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# On the plane strain problem for an isotropic mixture of two linear elastic solids

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THE PROBLEM of the existence and uniqueness of solutions, in the case of the first and second boundary value problems, for an isotropic mixture of two linear elastic solids is considered.

## 1. Introduction

THE PLANE strain problem for isotropic mixtures of two homogeneous compressible elastic solids has been considered by STEEL in papers [1 and 2]. The solution of equilibrium equations and displacement vectors have been expressed by means of complex potentials which present analogous properties with the complex potentials appearing in the classical theory of elasticity. However, an existence or uniqueness theorem, regarding this problem, has not yet been obtained.

The aim of the present paper is to solve (partially) this a problem. We note that in papers [3 and 4] other problems regarding the existence and the uniqueness of solutions in the theory of mixtures have been considered.

## 2. Statement of the problem

We consider the body, consisting of an isotropic mixture of two homogeneous compressible elastic solids, referred to a rectangular coordinate system $Ox_1x_2x_3$ and suppose that the deformations of the two solids hold in the $x_1Ox_2$-plane, so that (¹)

$$\omega_\alpha = \omega_\alpha(x_1, x_2), \quad \eta_\alpha = \eta_\alpha(x_1, x_2), \quad \alpha = 1, 2, \quad \omega_3 = \eta_3 = 0, \tag{2.1}$$

where $\omega_i$ and $\eta_i$ $(i = 1, 2, 3)$ are the components of the two displacement vectors.

The basic equations of the above mentioned theory, as given in [1, 5, 6] are:

— constitutive equations

$$\sigma_{(\alpha\beta)} \equiv \frac{1}{2}(\sigma_{\alpha\beta}+\sigma_{\beta\alpha}) = -\alpha_2\delta_{\alpha\beta}+(\lambda_1 e_{\gamma\gamma}+\lambda_3 g_{\gamma\gamma})\delta_{\alpha\beta}+2\mu_1 e_{\alpha\beta}+2\mu_3 g_{\alpha\beta}, \tag{2.2}$$

$$\pi_{(\alpha\beta)} \equiv \frac{1}{2}(\pi_{\alpha\beta}+\pi_{\beta\alpha}) = \alpha_2\delta_{\alpha\beta}+(\lambda_4 e_{\gamma\gamma}+\lambda_2 g_{\gamma\gamma})\delta_{\alpha\beta}+2\mu_3 e_{\alpha\beta}+2\mu_2 g_{\alpha\beta}, \tag{2.3}$$

$$\sigma_{[\beta\alpha]} \equiv \frac{1}{2}(\sigma_{\beta\alpha}-\sigma_{\alpha\beta}) = -\frac{1}{2}(\pi_{\beta\alpha}-\pi_{\alpha\beta}) \equiv \pi_{[\alpha\beta]} = \lambda_5(h_{\alpha\beta}-h_{\beta\alpha}) = 2\lambda_5 h_{[\alpha\beta]}, \tag{2.4}$$

(¹) Throughout this paper the indices denoted by small Greek letters take the values 1, 2. The convention of summing over repeated indices is adopted.

$$\pi_\alpha = \pi,\alpha, \quad \pi \equiv \alpha_2\left(\frac{\varrho_1}{\varrho}g_{\gamma\gamma} + \frac{\varrho_2}{\varrho}e_{\gamma\gamma}\right), \quad \varrho \equiv \varrho_1 + \varrho_2, \tag{2.5}$$

$$\sigma_{\alpha 3} = \sigma_{3\alpha} = \pi_{\alpha 3} = \pi_{3\alpha} = 0, \tag{2.6}$$

$$\sigma_{33} = -\alpha_2 + \lambda_1 e_{\gamma\gamma} + \lambda_3 g_{\gamma\gamma}, \tag{2.7}$$

$$\pi_{33} = \alpha_2 + \lambda_4 e_{\gamma\gamma} + \lambda_2 g_{\gamma\gamma}, \tag{2.8}$$

where $\sigma_{ij}$ and $\pi_{ij}$ are the components of partial stress tensors, $\pi_\alpha$ are the components of the diffusive force, $\alpha_2$, $\lambda_1$, $\mu_1$, $\lambda_5$, etc..., are material constants, $\varrho_1$, $\varrho_2$ are the initial mass — densities of the two elastic solids and where

$$e_{\alpha\beta} = \frac{1}{2}(\omega_{\alpha,\beta} + \omega_{\beta,\alpha}) \equiv \omega_{(\alpha,\beta)}, \quad g_{\alpha\beta} = \frac{1}{2}(\eta_{\alpha,\beta} + \eta_{\beta,\alpha}) \equiv \eta_{(\alpha,\beta)}, \quad h_{\alpha\beta} = \omega_{\beta,\alpha} + \eta_{\alpha,\beta}. \tag{2.9}$$

— equilibrium equations

$$\sigma_{\alpha\beta,\alpha} + F_\beta - \pi_\beta = 0, \quad \pi_{\alpha\beta,\alpha} + G_\beta + \pi_\beta = 0, \tag{2.10}$$

where $F_\beta = F_\beta(x_1, x_2)$ and $G_\beta = G_\beta(x_1, x_2)$ are the components of the two body forces (we have supposed, as usually, that $F_3 = G_3 = 0$).

— boundary conditions

a) for the first boundary value problem:

$$\omega_\alpha = \eta_\alpha = k_\alpha, \quad \text{on } L; \tag{2.11}$$

b) for the second boundary value problem:

$$(\sigma_{\alpha\beta} + \pi_{\alpha\beta})n_\beta = T_\alpha, \quad \omega_\alpha = \eta_\alpha, \quad \text{on } L; \tag{2.12}$$

$L$ being the boundary of the finite regular plane region $\Sigma$, which has been considered in the body.

With these boundary conditions, the global internal energy of the body is given by [4] (see also [5, 7, 8]):

$$W = 2\int_\Sigma U d\Sigma, \tag{2.13}$$

where

$$2U = \left(\lambda_1 - \frac{\varrho_2}{\varrho}\alpha_2\right)e_{\alpha\alpha}^2 + \left(\lambda_2 + \frac{\varrho_1}{\varrho}\alpha_2\right)g_{\alpha\alpha}^2 + 2\left(\lambda_3 - \frac{\varrho_1}{\varrho}\alpha_2\right)e_{\alpha\alpha}g_{\alpha\alpha} + \mu_1 e_{\alpha\beta}e_{\alpha\beta} + \mu_2 g_{\alpha\beta}g_{\alpha\beta} + 2\mu_3 e_{\alpha\beta}g_{\alpha\beta} - 2\lambda_5 h_{[\alpha\beta]}h_{[\alpha\beta]}. \tag{2.14}$$

We suppose that $U$ is a positive definite quadratic form. This assumption leads to the following inequalities which must be satisfied by the material constants [4, 5]:

$$\begin{gathered} \lambda_1 + \frac{2}{3}\mu_1 - \frac{\varrho_2}{\varrho}\alpha_2 > 0, \quad \lambda_2 + \frac{2}{3}\mu_2 + \frac{\varrho_1}{\varrho}\alpha_2 > 0, \\ \left(\lambda_3 + \frac{2}{3}\mu_3 - \frac{\varrho_1}{\varrho}\alpha_2\right)^2 < \left(\lambda_1 + \frac{2}{3}\mu_1 - \frac{\varrho_2}{\varrho}\alpha_2\right)\left(\lambda_2 + \frac{2}{3}\mu_2 + \frac{\varrho_1}{\varrho}\alpha_2\right), \\ \mu_1 > 0, \quad \mu_2 > 0, \quad \mu_3^2 < \mu_1\mu_2, \quad \lambda_5 < 0. \end{gathered} \tag{2.15}$$

**Uniquenes**

Consider the principle of virtual work in the linear theory of isotropic mixtures of two elastic solids, in the following form [4, 9]:

$$\int_L (\sigma_{\alpha\beta}+\pi_{\alpha\beta})\omega_\alpha n_\beta dL+\int_\Sigma (F_\alpha\omega_\alpha+G_\alpha\eta_\alpha)d\Sigma = 2\int_\Sigma U d\Sigma. \tag{3.1}$$

Let $\omega_\alpha^{(\varrho)}$, $\eta_\alpha^{(\varrho)}$ be two solutions of the boundary value problems which have been considered, and let us denote

$$\tilde{\omega}_\alpha = \omega_\alpha^{(1)}-\omega_\alpha^{(2)}, \quad \tilde{\eta}_\alpha = \eta_\alpha^{(1)}-\eta_\alpha^{(2)}. \tag{3.2}$$

From (2.3) to (2.6) and (2.10) to (2.12), we have

$$\int_\Sigma \tilde{U} d\Sigma = 0, \tag{3.3}$$

where $\tilde{U}$ is the quadratic form $U$ corresponding to the system (3.2).

The positive definiteness of $\tilde{U}$ involves that

$$\tilde{e}_{\alpha\beta} = \tilde{g}_{\alpha\beta} = \tilde{h}_{[\alpha\beta]} = 0. \tag{3.4}$$

From (3.4) and (2.12) it follows that

$$\tilde{\omega}_\alpha = \tilde{\eta}_\alpha = a\varepsilon_{\alpha\beta 3}x_\beta+b_\alpha, \quad a, b_\alpha = \text{const}, \tag{3.5}$$

such that the second boundary value problem admits in the hypotheses (2.15) a unique solution determined to within an additive rigid — displacement of the form (3.5).

From (3.4) and (2.11) it results that

$$\tilde{\omega}_\alpha = \tilde{\eta}_\alpha = 0, \tag{3.6}$$

so that the first boundary value problem admits at the most one solution in the hypotheses (2.15).

#### 4. Existence

Consider the body subjected to two different systems of elastic loads:

$$\mathscr{T}^{(\varrho)} \equiv \{F_\alpha^{(\varrho)}, G_\alpha^{(\varrho)}, T_\alpha^{(\varrho)}, k_\alpha^{(\varrho)}\} \tag{4.1}$$

and let $\mathscr{C}^{(\varrho)}$ be two distinct elastic configurations of the body:

$$\mathscr{C}^{(\varrho)} \equiv \{\omega_\alpha^{(\varrho)}, \eta_\alpha^{(\varrho)}\}. \tag{4.2}$$

The reciprocity theorem given in [8] enables us to write

$$\int_L (\sigma_{\alpha\beta}^{(1)}+\pi_{\alpha\beta}^{(1)})\omega_\alpha^{(2)}\eta_\beta dL+\int_\Sigma (F_\alpha^{(1)}\omega_\alpha^{(2)}+G_\alpha^{(1)}\eta_\alpha^{(2)})d\Sigma = 2\int_\Sigma U_{12} d\Sigma, \tag{4.3}$$

where

$$2U_{12} = \mu_1 e_{\alpha\beta}^{(1)}e_{\alpha\beta}^{(2)}+\mu_3(e_{\alpha\beta}^{(1)}g_{\alpha\beta}^{(2)}+e_{\alpha\beta}^{(2)}g_{\alpha\beta}^{(1)})+\mu_2 g_{\alpha\beta}^{(1)}g_{\alpha\beta}^{(2)}-2\lambda_5 h_{[\alpha\beta]}^{(1)}h_{[\alpha\beta]}^{(2)}+ \\ +\left(\lambda_1-\frac{\varrho_2}{\varrho}\alpha_2\right)e_{\gamma\gamma}^{(1)}e_{\gamma\gamma}^{(2)}+\left(\lambda_3-\frac{\varrho_1}{\varrho}\alpha_2\right)(e_{\gamma\gamma}^{(1)}g_{\gamma\gamma}^{(2)}+e_{\gamma\gamma}^{(2)}g_{\gamma\gamma}^{(1)})+\left(\lambda_2+\frac{\varrho_1}{\varrho}\alpha_2\right)g_{\gamma\gamma}^{(1)}g_{\gamma\gamma}^{(2)}. \tag{4.4}$$

The notations

$$u \equiv (\omega_1^{(1)}, \omega_2^{(1)}, \eta_1^{(1)}, \eta_2^{(1)}), \quad v \equiv (\omega_1^{(2)}, \omega_2^{(2)}, \eta_1^{(2)}, \eta_2^{(2)}),$$

$$t(u) \equiv (T_1^{(1)}, T_2^{(1)}, 0, 0), \quad t(v) \equiv (T_1^{(2)}, T_2^{(2)}, 0, 0), \tag{4.5}$$

$$U_{12} \equiv U(u, v), \quad U \equiv U(u, u),$$

permit us to write the relation (4.2) in the form:

$$\int_L v\, t(u) dL + \int_\Sigma vAu d\Sigma = 2 \int_\Sigma U(u, v) d\Sigma, \tag{4.6}$$

where $A$ is the Lamé's operator corresponding to our problem.

We have [4, 8]

$$U(u, v) = U(v, u) \tag{4.7}$$

so that from (4.6) we can infer:

$$\int_\Sigma (vAu - uAv) d\Sigma = \int_L [u\, t(v) - v\, t(u)] dL, \tag{4.8}$$

and

$$\int_\Sigma uAu d\Sigma = -\int_L u\, t(u) dL + 2 \int_\Sigma U d\Sigma. \tag{4.9}$$

From now on we consider homogeneous boundary conditions and apply the theory developed in [10] in order to obtain existence theorems for the two considered boundary value problems.

In view of the considered boundary conditions, from (4.9), we have:

$$\int_\Sigma uAu d\Sigma = 2 \int_\Sigma U d\Sigma. \tag{4.10}$$

Let $\mathring{H}_1(\Sigma)$ denoting the Hilbert space obtained by completation of $\mathring{C}^1(\Sigma)$ with respect to the scalar product

$$(u, v) = \int_\Sigma D_u^\varkappa D_v^\varkappa d\Sigma, \quad 0 \leqslant |\varkappa| \leqslant 1. \tag{4.11}$$

We are ready now to prove the following.

Theorem 1. *If $F_\alpha, G_\alpha \in C^\infty(\overline{\Sigma})$ and* (2.15) *are satisfied, then there exists one and only one solution of the first boundary value problem which belongs to $C^\infty(\overline{\Sigma})$.*

P r o o f. It is sufficient to show that [10]

$$2 \int_\Sigma U d\Sigma \geqslant c \|u\|^2, \tag{4.12}$$

where $c$ is a positive constant and $\|\cdot\|$ is the norm in $\mathring{H}_1(\Sigma)$.

The relations (2.15) enable us to write

$$U \geqslant c_1 \sum_{\alpha,\beta=1}^{2} (e_{\alpha\beta}^2 + g_{\alpha\beta}^2 + h_{[\alpha\beta]}^2), \quad c_1 > 0. \tag{4.13}$$

Using the first Korn's inequality three times as follows:

(4.14) $$\int_{\Sigma}\sum_{\alpha,\beta=1}^{2} e_{\alpha\beta}^{2}d\Sigma \geqslant c_2\|u^{(1)}\|^2, \quad u^{(1)} = (\omega_1, \omega_2, 0, 0),$$

(4.15) $$\int_{\Sigma}\sum_{\alpha,\beta=1}^{2} g_{\alpha\beta}^{2}d\Sigma \geqslant c_3\|u^{(2)}\|^2, \quad u^{(2)} = (0, 0, \eta_1, \eta_2),$$

(4.16) $$\int_{\Sigma} h_{[12]}^{2}d\Sigma \geqslant c_4\|u^{(3)}\|^2, \quad u^{(3)} = (\eta_1-\omega_1, \omega_2-\eta_2, 0, 0),$$

where $c_2, c_3, c_4$ are positive constants we are lead to (4.12).

Let us prove now

THEOREM 2. *In the hypotheses* (2.15) *the second boundary value problem has solutions belonging to* $C^{\infty}(\bar{\Sigma})$ *if, and only if,* $F_{\alpha}, G_{\alpha} \in C^{\infty}(\bar{\Sigma})$ *and the conditions of total equilibrium of applied forces* (see also [4])

(4.17) $$\int_{\Sigma}(F_{\alpha}+G_{\alpha})d\Sigma = 0, \quad \int_{\Sigma}\varepsilon_{\alpha\beta 3}x_{\alpha}(F_{\beta}+G_{\beta})d\Sigma = 0,$$

*are satisfied.*

Proof. Following [10], we consider the system

(4.18) $$Au+k_0u = F, \quad F \equiv (F_1, F_2, G_1, G_2),$$

where $k_0$ is any positive constant, with the homogeneous boundary conditions (2.12).

The inequality

(4.19) $$2\int_{\Sigma} Ud\Sigma + \int_{\Sigma} u^2 d\Sigma \geqslant c_5\|u\|^2, \quad c_5 > 0,$$

assures the existence of one, and only one $C^{\infty}$ — solution in $\bar{\Sigma}$, for the boundary value problem given by (4.18) and (2.12).

Taking into account (4.13) and using three times the second Korn's inequality for $u^{(1)}$, $u^{(2)}$, $u^{(3)}$, defined in (4.14) to (4.16), respectively, we are lead to (4.19).

The $C^{\infty}$ — solutions of the boundary value problem given by

(4.20) $$Au+k_0u-\lambda u = F$$

and (2.12) exists if, and only if,

(4.21) $$\int_{\Sigma} F\tilde{u}\, d\Sigma = 0,$$

where $\tilde{u}$ is any $C^{\infty}$ — solution of the homogeneous boundary value problem (4.20) and (2.12).

In the case when $k_0 = \lambda$, $\tilde{u}$ is given by (3.5) only, so that the assertion of the theorem follows.

Remark. Taking into account the results in [1 and 2] we can use the same method as in [11] to obtain two systems of integral equations, corresponding to the first and the second boundary valueproblems considered, for which Fredholm's basic theorem is valid. The existence and uniqueness of solutions of these systems have been proved in the Sec. 3 and 4 of the present paper.

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