condition of compatibility, in the one-dimensional context, has another form than $(2.17)_2$. This condition is expressed as follows ([6]). Suppose that functions $f(\cdot,\cdot)$, $\dot{f}(\cdot,\cdot)$ and $\partial_X f(\cdot,\cdot)$ defined on $\mathscr{B}\times[0,\infty)$ have jump discontinuities across the curve $\sum$ but are continuous functions everywhere else ([7]), then

$$\frac{\delta}{\delta t}[\![f]\!] = [\![\dot{f}]\!] + V[\![\partial_X f]\!], \tag{2.18}$$

where $\delta/\delta t$ is called *the displacement derivative*.

In view of the form of the evolution equation (2.4) and the continuity assumption of $\mathbf{a}$, we have for a time derivative of $\boldsymbol{\alpha}$:

$$[\![\dot{\boldsymbol{\alpha}}]\!] = \mathbf{a}(E^-, \boldsymbol{\alpha}) - \mathbf{a}(E^+, \boldsymbol{\alpha}). \tag{2.19}$$

The kinematical condition (2.18) applied to this case gives

$$[\![\dot{\boldsymbol{\alpha}}]\!] + V[\partial_X \boldsymbol{\alpha}] = \mathbf{0} \quad \text{or} \quad [\![\dot{\boldsymbol{\alpha}}]\!] = -V[\partial_X \boldsymbol{\alpha}]. \tag{2.20}$$

Furthermore, in a dynamic process with a shock wave the law of motion (2.5) holds on either side of the curve $\sum$ and across it, in addition to $(2.17)_1$, we have

$$[\![\partial_X T]\!] = \varrho[\![\dot{v}]\!], \tag{2.21}$$

where the continuity of the body force b was assumed.

The Eqs. $(2.17)_{1,2}$ imply the well-known result

$$\varrho V^2 = \frac{[\![T]\!]}{[\![E]\!]} \tag{2.22}$$

for the velocity of the shock wave, while (2.18) with $f = E$ and $f = v$ in conjunction with (2.21) yield the relation

$$2\sqrt{V}\frac{\delta}{\delta t}(\sqrt{V}[\![E]\!]) = V^2[\![\partial_X E]\!] - \frac{1}{\varrho}[\![\partial_X T]\!]. \tag{2.23}$$

([6]) Cf. for example Chen [12].

([7]) In [13] the precise conditions of regularity in term of the function f are given to verify the Eq. (2.18), Cf. also [12]. Here $\mathscr{B}$ denotes the body.

In the present study the jump in the strain is the basic variable under consideration, so that the jump $[\![E]\!]$ will be called *the amplitude* of the shock wave. It may be treated as a function of time only.

It should be noticed that the Eq. (2.23) is independent of any constitutive equations and must be fulfilled by the amplitude of each shock wave.

3. Amplitude equation

In this section we derive general and explicit expressions for the change in amplitude of one-dimensional shock waves propagating through arbitrary homogeneous materials with internal state variables. This derivation will be done under the special assumptions concerning the region ahead of the wave. We will assume that the wave propagates through the material which has been in homogeneous states of equilibrium or non-equilibrium.

In the paper [6] the definition of the homogeneous equilibrium state of the material under consideration was given. In reference to this we say that the body (material) is in a *homogeneous equilibrium state* $(E_0, \boldsymbol{\alpha}_0)$ if

$$(3.1) \qquad \mathbf{a}(E_0, \boldsymbol{\alpha}_0) = \mathbf{0}, \quad \partial_X E_0 = \dot{E}_0 = 0 \quad \text{and} \quad \partial_X \boldsymbol{\alpha}_0 = \mathbf{0} \quad \text{for} \quad X \in \mathscr{B}.$$

The condition $(3.1)_1$ means that the pair $(E_0, \boldsymbol{\alpha}_0)$ is an equilibrium point of the evolution equation (2.4). In other words, the constant function $\boldsymbol{\alpha}(t) \equiv \boldsymbol{\alpha}_0$, where $t \in [0, \infty)$, is the solution of the Eq. $(3.1)_1$ with the initial value $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0$.

The notion of *the homogeneous non-equilibrium state* introduced in [14] tells that the pair $\big(E(X,t), \boldsymbol{\alpha}(X,t)\big)$ forms such a state if

$$(3.2) \qquad \begin{aligned} &E(X,t) = E_0 + wt, \quad \dot{E}(X,t) = w, \quad \partial_X E(X,t) = 0, \quad \partial_X \boldsymbol{\alpha}(X,t) = \mathbf{0}, \\ &\dot{\boldsymbol{\alpha}}(X,t) = \mathbf{a}\big(E_0 + wt, \boldsymbol{\alpha}(X,t)\big), \quad \text{for} \quad X \in \mathscr{B}, \; w = \text{const}. \end{aligned}$$

Now, we can try to derive the expression for the change in the amplitude of the wave. This expression is based on the amplitude equation (2.23). In that equation the jump of $\partial_X T$ takes place. In view of our constitutive assumption (2.3), we have

$$(3.3) \qquad [\![\partial_X T]\!] = [\![\partial_E \mathscr{T} \partial_X E]\!] + [\![\partial_{\boldsymbol{\alpha}} \mathscr{T} \cdot \partial_X \boldsymbol{\alpha}]\!].$$

Substitution of (3.3) into (2.23) yields

$$(3.4) \qquad 2V \frac{\delta [\![E]\!]}{\delta t} + [\![E]\!] \frac{\delta V}{\delta t} = V^2 [\![\partial_X E]\!] - \frac{1}{\varrho} \{[\![\partial_E \mathscr{T} \partial_X E]\!] + [\![\partial_{\boldsymbol{\alpha}} \mathscr{T} \cdot \partial_X \boldsymbol{\alpha}]\!]\}.$$

In this equation the displacement derivative of the wave velocity V occurs. By an additional calculation this derivative may be determined by $\frac{\delta}{\delta t}[\![E]\!]$. Let us compute

$$(3.5) \qquad \begin{aligned} \frac{\delta [\![T]\!]}{\delta t} = [\![\dot{T}]\!] + V [\![\partial_X T]\!] = [\![\partial_E \mathscr{T}(E, \boldsymbol{\alpha})]\!] \frac{\delta E^+}{\delta t} + \partial_E \mathscr{T}(E^-, \boldsymbol{\alpha}) \frac{\delta [\![E]\!]}{\delta t} \\ + [\![\partial_{\boldsymbol{\alpha}} \mathscr{T}(E, \boldsymbol{\alpha})]\!] \cdot \frac{\delta \boldsymbol{\alpha}^+}{\delta t} + \partial_{\boldsymbol{\alpha}} \mathscr{T}(E^-, \boldsymbol{\alpha}) \cdot \frac{\delta [\![\boldsymbol{\alpha}]\!]}{\delta t}. \end{aligned}$$

Now, let us assume that the wave is propagating in the direction of increasing X. This implies the positive velocity $V(t)$.

We suppose additionally, that the shock wave advances into a body being either in a homogeneous non-equilibrium or in a homogeneous equilibrium state; then the body is in one of those states and we have

$$\partial_X E^+ = 0, \quad \partial_X \boldsymbol{\alpha}^+ = \mathbf{0} \quad \text{or} \quad \partial_X E^+ = \dot{E}^+ = 0 \quad \text{and} \quad \partial_X \boldsymbol{\alpha}^+ = \dot{\boldsymbol{\alpha}}^+ = \mathbf{0}, \tag{3.6}$$

respectively. In the first case $\frac{\delta E^+}{\delta t} = \dot{E}^+$ and $\frac{\delta \boldsymbol{\alpha}}{\delta t} = \dot{\boldsymbol{\alpha}}^+$. In view of Theorem 1

$$[\![\boldsymbol{\alpha}]\!] = \mathbf{0} \quad \text{and} \quad \frac{\delta}{\delta t}[\![\boldsymbol{\alpha}]\!] = \mathbf{0}. \tag{3.7}$$

Hence, for the non-equilibrium state, we have

$$\frac{\delta [\![T]\!]}{\delta t} = \partial_E \mathscr{T}(E^-, \boldsymbol{\alpha}) \frac{\delta [\![E]\!]}{\delta t} + [\![\partial_E \mathscr{T}(E, \boldsymbol{\alpha})]\!] \dot{E}^+ + [\![\partial_{\boldsymbol{\alpha}} \mathscr{T}(E, \boldsymbol{\alpha})]\!] \cdot \dot{\boldsymbol{\alpha}}^+. \tag{3.8}$$

In the case of the equilibrium state $(E_0, \boldsymbol{\alpha}_0)$

$$\frac{\delta [\![T]\!]}{\delta t} = \partial_E \mathscr{T}(E^-, \boldsymbol{\alpha}) \frac{\delta [\![E]\!]}{\delta t}. \tag{3.9}$$

If we notice that, by (2.22),

$$\frac{\delta [\![T]\!]}{\delta t} = \varrho V^2 \frac{\delta [\![E]\!]}{\delta t} + 2\varrho V [\![E]\!] \frac{\delta V}{\delta t}$$

or

$$\frac{\delta V}{\delta t} = \frac{1}{2\varrho V [\![E]\!]} \left(\frac{\delta [\![T]\!]}{\delta t} - \varrho V^2 \frac{\delta [\![E]\!]}{\delta t} \right), \tag{3.10}$$

then we obtain the proof of the following

Lemma. *The velocity of the shock wave propagating into a material in a homogeneous non-equilibrium state obeys the equation*:

$$\frac{\delta V}{\delta t} = \frac{1}{2\varrho V [\![E]\!]} \Big\{ \big(\partial_E \mathscr{T}(E^-, \boldsymbol{\alpha}) - \varrho V^2\big) \frac{\delta [\![E]\!]}{\delta t} + [\![\partial_E \mathscr{T}(E, \boldsymbol{\alpha})]\!] \dot{E}^+ + [\![\partial_{\boldsymbol{\alpha}} \mathscr{T}(E, \boldsymbol{\alpha})]\!] \cdot \dot{\boldsymbol{\alpha}}^+ \Big\}. \tag{3.11}$$

When the material ahead of the wave is in the homogeneous equilibrium state $(E_0, \boldsymbol{\alpha}_0)$, *then*

$$\frac{\delta V}{\delta t} = \frac{\partial_E \mathscr{T}(E^-, \boldsymbol{\alpha}_0) - \varrho V^2}{2\varrho V [\![E]\!]} \frac{\delta [\![E]\!]}{\delta t}. \tag{3.12}$$

Let us notice that in the second case for $\varrho V^2 = \partial_E \mathscr{T}(E^-, \boldsymbol{\alpha}_0)$ we have $\delta V / \delta t = 0$. Such a situation will not be treated here. It takes place in the case of a linear in E constitutive function $\mathscr{T}$, cf. Kosiński [6, 7].

Now we are able to derive the explicit expression for the change of $[\![E]\!]$ in time.

THEOREM 2. *The amplitude* $[\![E]\!]$ *of a shock wave propagating into the material satisfies the equation*

$$\frac{\delta[\![E]\!]}{\delta t} = \frac{2V}{\frac{4\varrho V^2}{r}-1}(\partial_X E^- - \lambda), \tag{3.13}$$

where V *is given by* (3.10) *and* (2.22), *and the form of the quantity* λ *depends on the state ahead of the wave* (*in* + *region*), *i.e. if the material is in a homogeneous non-equilibrium state, then*

$$\lambda \equiv \frac{1}{r}\left\{\frac{1}{2V}\left([\![\partial_E \mathscr{T}(E, \boldsymbol{\alpha})]\!]\dot{E}^+ + [\![\partial_{\boldsymbol{\alpha}} \mathscr{T}(E, \boldsymbol{\alpha})]\!]\cdot \dot{\boldsymbol{\alpha}}^+\right) + \partial_{\boldsymbol{\alpha}} \mathscr{T}(E^-, \boldsymbol{\alpha})\cdot \partial_X \boldsymbol{\alpha}^-\right\}, \tag{3.14}$$

but if the material is in the homogeneous equilibrium state $(E_0, \boldsymbol{\alpha}_0)$, *then*

$$\lambda \equiv -\frac{1}{rV}\partial_{\boldsymbol{\alpha}} \mathscr{T}(E^-, \boldsymbol{\alpha}_0)\cdot \mathbf{a}(E^-, \boldsymbol{\alpha}_0). \tag{3.15}$$

In both cases r *is given by*

$$r = \varrho V^2 - \partial_E \mathscr{T}(E^-, \boldsymbol{\alpha}). \tag{3.16}$$

Proof. The equations (3.4), (3.8) and (3.11) for the non-equilibrium case give

$$2V\frac{\delta[\![E]\!]}{\delta t} + \frac{1}{2\varrho V}\left(\partial_E \mathscr{T}(E^-, \boldsymbol{\alpha}) - \varrho V^2\right)\frac{\delta[\![E]\!]}{\delta t} = \left(V^2 - \frac{1}{\varrho}\partial_E \mathscr{T}(E^-, \boldsymbol{\alpha})\right)\partial_X E^-$$
$$-\frac{1}{\varrho}\partial_{\boldsymbol{\alpha}} \mathscr{T}(E^-, \boldsymbol{\alpha})\cdot \partial_X \boldsymbol{\alpha}^- - \frac{1}{2\varrho V}\{[\![\partial_E \mathscr{T}(E, \boldsymbol{\alpha})]\!]\dot{E}^+ + [\![\partial_{\boldsymbol{\alpha}} \mathscr{T}(E, \boldsymbol{\alpha})]\!]\cdot \dot{\boldsymbol{\alpha}}^+\}.$$

Then

$$\frac{\delta[\![E]\!]}{\delta t} = \frac{2V}{3\varrho V^2 + \partial_E \mathscr{T}(E^-, \boldsymbol{\alpha})}\Big\{(\varrho V^2 - \partial_E \mathscr{T}(E^-, \boldsymbol{\alpha}))\partial_E E^- - \partial_{\boldsymbol{\alpha}} \mathscr{T}(E^-, \boldsymbol{\alpha})\cdot \partial_X \boldsymbol{\alpha}^-$$
$$-\frac{1}{2V}\left([\![\partial_E \mathscr{T}(E, \boldsymbol{\alpha})]\!]\dot{E}^+ + [\![\partial_{\boldsymbol{\alpha}} \mathscr{T}(E, \boldsymbol{\alpha})]\!]\cdot \dot{\boldsymbol{\alpha}}^+\right)\Big\}.$$

Introducing the symbol r, after some calculations we obtain (3.13) with λ given by (3.14). In the equilibrium ahead of the wave, the relations (3.1) and (2.20) imply another form of λ, given by (3.15).

Let us notice that, assuming the strain and the internal variables in the homogeneous non-equilibrium state given by (3.2), the relation (3.14) may be written as follows

$$\lambda \equiv \frac{1}{r}\Big\{\frac{1}{2V}\left(\partial_E \mathscr{T}(E^-, \boldsymbol{\alpha}) - \partial_E \mathscr{T}(E_0 + wt, \boldsymbol{\alpha})\right)w \tag{3.17}$$
$$+\left(\partial_{\boldsymbol{\alpha}} \mathscr{T}(E^-, \boldsymbol{\alpha}) - \partial_{\boldsymbol{\alpha}} \mathscr{T}(E_0 + wt, \boldsymbol{\alpha})\right)\cdot \mathbf{a}(E_0 + wt, \boldsymbol{\alpha})$$
$$-\frac{1}{V}\partial_{\boldsymbol{\alpha}} \mathscr{T}(E^-, \boldsymbol{\alpha})\cdot \left(\mathbf{a}(E_0 + wt, \boldsymbol{\alpha}) - \mathbf{a}(E^-, \boldsymbol{\alpha})\right)\Big\},$$

where we have used (2.20) and

$$\frac{1}{V}(\dot{\boldsymbol{\alpha}}^+ - \dot{\boldsymbol{\alpha}}^-) = \partial_X \boldsymbol{\alpha}^-.$$

Now, our aim is to discuss the local, in time, behaviour of the amplitude $[\![E]\!]$. It is evident that this behaviour depends on the sign of r and on the magnitude of $\partial_X E^-$ in comparison with λ. By the Eq. (3.13) we establish the existence of *a critical*(8) *strain gradient* λ in the propagation of the shock wave in the material under consideration. To discuss the behaviour of $[\![E]\!]$ on the wave we consider two cases, i.e. compressive and expansive shock waves.

Case A. *Compressive shock waves*

A compressive shock wave is defined by the conditions

$$[\![E]\!] < 0 \quad \text{and} \quad E^+ \leqslant 0. \tag{A.1}$$

From the condition of the hyperbolicity of our system of equations [cf. (2.11)] we have

$$\partial_E \mathcal{T}(E, \boldsymbol{\alpha}) > 0. \tag{A.2}$$

We assume, additionally, that for each value of $\boldsymbol{\alpha}$ the strain-stress relation $T = \mathcal{T}(E, \boldsymbol{\alpha})$ in compression is concave from below, i.e.

$$\partial_E^2 \mathcal{T}(E, \boldsymbol{\alpha}) < 0 \quad \text{for} \quad E \leqslant 0. \tag{A.3}$$

Then, by (2.22) and

$$[\![T]\!] = \mathcal{T}(E^-, \boldsymbol{\alpha}) - \mathcal{T}(E^+, \boldsymbol{\alpha}), \tag{A.4}$$

we have

$$r = \varrho V^2 - \partial_E \mathcal{T}(E^-, \boldsymbol{\alpha}) < 0. \tag{A.5}$$

This fact implies that the sign of the right-hand side of (3.13) depends on the magnitudes of $\partial_X E^-$ and λ. Since $[\![E]\!] < 0$, then $|[\![E]\!]| = -[\![E]\!]$. Hence in view of the positive wave velocity, $V > 0$, we can formulate the following theorem(9).

THEOREM 3. *In the propagation of a compressive shock wave in the material under consideration the assumptions* (A.2) *and* (A.3) *imply*:

1) *if at a given time* $\partial_X E^- > \lambda$, *then at that time and immediately behind the wave the jump magnitude in the strain* $|[\![E]\!]|$ *is increasing, i.e.* $\dfrac{\delta}{\delta t}|[\![E]\!]| > 0$;

2) *if* $\partial_X E^- < \lambda$, *then the jump* $|[\![E]\!]|$ *is decreasing, i.e.* $\dfrac{\delta}{\delta t}|[\![E]\!]| < 0$;

3) *at any instant* $\partial_X E^- = \lambda$ *if, and only if, the jump* $[\![E]\!]$ *remains constant, i.e.* $\dfrac{\delta}{\delta t}[\![E]\!] = 0$.

Let us notice that the critical strain gradient λ in both cases (3.14) and (3.15) is a function of time. Further, the statement 3) of the preceding theorem does not imply that the wave has a constant amplitude over some period of time, for λ may increase or decrease during this period. Clearly, the wave can have a constant amplitude over some period of time only if $\dfrac{\delta}{\delta t}(\partial_X E^- - \lambda) = 0$ during this period.

Case B. *Expansive shock waves*

(8) Like in materials with memory, cf. CHEN [12].

(9) In the theorem given by CHEN and GURTIN in [4] only the case of the positive critical strain gradient λ was considered.

For such waves the amplitude is positive:

$$\llbracket E \rrbracket > 0 \quad \text{and} \quad E^+ \geqslant 0. \tag{B.1}$$

We assume that for each value of $\boldsymbol{\alpha}$ the relation $T = \mathscr{T}(E, \boldsymbol{\alpha})$ is concave from above([10]), as the function of E only, i.e.

$$\partial_E^2 \mathscr{T}(E, \boldsymbol{\alpha}) > 0 \quad \text{for} \quad E \geqslant 0. \tag{B.2}$$

If (A.2) is used, then (A.5) is still valid and we have

THEOREM 4. *In the propagation of an expansive shock wave the assumptions* (A.2) *and* (B.2) *imply*:

1) *if* $\partial_X E^- > \lambda$, *then the jump in the strain is decreasing, i.e.* $\dfrac{\delta}{\delta t}\llbracket E \rrbracket < 0$;

2) *if* $\partial_X E^- < \lambda$, *then the jump is increasing, i.e.* $\dfrac{\delta}{\delta t}\llbracket E \rrbracket > 0$;

3) *the jump remains constant if, and only if,* $\partial_X E^- = \lambda$.

The properties of the wave given in the preceding theorems are valid under the assumption that the material ahead of the wave is in the homogeneous equilibrium or non-equilibrium states. But in the case of the equilibrium the similar properties may be formulated for the wave velocity([11]).

In the previous paper [5] the amplitude equation and the general form of its solution in the case of the infinitesimal shock wave were furnished. There it was shown that the limit value of the critical strain gradient with the amplitude $\llbracket E \rrbracket$ tending to zero is equal to twice the critical amplitude of an acceleration wave([12]).

4. Shock wave formation

It is a well-known fact that, in the case of a linear system of hyperbolic equations, shock waves cannot occur spontaneously. They may happen only by externally imposed impacts. Furthermore, the shock wave, i.e. the discontinuities in the solution $\mathbf{U}$ of the equations, are propagated only along the characteristics of the system, like the discontinuities of its first derivatives (we call them acceleration waves). Quite another situation takes place in the case of non-linear (quasi-linear, say) equations. Here acceleration waves propagate along the characteristics, too, but the shock waves may occur not only under the discontinuous initial conditions. Perhaps one of the most striking features of non-linear hyperbolic systems is that even when starting from analytic initial data, a discontinuity can develop in the derivative([13]) of the solution $\mathbf{U}$ and can then tend to an actual jump

([10]) Here (B.2) is the necessary condition for the existence of the expansive shock wave, like (A. 3) was the condition for the compressive wave. The negative r denotes that the shock wave velocity is subsonic with respect the rear of the wave.

([11]) This was remarked in [5], where the case of the equilibrium state ahead of the wave was considered.

([12]) Cf. CHEN and GURTIN [4, 15].

([13]) It is a Lipschitz discontinuity in the derivative of $\mathbf{U}$ normal to the wave front (characteristic). Cf. [11].

discontinuity in **U** itself. This jump discontinuity, called the shock wave, propagates in a totally different manner from the derivative discontinuity.

The solution **U** is a function of t and X. Thus, in view of the paper by Jeffrey [16], at some critical time t_c and at some critical distance X_c the solution ceases to be Lipschitz continuous on the characteristic and finite jump or shocklike discontinuity appears in **U**.

If the solution loses the Lipschitz continuity it means that at least one of its derivatives goes to the infinity. This condition used([14]) in [16] enabled Jeffrey to obtain analytic expressions determining the critical time t_c in terms of the eigenvector gradient of the matrix **A** cf. (2.11) and the functions of jumps.

Now, there is a place in which our previous investigations of the behaviour of acceleration waves in the material with internal state variables may be helpful in the derivation of the conditions of the shock wave formation. In [5, 14, 17] we have established the existence of the finite time t_∞ at which the amplitude of the acceleration wave becomes the infinity. The fact that an acceleration wave will have infinite amplitude within a finite time means that a shock wave is produced. So the conditions derived in [5, 14, 17] concerning unbounded growing of the amplitude of the acceleration waves give us the simple expression for the critical time. In the next section we use the results of [17] to this problem.

5. Wave in a material of grade two

In this section we would like to give an example of the shock wave propagation. It will be done in the case of expansive waves in a material of grade two. In [14, 17] such a material was considered. In reference to this paper we postulate the following form of the constitutive function $\mathcal{T}$ and the preparation function $\mathbf{a}$ in the right-hand side of the evolution equation:

$$\begin{aligned} \mathcal{T}(E, \boldsymbol{\alpha}) &\equiv b_1 E + \mathbf{b}_2 \boldsymbol{\alpha} + b_3 E^2 + b_0, \\ \mathbf{a}(E, \boldsymbol{\alpha}) &\equiv \mathbf{c}_1 E + \mathbf{c}_2 \boldsymbol{\alpha} + \mathbf{c}_0, \end{aligned} \tag{5.1}$$

where b_i, $\mathbf{c}_j$ are some physical (material) constants. For them we have the following inequalities

$$b_1 > 0, \quad b_3 > 0.$$

Hence the assumptions (A.2) and (B.2) are satisfied

$$\partial_E \mathcal{T}(E, \boldsymbol{\alpha}) = b_1 + 2b_3 E > 0, \quad \partial_E^2(E, \boldsymbol{\alpha}) = b_3 > 0 \quad \text{for} \quad E \geqslant 0. \tag{5.2}$$

The wave velocity V obeys the equation

$$\varrho V^2 = \frac{[\![T]\!]}{[\![E]\!]} = b_1 + b_3[\![E]\!] + 2b_3 E^+. \tag{5.3}$$

For r we have

$$r = \varrho V^2 - \partial_E \mathcal{T}(E^-, \boldsymbol{\alpha}) = -b_3[\![E]\!] < 0. \tag{5.4}$$

([14]) In [16] this condition was formulated in term of the vanishing Jacobian of the transformation of (t, X) variables into (t', φ) variables, where $\varphi = 0$ was the wavefront. This condition, geometrically, denotes that the family of characteristics $\varphi =$ constant intersect at a cusp.

The velocity V changes according to [cf. (3.11)]

$$\frac{\delta V}{\delta t} = \frac{b_3}{2\varrho V}\left(\frac{\delta[\![E]\!]}{\delta t} + 2\dot{E}^+\right). \tag{5.5}$$

Since

$$\lambda = -\frac{1}{V}\left(\dot{E}^+ - \left(\frac{\mathbf{c}_1 \cdot \mathbf{b}_2}{b_3}\right)\right), \tag{5.6}$$

the amplitude equation has the form

$$\frac{\delta[\![E]\!]}{\delta t} = -\frac{2Vb_3[\![E]\!]}{4b_1 + 3b_3 E^+ + 5b_3 E^-}\left(\partial_X E^- + \frac{1}{V}\left(\dot{E}^+ - \frac{\mathbf{c}_1 \cdot \mathbf{b}_2}{b_3}\right)\right). \tag{5.7}$$

Hence we can formulate the following

THEOREM 5. *In the material of grade two the amplitude of the expansive shock wave undergoes the following behaviours*:

1) *if* $\partial_X E^- > \frac{1}{V}\left(\frac{\mathbf{c}_1 \cdot \mathbf{b}_2}{b_3} - w\right)$, *then* $\delta[\![E]\!]/\delta t < 0$;

2) *if* $\partial_X E^- < \frac{1}{V}\left(\frac{\mathbf{c}_1 \cdot \mathbf{b}_2}{b_3} - w\right)$, *then* $\delta[\![E]\!]/\delta t > 0$;

3) *if* $\mathbf{c}_1 \cdot \mathbf{b}_2 < 0$, *then* $\delta[\![E]\!]/\delta t$ *is always smaller than zero provided that the strain gradient* $\partial_X E^-$ *is not negative;*

4) *if* $\partial_X E^- = \frac{1}{V}\left(\frac{\mathbf{c}_1 \cdot \mathbf{b}_2}{b_3} - w\right)$, *then* $\delta[\![E]\!]/\delta t = 0$.

In the above statements we used the notation $\dot{E}^+ = w$ [cf. (3.2)] and the inequality $w \geqslant 0$.

Now we give the expressions for the critical time t_c and the critical distance X_c in the case under consideration. According to Theorem 2 in [17] for the acceleration wave propagating through the material of grade two being in the homogeneous equilibrium state $(\mathbf{E}_0, \boldsymbol{\alpha}_0)$, the amplitude $[\![\ddot{x}]\!](t) \equiv a(t)$ goes to minus infinity in the finite time $[0, t_c]$ if either

a) $\mathbf{b}_2 \cdot \mathbf{c}_1 < 0$ and $a(0) < \frac{\mathbf{b}_2 \cdot \mathbf{c}_1}{2b_3}\sqrt{\frac{b_1 + 2b_3 E_0}{\varrho}}$ or

b) $\mathbf{b}_2 \cdot \mathbf{c}_1 > 0$ and $a(0) < 0$.

For both cases the critical time t_c is given by

$$t_c = \frac{2(b_1 + 2b_3 E_0)}{\mathbf{b}_2 \cdot \mathbf{c}_1} \ln\left(1 - \frac{\mathbf{b}_2 \cdot \mathbf{c}_1}{2a(0)b_3}\sqrt{\frac{b_1 + 2b_3 E_0}{\varrho}}\right). \tag{5.8}$$

Here $a(0)$ denotes the initial value of the amplitude of the acceleration wave. Because the critical distance X_c fulfils the formula

$$X_c = X_0 + \sqrt{\frac{b_1 + 2b_3 E_0}{\varrho}}\, t_c, \tag{5.9}$$

where X_0 is the material point at which the acceleration wave is to be found at time $t = 0$ and the square root is the wave velocity, we have

$$X_c = X_0 + \frac{2}{\mathbf{b}_2 \cdot \mathbf{c}_1} \sqrt{\frac{(b_1 + 2b_3 E_0)^3}{\varrho}} \ln\left(1 - \frac{\mathbf{b}_2 \cdot \mathbf{c}_1}{2a(0) b_3} \sqrt{\frac{b_1 + 2b_3 E_0}{\varrho}}\right). \tag{5.10}$$

The statement a) above tells that the shock wave will be formed whenever the initial amplitude $a(0)$ is less than some value equal to the critical amplitude for acceleration waves. The statement b) pronounces, however, that each expansive acceleration wave turns out, in the finite time, to be a shock wave.

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