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Structure of equations and estimation of solutions in non-linear shell theory

Z. F. BACZYŃSKI (WARSZAWA)

IN THIS paper a new approach to non-linear shell theory is discussed based on a three-dimensional constrained continuum theory which is being developed in WOŹNIAK's papers [1, 2, 3]. The constraints are introduced in such a way that the whole boundary-initial value problem can be formulated in terms of functions depending on a coordinate Z belonging to a flat surface and on the time t. It is shown that we are able to give a positive answer whether or not the present theory is convenient to attack some particular problems.

Autor omawia w pracy nowe podejście do nieliniowej teorii powłok, oparte na trójwymiarowej teorii kontinuum z więzami, rozwijanej przez WOŹNIAKA w pracach [1, 2, 3]. Więzy są tak wprowadzone, że całkowity problem początkowo-brzegowy może być sformułowany za pomocą funkcji zależnych od współrzędnej $\mathbf{Z}$, odnoszącej się do pewnej płaskiej powierzchni, zwanej powierzchnią podstawową, i od czasu t. W pracy wykazano, że można odpowiedzieć pozytywnie na pytanie, czy obecna teoria nadaje się do rozwiązania pewnych szczególnych problemów.

В настоящей работе обсуждается новый подход к нелинейной теории оболочек, который опирается на трехмерной теории континуум со связями развиваемой Возняком в работах [1, 2, 3]. Связи введены так, что полная начально-краевая задача может быть сформулирована при помощи функций зависящих от координаты $\mathbf{Z}$, принадлежащей к некоторой плоской поверхности (называемой основной поверхностью), и от времени t. В работе показано, что можно положительно ответить на вопрос годится ли настоящая теория для решения некоторых частных задач.

1. General assumptions

WE ASSUME that the region of the shell or shell-like body under consideration in reference configuration $\mathring{\varkappa}$ is given by the relations

$$\mathring{\varkappa}(B) = F\times\Pi, \quad \partial\mathring{\varkappa}(B) = (\partial F\times\Pi)\cup(\partial\Pi\times F), \quad X^3\in F, \quad \mathbf{Z}\in\Pi \tag{1.1}$$

and that each material element $F\times\{\mathbf{Z}\}$ is the rectilinear element. Parametrizing the shell region by material coordinates $\{X^\alpha\}$ we have the relations

$$X^\alpha = \delta^\alpha_K Z^K + \delta^\alpha_3 X^3, \quad \mathbf{Z} = (Z^1, Z^2)\in\Pi, \quad \alpha\in\{1,2,3\} \quad K\in\{1,2\}, \tag{1.2}$$

which means that the material coordinates $\{X^\alpha\}$ and the orthogonal Cartesian coordinates $\{x^\alpha\}$ coincide in a physical space when the shell is referred to $\mathring{\varkappa}$ (Fig. 1). Treating the shell as a three-dimensional constrained continuum we assume that the deformation $\boldsymbol{\chi}$ is restricted by the integrable constraints of the form

$$(\chi^k_{,\alpha}e^\alpha)_{,\beta}e^\beta = 0, \quad e^\alpha_{,\beta}e^\beta = 0, \quad \mathbf{e}\cdot\mathbf{e} = 1, \tag{1.3}$$

where the vector $\mathbf{e}$ denotes a direction of material line element for which the hypothesis is established. The constraint equations (1.3) correspond to the hypothesis of uniform strain of each material element.

A general solution of the constraint equation (1.3) has the form

$$\chi^k(\mathbf{Z}, X^3, t) = \psi^k(\mathbf{Z}, t) + d^k(\mathbf{Z}, t)X^3, \quad k \in \{1, 2, 3\}, \tag{1.4}$$

where the vectorial functions $\boldsymbol{\psi}$ and $\mathbf{d}$ are unknown functions, and ψ^k represents the motion of the fundamental shell surface while d^k are directors of the fundamental shell surface (Fig. 1).

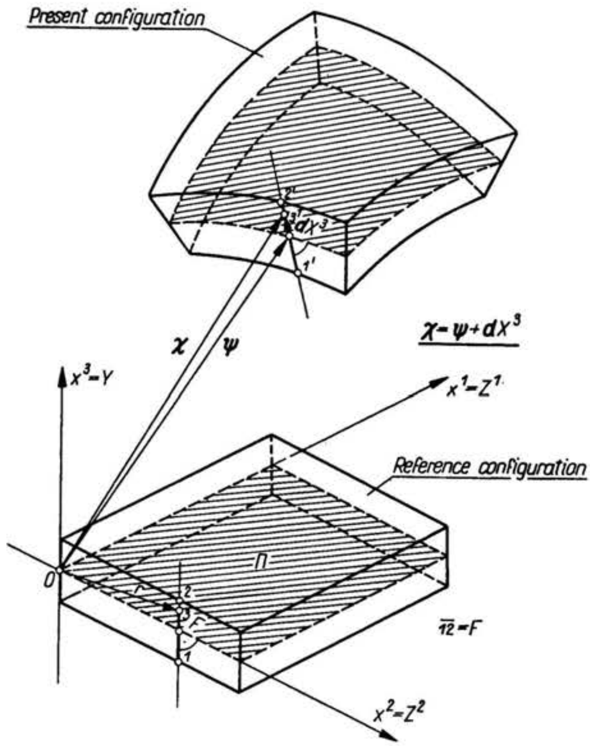

FIG. 1. The shell element related to the spatial and material coordinate system.

We assume that the constitutive function for elastic material has the form

$$T^{\alpha\beta} = 2\varrho \frac{\partial e}{\partial c_{\alpha\beta}}, \quad e = e(X^\alpha, c_{\alpha\beta}), \quad \alpha, \beta \in \{1, 2, 3\}, \tag{1.5}$$

where the elastic potential e depends on the material coordinates and the Green deformation tensor in the convected coordinate system $\{X^\alpha\}$. The Green deformation tensor is expressed by

$$c_{\alpha\beta} = \chi^k_{,\alpha}\chi^l_{,\beta}\delta_{kl}, \quad k, l \in \{1, 2, 3\}. \tag{1.6}$$

If the reference configuration $\mathring{\varkappa}$ at the time t_0 is given, and no forces are acting in the region of the shell, then we can assume that the Green-Saint Venant deformation tensor has the form

$$e_{\alpha\beta} = \frac{1}{2}(c_{\alpha\beta} - g_{\alpha\beta}), \tag{1.7}$$

where $g_{\alpha\beta}$ is the metric tensor of natural state. Assuming the elastic potential of the material as a quadratic form, we have from (1.5) the constitutive equation

$$T^{\alpha\beta} = \frac{1}{2}A^{\alpha\beta\gamma\delta}(c_{\gamma\delta} - g_{\gamma\delta}), \quad \alpha, \beta, \gamma, \delta \in \{1, 2, 3\}, \tag{1.8}$$

where $A^{\alpha\beta\gamma\delta}$ is the tensor of elastic moduli of the shell. In the Eq. (1.8) the tensor of elastic moduli as well as the metric tensor $g_{\gamma\delta}$ are known at each point of the shell region, if the geometry and material of the shell are given.

2. Basic equations of non-linear shell theory (in the case of small strains)

If we take into account that the shell deformation is described by the general solution (1.4), and we compute the Green deformation tensor, then from the constitutive equation (1.8) we get the expressions for the components of the Cauchy extra-stress tensor in the form:

$$\begin{aligned} T^{\alpha\beta} = {} & A^{\alpha\beta MN}[e_{MN} + e'_{MN}X^3 + e''_{MN}(X^3)^2] + A^{\alpha\beta 33}e_{33} \\ & + 2A^{\alpha\beta M3}(e_{M3} + e'_{M3}X^3), \quad \alpha, \beta \in \{1, 2, 3\}, \quad M, N \in \{1, 2\}, \end{aligned} \tag{2.1}$$

where the deformation measures $e_{\gamma\delta}$ are defined by the geometrical relations:

$$\begin{aligned} & e_{MN} = \frac{1}{2}(\psi^m_{,M}\psi^n_{,N}\delta_{mn} - a_{MN}), \quad e_{M3} = \frac{1}{2}(\psi^m_{,M}d^n\delta_{mn} - a_{M3}), \\ & e_{33} = \frac{1}{2}(d^m d^n\delta_{mn} - a_{33}), \quad e'_{MN} = \frac{1}{2}(\psi^m_{,M}d^n_{,N} + \psi^n_{,N}d^m_{,M})\delta_{mn} + b_{MN}, \\ & e'_{M3} = \frac{1}{2}(d^m_{,M}d^n\delta_{mn} + b_{M3}), \quad e''_{MN} = \frac{1}{2}(d^m_{,M}d^n_{,N}\delta_{mn} - c_{MN}), \end{aligned} \tag{2.2}$$

and $a_{\gamma\delta}$, $b_{\gamma\delta}$, $c_{\gamma\delta}$ characterize the shell metric $g_{\gamma\delta}$ in the reference configuration $\mathring{\varkappa}$ according to the formula

$$g_{\gamma\delta} = a_{\gamma\delta} - 2b_{\gamma\delta}X^3 + c_{\gamma\delta}(X^3)^2. \tag{2.3}$$

Summing up, we see that using the quoted axioms, we get basic equations of our theory in a convected coordinate system [4].

The equations of motion have the following form:

$$H^{K\alpha}|_K + f^\alpha = i^\alpha, \quad \overline{H}^{K\alpha}|_K + \overline{h}^\alpha + \overline{f}^\alpha = \overline{i}^\alpha, \tag{2.4}$$

where $H^{K\alpha}|_K$ and $\overline{H}^{K\alpha}|_K$ are the covariant derivatives of $H^{K\alpha}$ and $\overline{H}^{K\alpha}$ with the metric $a_{\alpha\beta} = a^m_\alpha a^n_\beta \delta_{mn}$.

All quantities in the Eq. (2.4) are related to the present configuration $\varkappa$ and they have the following sense:

Quantities i^α, $\bar{i}^\alpha$ denote the inertia forces and they are determined by the kinetic energy K

$$(2.5)\qquad i^\alpha \equiv \frac{1}{\sqrt{a}}\left(\frac{\partial K}{\partial \dot{\psi}^k}\right)_{,t} a_k^\alpha, \quad \bar{i}^\alpha \equiv \frac{1}{\sqrt{a}}\left(\frac{\partial K}{\partial \dot{d}^k}\right)_{,t} a^{\alpha k}, \quad a \equiv \det(a_\alpha^k a_\beta^l \delta_{kt}),$$
$$a^{\alpha k}(\psi_k + d_k X^3)_{,\beta} = \delta_\beta^\alpha,$$

where the kinetic energy has the form

$$(2.6)\qquad K \equiv \frac{1}{2}\left(\int_{Y^-}^{Y^+} \mathring{\varrho}\dot{\psi}^k\dot{\psi}_k dX^3 + 2\int_{Y^-}^{Y^+} \mathring{\varrho}\dot{\psi}^k\dot{d}_k X^3 dX^3 + \int_{Y^-}^{Y^+} \mathring{\varrho}\dot{d}^k\dot{d}_k (X^3)^2 dX^3\right).$$

Quantities f^α, $\bar{f}^\alpha$ represent the generalized densities of external loads, and they are expressed by

$$(2.7)\qquad f^\alpha \equiv \int_{Y^-}^{Y^+} \mathring{\varrho} b^\alpha dX^3 + (\mathring{p}^\alpha)^+ - (\mathring{p}^\alpha)^-, \qquad \bar{f}^\alpha \equiv \int_{Y^-}^{Y^+} \mathring{\varrho}^\alpha X^3 dX^3 + [(\mathring{p}^\alpha)^+ - (\mathring{p}^\alpha)^-]X^3.$$

Quantities $H^{K\alpha}$, $\bar{H}^{K\alpha}$, $\bar{h}^\alpha$ represent generalized shell forces given by the constitutive equations

$$(2.8)\qquad \begin{aligned} H^{K\alpha} &\equiv (J_0^{K\alpha\beta\gamma} - J_1^{KP\beta\gamma} b_P^\alpha) e_{\beta\gamma} + (J_1^{K\alpha\beta\gamma} - J_2^{KP\beta\gamma} b_P^\alpha) e'_{\beta\gamma} + (J_2^{K\alpha MN} - J_3^{KPMN} b_P^\alpha) e''_{MN},\\ \bar{H}^{K\alpha} &\equiv (J_1^{K\alpha\beta\gamma} - J_2^{KP\beta\gamma} b_P^\alpha) e_{\beta\gamma} + (J_2^{K\alpha\beta\gamma} - J_3^{KP\beta\gamma} b_P^\alpha) e'_{\beta\gamma} + (J_3^{K\alpha MN} - J_4^{KPMN} b_P^\alpha) e''_{MN},\\ \bar{h}^\alpha &\equiv (J_0^{\alpha\beta\gamma} - J_1^{P\beta\gamma} b_P^\alpha) e_{\beta\gamma} + (J_1^{\alpha\beta\gamma} - J_2^{P\beta\gamma} b_P^\alpha) e'_{\beta\gamma} + (J_2^{\alpha MN} - J_3^{PMN} b_P^\alpha) e''_{MN} \end{aligned}$$

(summing up one should assume $e'_{33} \equiv 0$), where the modified tensors of elastic moduli are defined by the integrals

$$(2.9)\qquad \begin{aligned} J_n^{K\alpha\beta\gamma} &\equiv \frac{1}{\sqrt{a}} \int_{Y^-}^{Y^+} \sqrt{g} A^{K\alpha\beta\gamma}(\mathbf{X}, t)(X^3)^n dX^3, \qquad n = 0, 1, 2, 3, 4.\\ J_n^{\alpha\beta\gamma} &\equiv \frac{1}{\sqrt{a}} \int_{Y^-}^{Y^+} \sqrt{g} A^{\alpha\beta\gamma}(\mathbf{X}, t)(X^3)^n dX^3, \end{aligned}$$

and the curvature of the middle surface of the shell is characterized by the relation

$$(2.10)\qquad b_P^\alpha \equiv -d_{k,P} a^{\alpha k}.$$

To formulate a boundary-initial value problem we can prescribe the boundary conditions in the following manner: If a boundary of the shell is composed of the part $\overset{*}{\partial}\Pi$ on which a load is given, and of a part $\tilde{\partial}\Pi$ with a motion, then we have:

(i) for $\mathbf{Z} \in \overset{*}{\partial}\Pi$

$$(2.11)\qquad \begin{aligned} H^{K\alpha} n_K &= \overset{*}{p}^\alpha, \qquad \overset{*}{p}^\alpha \equiv \frac{1}{\sqrt{a}} \int_{Y^-}^{Y^+} p_k(\mathbf{X}, t) a^{\alpha k} dX^3,\\ \bar{H}^{K\alpha} n_K &= \overset{*}{\bar{p}}^\alpha, \qquad \overset{*}{\bar{p}}^\alpha \equiv \frac{1}{\sqrt{a}} \int_{Y^-}^{Y^+} p_k(\mathbf{X}, t) a^{\alpha k} X^3 dX^3; \end{aligned}$$

(ii) for $\mathbf{Z} \in \tilde{\partial}\Pi$

$$\psi^k = \tilde{\psi}^k(\mathbf{X}, t), \quad d^k = \tilde{d}^k(\mathbf{X}, t) \quad \text{i.e.} \quad \tilde{\chi}^k = \tilde{\psi}^k + \tilde{d}^k X^3. \tag{2.12}$$

Considering a dynamical problem, we have to introduce some information about functions $\boldsymbol{\psi}$, and $\mathbf{d}$, as well as their time derivatives at a given time instant. If the functions $\boldsymbol{\psi}$, and $\mathbf{d}$ are obtained from a solution of the appropriate boundary-initial value problem, then by means of the formula

$$\boldsymbol{\chi} = \boldsymbol{\psi} + \mathbf{d}X^3 \tag{2.13}$$

the motion at each point of the shell region and for time is described.

3. The formal procedure for finding solutions to boundary-initial value problems

A formal procedure for finding a solution to the boundary-initial value problem takes into account the following steps.

(i) Unknown functions $\boldsymbol{\psi}$, $\mathbf{d}$ are to be obtained by solving a suitable boundary-initial value problem described by the following equations: equations of motion for generalized shell forces (2.4); constitutive equations for generalized shell forces (2.8); geometrical equations for strain measures (2.2); and boundary-initial conditions in terms of generalized forces and motions (2.11), (2.12) as well as in terms of initial values of generalized coordinates and their time derivatives.

(ii) Once the vector functions $\boldsymbol{\psi}$, $\mathbf{d}$ have been found, the motion of the shell is given by (2.13).

(iii) Using the constitutive equations for the components of the Cauchy convected extra-stress tensor (2.1), we determine the stress field at each point of the shell region and for arbitrary time.

(iv) Finally, we determine the body reaction field $\mathbf{r}$, and the surface reaction field $\mathbf{s}$, using the formulae:

$$r^\alpha = \frac{\varrho}{\sqrt{g}} \ddot{\chi}_k g^{\alpha k} - \left(\frac{\varrho}{\sqrt{g}} b_k g^{\alpha k} + T^{\beta\alpha}\|_\beta \right) \quad \underline{\text{in } F \times \Pi}, \tag{3.1}$$

$$s^\alpha = T^{K\alpha} n_K - p^\alpha \quad \underline{\text{on } F \times \partial\Pi}, \tag{3.2}$$

$$s^\alpha = T^{\beta\alpha} n_\beta - p^\alpha \quad \underline{\text{on } \partial F \times \Pi}, \tag{3.3}$$

where all quantities are related to the present configuration $\varkappa$. Moreover $T^{\beta\alpha}\|_\beta$ denotes the classical covariant derivatives of $T^{\beta\alpha}$ at an arbitrary point of the shell region in a three-dimensional Euclidean space with the metric $g_{\alpha\beta}$, such that

$$[T^{\beta\alpha}\|_\beta]_{X=0} = T^{\beta\alpha}|_\beta, \tag{3.4}$$

and $T^{\beta\alpha}|_\beta$ is the covariant derivative of $T^{\beta\alpha}$ with the metric $a_{\alpha\beta}$.

4. Estimation of the solutions (criteria of "error")

To obtain an estimation of "error" of approximation of the dynamical process for the constrained shell, and to compare this with the analogical results for unconstrained shell, we have to introduce some norms in the load space $\{\mathbf{b}, \mathbf{p}\}$. Proximity of the dynamical process will be given by proximity of the introduced norms in the load space with the "error" defined by

$$\text{"error"} = \frac{\|\{\mathbf{r}, \mathbf{s}\}\|}{\|\{\mathbf{b}, \mathbf{p}\}\|}. \tag{4.1}$$

If the "error" is given *a priori*, the following inequality (an estimate criterion) can be proposed [5]:

$$\frac{\|\{\mathbf{r}, \mathbf{s}\}\|}{\|\{\mathbf{b}, \mathbf{p}\}\|} \leqslant \text{"error"}, \tag{4.2}$$

where **b** and **p** are given body and surface loads, and **r**, and **s** stand for the body and surface reaction forces of the constraints.

The set of elements $\{\mathbf{b}, \mathbf{p}\}$ one can identify with a linear, normalized, and complete vectorial space (Banach space). The norms in such a load space can be defined in many ways according to the character of loads and reaction forces of the constraints. Let us mention some of the most simple definitions of the norms. The first one can be introduced using the vectorial densities of the loads and the found reactions of constraints, i.e.

$$\|\{\mathbf{b}, \mathbf{p}\}\| = \Big(\int_{\overset{c}{\varkappa}(B)} \alpha|\mathbf{b}|^2 dV + \int_{\partial\overset{\circ}{\varkappa}(B)} \beta|\mathbf{p}|^2 dS\Big)^{\frac{1}{2}}, \tag{4.3}$$

where α, β are suitable weight functions of the norms. The second one can be defined by means of a maximum of the absolute value of the loads and found constraint reactions i.e.

$$\|\{\mathbf{b}, \mathbf{p}\}\| = \bar{\alpha} \cdot \underset{\mathbf{X}\in\overset{\circ}{\varkappa}(B)}{\mathrm{Max}} |\mathbf{b}| + \bar{\beta} \cdot \underset{\mathbf{X}\in\partial\overset{\circ}{\varkappa}(B)}{\mathrm{Max}} |\mathbf{p}|, \tag{4.4}$$

where $\bar{\alpha}, \bar{\beta}$ are suitable weight functions of the norms.

5. Illustration of the proposed theory (example)

Let us consider an infinite cylinder of arbitrary thickness $(R^+ - R^-)$, where R^+ is the external and R^- is the internal radius of the cylinder. Reference configuration $\overset{\circ}{\varkappa}(B)$ is given in x^α or X^α $(\alpha = 1, 2, 3)$ coordinate systems (Fig. 2), and the cylindrical region $F \times \Pi$ is described by the inequalities

$$-\infty \leqslant X^1 \leqslant +\infty, \quad 0 < X^2 \leqslant 2\pi, \quad 0 \leqslant X^3 \leqslant R^+ - R^-. \tag{5.1}$$

The cylinder is made of a homogeneous, orthotropic, elastic material, subject to a shear deformation in the directions of X^1 and X^2 such that relative displacements of the cylindrical shell are:

$$u^1 = (R^+ - R^-)\gamma^{31}, \quad u^2 = (R^+ - R^-)\gamma^{32}, \quad u^3 = 0, \tag{5.2}$$

where γ^{31}, γ^{32} are given angles of the shear deformations.

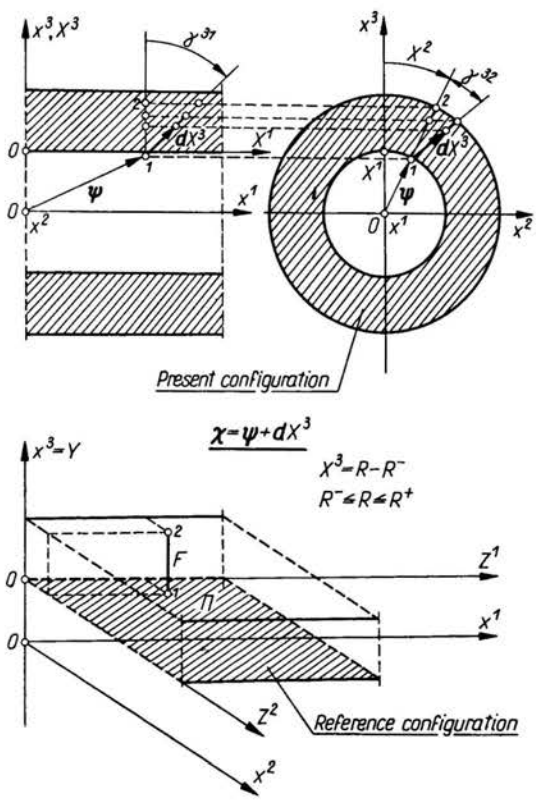

FIG. 2. The infinite cylinder subjected to the shear deformation.

The problem is to be formulated with the non-linear shell theory of small strains and a statical process $\{\boldsymbol{\chi}, \mathbf{T}\}$ characterized by the deformation function and the Cauchy extra-stress tensor is to be found. Some practical criteria of the estimation of the solutions obtained will also be discussed.

Assuming that the cylindrical surface $R = R^-$, $(X^3 = 0)$, coincides with the fundamental surface

$$x^1 = X^1, \quad x^2 = R^- \sin X^2, \quad x^3 = R^- \cos X^2 \tag{5.3}$$

we obtain, immediately, a solution of the boundary value problem described by the Eqs. (2.2), (2.4), (2.8), and the conditions (2.11), (2.12), in the form:

$$\begin{aligned} &\psi^1 = X^1, \quad \psi^2 = R^- \sin X^2, \quad \psi^3 = R^- \cos X^2, \\ &d^1 = \gamma^{31}, \quad d^2 = \sin X^2 + \gamma^{32}\cos X^2, \quad d^3 = \cos X^2 - \gamma^{32} X^2. \end{aligned} \tag{5.4}$$

Using (2.13) we find the deformation functions

$$\begin{aligned} &\chi^1 = X^1 + \gamma^{31} X^3, \quad \chi^2 = R^- \sin X^2 + (\sin X^2 + \gamma^{32}\cos X^2) X^3, \\ &\chi^3 = R^- \cos X^2 + (\cos X^2 - \gamma^{32} \sin X^2) X^3. \end{aligned} \tag{5.5}$$

Now, the definitions (2.2), (2.3), and (5.4) lead to the deformation measures:

$$e_{11} = e_{22} = e_{12} = 0, \quad e_{33} = \frac{1}{2}[(\gamma^{31})^2 + (\gamma^{32})^2],$$

(5.6)
$$e_{31} = \frac{1}{2}\gamma^{31}, \quad e_{32} = \frac{1}{2}R^-\gamma^{32},$$

$$e'_{11} = e'_{22} = e'_{12} = e'_{13} = e'_{23} = 0, \quad e''_{11} = e''_{12} = 0, \quad e''_{22} = \frac{1}{2}(\gamma^{32})^2.$$

The constitutive equations (2.1), and (5.6), together with the definition of elastic moduli for cylindrical orthotropy imply the following form of the Cauchy extra-stress tensor:

(5.7)
$$\begin{bmatrix} T^{11} \\ T^{22} \\ T^{33} \\ T^{23} \\ T^{13} \\ T^{12} \end{bmatrix} = \begin{bmatrix} A^{1111} & A^{1122} & A^{1133} & 0 & 0 & 0 \\ A^{2211} & A^{2222} & A^{2233} & 0 & 0 & 0 \\ A^{3311} & A^{3322} & A^{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & A^{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & A^{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & A^{1212} \end{bmatrix} \begin{bmatrix} e^*_{11} \\ e^*_{22} \\ e^*_{33} \\ e^*_{23} \\ e^*_{13} \\ e^*_{12} \end{bmatrix},$$

where $e^*_{11}, \ldots, e^*_{12}$ are new deformation measures

(5.8)
$$\begin{aligned} e^*_{MN} &\equiv e_{MN} + e'_{MN}X^3 + e''_{MN}(X^3)^2, \\ e^*_{M3} &\equiv 2e_{M3} + e'_{M3}X^3, \quad e^*_{33} \equiv e_{33}. \end{aligned}$$

The components of the Cauchy extra-stress tensor can also be written in the form:

(5.9)
$$\begin{aligned} T^{11} &= \frac{1}{2}A^{1122}(\gamma^{32}X^3)^2 + \frac{1}{2}A^{1133}[(\gamma^{31})^2 + (\gamma^{32})^2], \\ T^{22} &= \frac{1}{2}A^{2222}(\gamma^{32}X^3)^2 + \frac{1}{2}A^{2233}[(\gamma^{31})^2 + (\gamma^{32})^2], \\ T^{33} &= \frac{1}{2}A^{3322}(\gamma^{32}X^3)^2 + \frac{1}{2}A^{3333}[(\gamma^{31})^2 + (\gamma^{32})^2], \\ T^{12} &= 0, \quad T^{13} = A^{1313}\gamma^{31}, \quad T^{23} = A^{2323}\gamma^{32}R^-. \end{aligned}$$

The introduced constraints of the form (1.3) produce the body and surface reaction forces, which can be calculated from (3.1) and (3.3). In the static case, we obtain

(5.10)
$$r^\alpha = -T^{\beta\alpha}\|_\beta,$$

(5.11)
$$s^\alpha = T^{\beta\alpha}n_\beta, \quad \alpha, \beta \in \{1, 2, 3\}.$$

The components of the body reaction forces assume the form:

(5.12)
$$\begin{aligned} r^1 &= -R^{-1}(A^{1313}\gamma^{31} + A^{2323}\gamma^{32}R^-), \quad r^2 = -3R^{-1}A^{2323}\gamma^{32}R^-, \\ r^3 &= -A^{3322}(\gamma^{32})^2(R-R^-) - \frac{1}{2}R^{-1}\{A^{3322}(\gamma^{32})^2(R-R^-)^2 \\ &\quad + A^{3333}[(\gamma^{31})^2 + (\gamma^{32})^2]\} + \frac{1}{2}R\{A^{2222}(\gamma^{32})^2(R-R^-)^2 + A^{2233}[(\gamma^{31})^2 + (\gamma^{32})^2]\}; \end{aligned}$$

while the components of surface reaction forces are:

$$(s^1)^+ = (s^1)^- = A^{1313}\gamma^{13}, \quad (s^2)^+ = (s^2)^- = A^{2323}\gamma^{32}R^-,$$

(5.13)
$$(s^3)^+ = \frac{1}{2}A^{3322}(\gamma^{32})^2(R^+-R^-)^2 + \frac{1}{2}A^{3333}[(\gamma^{31})^2+(\gamma^{32})^2],$$

$$(s^3)^- = \frac{1}{2}A^{3333}[(\gamma^{31})^2+(\gamma^{32})^2].$$

In the above formulae γ^{31}, γ^{32}, are given angles of the shear deformations. All quantities with index plus $(\ldots)^+$ are related to the external cylindrical surface, and with minus $(\ldots)^-$ to the internal cylindrical surface.

To estimate the accuracy of the solution, we can apply the criterion (4.1) in the following form:

(5.14)
$$\text{“error”} = \frac{\|\{\mathbf{r}, \mathbf{0}\}\|}{\|\{\mathbf{0}, \mathbf{s}\}\|},$$

where the norms of the load space $\|\{\mathbf{r}, \mathbf{s}\}\|$ are given by

(5.15)
$$\|\{\mathbf{r}, \mathbf{0}\}\| = (R^+-R^-)\operatorname*{Max}_{X\in\overset{\circ}{\varkappa}(B)} |\mathbf{r}|, \quad \|\{\mathbf{0}, \mathbf{s}\}\| = \operatorname*{Max}_{X\in\partial\overset{\circ}{\varkappa}(B)} |\mathbf{s}|.$$

In the case of isotropy and of the shear deformation of the cylinder in the direction X^1 only, i.e. if $\gamma^{31} \neq 0$ and $\gamma^{32} = 0$, we have a practical criterion

(5.16)
$$\text{“error”} = \frac{R^+-R^-}{R^-}\,\frac{\mu\sqrt{1+(\gamma^{31})^2}}{\sqrt{\mu^2+\frac{1}{4}(\lambda+2\mu)^2(\gamma^{31})^2}},$$

where λ, μ are Lame constants.

For a very small angle of shear deformation, i.e. when $\gamma^{31} \approx 0$, the criterion (5.16) becomes:

(5.17)
$$\text{“error”} = \frac{R^+-R^-}{R^-},$$

where (R^+-R^-) is the shell thickness, and R^- is the main curvature radius of the fundamental shell surface.

In the case of isotropy and of the shear deformation of the cylinder in the direction X^2 only, i.e. if $\gamma^{31} = 0$ and $\gamma^{32} \neq 0$ we have:

(5.18)
$$\text{“error”} = \frac{R^+-R^-}{R^-}\,\frac{\mu\sqrt{9+(R^-)^{-2}+(\gamma^{32})^2}}{\sqrt{\mu^2+\frac{1}{4}(\lambda+2\mu)^2(\gamma^{32})^2}}.$$

Neglecting the value $(R^-)^{-2}$ in (5.18) as very small, and assuming that the angle of shear deformation γ^{32} is very small, we arrive at:

(5.19)
$$\text{“error”} = 3\frac{R^+-R^-}{R^-}.$$

As one can see from the above analysis, the accuracy of the solution depends on: the ratio of the shell thickness to the main curvature radius of the fundamental shell surface; on the elastic moduli; and on the load system, and distribution of the constraint reactions.

6. Conclusions and final remarks

The new approach regarding the foundation and formulation of the general non-linear theory of shells and shell-like bodies allows us to describe boundary-initial value problems of shell theory treated as problems of a three-dimensional constrained continuum, and to give full and exact information about the dynamical process of the shell. By introducing some constraints characterizing kinematics of shell, the boundary-initial value problem has been formulated in terms of functions depending on arguments $\{\mathbf{Z}, t\}$, i.e. depending on the point of a flat region Π in reference configuration and on time. The following conclusions are to be drawn from our consideration:

The treatment of a shell as a three-dimensional constrained continuum means, that each boundary-initial value problem for the shell is consistent with the appropriate problem of the three-dimensional theory. The dynamical process of the boundary-initial value problem is uniquely defined by geometry, material, and loads of the shell treated as a three-dimensional body.

When a solution to the boundary-initial value problem is found, we are able to estimate its "error" in a direct manner, and to give the answer, whether or not the present theory is adequate to attack the formulated problem.

It is easy to notice that if the reaction forces of constraints $\{\mathbf{r}, \mathbf{s}\}$ are fading out in the considered region of the shell, then the solution obtained for the dynamical process $\{\boldsymbol{\chi}, \mathbf{T}\}$ is exact in the sense of a solution to a boundary-initial value problem of three-dimensional classical continuum. Also, the solution obtained seems to be exact in the sense of a three-dimensional classical continuum, if the system $\{\mathbf{b}+\mathbf{r}, \mathbf{p}+\mathbf{s}\}$ is to be treated as a system of loads.

It also follows that, if the reaction forces of constraints $\{\mathbf{r}, \mathbf{s}\}$ are sufficiently small in comparison to the loads acting on the shell, then the solution obtained from a three-dimensional constrained continuum is a good approximation to that of a three-dimensional unconstrained continuum.

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