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A note on some crack problems in a variable modulus strip

C. ATKINSON (LONDON)

Two distinct problems are considered which have certain mathematical features in common. The first problem is that of a semi-infinite crack in a strip which has elastic moduli which vary in a direction perpendicular to the crack direction. The strip is subjected to a certain time dependent loading and a small time solution is obtained for a general variation in moduli. The second problem considered is a crack in a strip of a micropolar elastic solid with moduli which vary with position. In this case time-independent loading is considered and results for the variation of the energy release rate at the crack tip are obtained in terms of quite general variations in the moduli.

W pracy rozważane są dwa odrębne zagadnienia, lecz mające pewne wspólne cechy matematyczne. Problem pierwszy dotyczy półnieskończonej szczeliny w warstwie, której moduły sprężyste zmieniają się w kierunku prostopadłym do kierunku szczeliny. Warstwa poddana jest pewnemu obciążeniu zależnemu od czasu. Rozwiązanie dla krótkiego czasu zostało otrzymane dla ogólnej zmiany modułów. Drugim z rozważanych zagadnień jest szczelina umieszczona w sprężystym mikropolarnym ciele stałym z modułami zmieniającymi się wraz z położeniem. W tym przypadku przyjęto obciążenie niezależne od czasu, a wyniki dla prędkości zmiany energii uwolnionej na końcu szczeliny otrzymano dla całkiem ogólnych zmian modułów.

В работе рассмотрены две отдельные задачи, но имеющие некоторые общие математические свойства. Первая задача касается полубесконечной щели в слое, модули упругости которого изменяются в направлении перпендикулярном к направлению щели. Слой подвергнут некоторой нагрузке зависящей от времени. Решение для короткого отрезка времени получено для общего изменения модулей. Второй из рассматриваемых задач является щель помещена в упругом микрополярном твердом теле с модулями изменяющимися совместно с положением. В этом случае приняты нагрузки независящие от времени, а результаты для скорости изменения энергии освобожденной на конце щели получены для совсем общих изменений модулей.

1. Introduction

In this paper two problems are considered, both problems concerning cracks in strips. In the first problem there is a semi-infinite crack in an infinite strip of width $2h$ (Fig. 1). Conditions of plane-strain are assumed to exist so that the displacements do not vary in the X_3 direction. The strip consists of isotropic elastic material and the elastic moduli are functions of X_2 the crack lying on the $X_2 = 0$ plane. The problem when such a crack moves uniformly, the sides of the strip being loaded by a time-independent fixed displace-

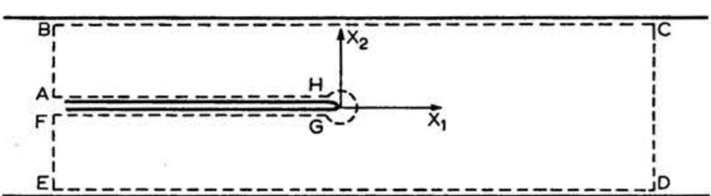

Fig. 1.

ment, has been considered in [1] where general formulae for the energy flow into the crack tip have been obtained. Here we consider the effect on a stationary crack of a *time-dependent* loading on the sides of the strip. Such a problem for a constant moduli elastic strip has been considered by NILSSON [2] by the use of a certain path independent integral It has been shown in [1] that a similar path independent integral can be derived for the variable moduli case and this is used in Sec. 2 to obtain the crack tip stress field for small times.

The second problem is considered for a strip of micropolar elastic material with a crack situated as in Fig. 1. This problem has been considered in [3] for time-independent loading of a constant moduli strip. In Sec. 3 we consider the situation when the moduli are functions of X_2.

2. A transient crack problem in a variable moduli elastic strip

Our approach to this problem is similar to that given in [2], so we consider only briefly the initial equations. We consider the integral

$$\overline{I} = \int\limits_{C} \left(\left[\overline{V} + \frac{1}{2}\varrho p^2 \overline{u}_i \overline{u}_i\right]\delta_{1j} - \overline{\sigma}_{ij}\frac{\partial \overline{u}_i}{\partial X_1}\right) dS_j, \tag{2.1}$$

where C is to be a contour embracing the crack tip. The bars denote Laplace functions transformed over time, p being the transform variable. $\overline{V}$ is defined as

$$\overline{V}(\overline{\varepsilon}_{ij}) = \frac{1}{2}\overline{\sigma}_{ij}\overline{\varepsilon}_{ij} \tag{2.2}$$

with the property that $\overline{\sigma}_{ij} = \dfrac{\partial \overline{V}}{\partial \overline{\varepsilon}_{ij}}$. σ_{ij} and ε_{ij} are the usual tensor components of stress and strain, u_i are the displacement components and X_i Cartesian coordinates. The density ϱ and the elastic moduli are considered to be functions of X_2 and the crack lies on the axis $X_2 = 0$, $X_1 < 0$. The integral $\overline{I}$ can be shown to be path independent (see [1]) when density and moduli vary only in the X_2 direction.

The Laplace transformed stress-strain relations can be written

$$\overline{\sigma}_{kk} = p\overline{G}_2\overline{\varepsilon}_{kk}, \quad \overline{\sigma}_{ij} - \frac{1}{3}\overline{\sigma}_{kk}\delta_{ij} = p\overline{G}_1\left(\overline{\varepsilon}_{ij} - \frac{1}{3}\overline{\varepsilon}_{kk}\delta_{ij}\right) \tag{2.3}$$

and

$$\overline{\varepsilon}_{ij} = \frac{1}{2}(\overline{u}_{i,j} + \overline{u}_{j,i}).$$

We have written equations (2.3) as they would apply to a linear isotropic viscoelastic material with moduli which are functions of X_2 the analysis of this section would apply to such media, although detailed attention will only be given to the elastic case. The Laplace transformed viscoelastic moduli are related to the usual elastic moduli by

$$\mu = \frac{p\overline{G}_1}{2}, \quad \nu = \frac{\overline{G}_2 - \overline{G}_1}{2\overline{G}_2 + \overline{G}_1}, \tag{2.4}$$

where μ is the shear modulus and ν Poisson's ratio. Near the crack tip with polar coordinates (r, θ) centered on the crack tip the variations of moduli and density will be functions of $X_2 = r\sin\theta$, so for small r these functions can be expanded by Taylor's theorem and the local results given in [2] should be valid for this case also with the moduli replaced by their values at $X_2 = 0$; hence

$$
\begin{aligned}
\bar{\sigma}_{ij} &= \bar{K}_1(p)(2\pi r)^{-1/2} f_{ij}(\theta) \qquad \text{as} \quad r \to 0, \\
\bar{u}_i &= \frac{\bar{K}_1(p)}{\mu_0}\left(\frac{2r}{\pi}\right)^{1/2} g_i(\theta, \nu_0) \qquad \text{as} \quad r \to 0,
\end{aligned} \tag{2.5}
$$

where the subscript zero denotes that μ and ν are evaluated at $X_2 = 0$. By a similar argument the integral $\bar{I}$ evaluated around a small contour enclosing the crack tip gives

$$
\begin{aligned}
\bar{I} &= \bar{K}_1^2(p)\frac{(\bar{G}_{20} + 2\bar{G}_{10})}{p\bar{G}_{10}(2\bar{G}_{20} + \bar{G}_{10})} \quad \text{plane strain}, \\
\bar{I} &= \frac{K_{\bar{1}11}^2(p)}{p\bar{G}_{10}} \quad \text{anti-plane strain},
\end{aligned} \tag{2.6}
$$

where the subscripts zero mean the same as above; in Eq. (2.5) f_{ij} and g_i are known functions and $\bar{K}_1(p)$ is to be determined.

Having obtained $\bar{I}$ when taken around a small contour enclosing the crack tip in terms of an unknown function $\bar{K}_1$, the approach is to deform the contour into a large contour in such a way that the path independence can be used to relate the near field integral (2.6) to a far field integral, which can be evaluated more easily. The chosen contour is shown in Fig. 1 as a dotted line.

For the strip problem we take the same boundary conditions as in [2], i.e.

$$
\begin{aligned}
&u_1 = 0, \quad u_2 = \pm u_0 q(t) \quad \text{on} \quad X_2 = \pm h, \\
&\sigma_{22} = \sigma_{12} = 0 \quad \text{on} \quad X_2 = 0, \quad X_1 \leqslant 0,
\end{aligned} \tag{2.7}
$$

where u_0 is a constant and $q(t)$ a dimensionless function of time which is zero for $t < 0$. The boundary conditions of $(2.7)_2$ are of course just those of a stress-free crack. Using these boundary conditions and taking the integral $\bar{I}$ around the contour shown in Fig. 1 we see that the only non-zero contributions are from the small contour around the tip [Eq. (2.6)] and from the vertical strips at $X_1 = \pm\infty$ which remain to be calculated. Along these vertical strips we assume the stress field to be so far removed from the crack tip that all derivatives with respect to X_1 are zero. The transformed equations of motion then simplify to

$$
\bar{\sigma}_{12,2} = \varrho p^2 \bar{u}_1, \quad \bar{\sigma}_{22,2} = \varrho p^2 \bar{u}_2 \tag{2.8}
$$

with the stress-strain relations becoming

$$
\bar{\sigma}_{12} = \frac{p\bar{G}_1}{2}\bar{u}_{1,2}, \quad \bar{\sigma}_{22} = \frac{(\bar{G}_2 + 2\bar{G}_1)}{3} p\bar{u}_{2,2} \tag{2.9}
$$

and when taken together these two sets of equations become

$$
\frac{d}{dX_2}\left(\bar{G}_1 \frac{d\bar{u}_1}{dX_2}\right) = 2\varrho p\bar{u}_1, \tag{2.10}
$$

(2.11)
$$\frac{d}{dX_2}\left(\frac{(\bar{G}_2+2\bar{G}_1)}{3}\frac{d\bar{u}_2}{dX_2}\right) = \varrho p\bar{u}_2.$$

The boundary conditions on the vertical strips on which these equations are valid are:

(2.12)
$$\text{At } X_1 = +\infty, \quad \bar{u}_1 = 0, \quad \bar{u}_2 = \pm u_0\bar{q}(p) \quad \text{on} \quad X_2 = \pm h,$$
$$\text{At } X_1 = -\infty \quad \bar{u}_1 = 0, \quad \bar{u}_2 = \pm u_0\bar{q}(p) \quad \text{on} \quad X_2 = \pm h,$$

(2.13)
$$\bar{u}_{1,2} = 0, \quad \bar{u}_{2,2} = 0 \quad \text{on} \quad X_2 = \pm 0.$$

The two sets of boundary conditions (2.13) are for the strip at $X_1 = -\infty$ separated by the crack, the derivative boundary conditions coming from the stress free crack conditions and the Eqs. (2.9). The boundary conditions and the Eq. (2.10) for $\bar{u}_1$ are satisfied identically for $\bar{u}_1 = 0$. The solution of (2.11), however, subject to the boundary conditions (2.12) or (2.13), is not so straightforward and so we consider solutions for small times, i.e. for large p.

2.1. Small time solutions

If in (2.11) we write

(2.14)
$$f(X_2) = \frac{(\bar{G}_2+2\bar{G}_1)p}{3}$$

and make the substitution

(2.15)
$$\phi = [f(X_2)]^{1/2}\bar{u}_2,$$

then (2.11) becomes

(2.16)
$$\frac{d^2\phi}{dX_2^2} = \phi\left[\frac{\varrho p^2}{f} + \frac{1}{2f^{1/2}}\frac{d}{dX_2}\left(\frac{f'}{f^{1/2}}\right)\right],$$

where the dash denotes differentiation with respect to X_2 and both ϱ and f are functions of X_2. For the elastic problems f is independent of p and is from (2.4),

(2.17)
$$f(X_2) = \frac{2(1-\nu)\mu}{(1-2\nu)}.$$

The solution of (2.16) for large p can be found by writing

(2.18)
$$\phi = \exp\left[p\sum_{N=0}^{\infty} p^{-N}g_N(X_2)\right],$$

then substituting in (2.16) and equating like powers of p. This procedure is standard and is outlined for example in NAYFEH [4]. Using a shorthand notation for the right-hand side of (2.16) as:

(2.19)
$$p^2\phi[q_0(X_2)+p^{-2}q_2(X_2)],$$

where

$$q_0(X_2) = \frac{\varrho}{f}; \quad q_2(X_2) = \frac{1}{2f^{1/2}}\frac{d}{dX_2}\left(\frac{f'}{f^{1/2}}\right),$$

one gets from the above procedure differential equations for the functions $g_N(X_2)$ which have the following solutions (apart from arbitrary constants)

$$
(2.20)\qquad
\begin{aligned}
g_0 &= \pm \int_0^{X_2} q_0^{1/2}\, dX_2, \quad g_1 = -\frac{1}{2}\log(g_0'),\\
g_2 &= \int_0^{X_2} \frac{[q_2-(g_1')^2-g_1'']}{2g_0'}\, dX_2 \quad \text{etc.}
\end{aligned}
$$

It remains to use these expressions to find solutions of (2.16) subject to the boundary conditions (2.12) and (2.13) and then to evaluate the integral $\bar{I}$ along the vertical strips at $X_2 = \pm\infty$. After some algebra we obtain for the case when p and f are symmetric functions of X_2 the expression

$$
(2.21)\qquad
\begin{aligned}
\bar{I} = u_0^2\bar{q}^2[\varrho(h)f(h)]^{1/2}[p(h)f(h)]^{1/2}\Bigg\{&\frac{2p}{\mathrm{sh}[2pg_0(h)]} + \frac{1}{2}\left[\left(\frac{\varrho'}{\varrho}+\frac{f'}{f}\right)\frac{1}{g_0'}\right]_0 \frac{1}{\mathrm{sh}^2 2pg_0(h)}\\
&-\frac{1}{2}\left[\left(\frac{\varrho'}{\varrho}+\frac{f'}{f}\right)\frac{1}{g_0'}\right]_h \frac{\mathrm{ch}\,2pg_0(h)}{\mathrm{sh}^2 2pg_0(h)}\\
&-\left[\int_0^h \frac{1}{8g_0'(x_2)}\left(\frac{\varrho'}{\varrho}+\frac{f'}{f}\right)^2_{x_2} dx_2\right]\frac{\mathrm{ch}\,2pg_0(h)}{\mathrm{sh}^2 2pg_0(h)} + \dots\Bigg\}.
\end{aligned}
$$

Higher order terms in the curly brackets will include terms like $\frac{1}{p}\exp(-2pg_0(h))$ for large p. Note that $\left(\frac{\varrho'}{\varrho}+\frac{f'}{f}\right)_h$ means that the expression inside the bracket is to be evaluated at $X_2 = h$ with a similar interpretation when the subscript h is replaced by zero.

Finally, substituting the above result into (2.6) gives a corresponding result for $K_1(p)$ and once $q(t)$ is specified $K_1(t)$ can be obtained using the Laplace inversion theorem. If following [2] we take as an example a linearly increasing time function

$$
(2.22)\qquad q(t) = \frac{t}{t_0}, \quad \text{hence} \quad \bar{q}(p) = \frac{1}{t_0p^2},
$$

where t_0 is a constant, the following result is obtained for $K_1(t)$;

$$
(2.23)\qquad
\begin{aligned}
K_1(t) = \left(\frac{2\mu(0)}{1-\nu(0)}\right)^{1/2} u_0\frac{[\varrho(h)f(h)]^{1/4}}{t_0}\frac{4}{\pi^{1/2}}\Bigg\{&(t-g_0(h))^{1/2}\\
&-\frac{1}{3}\left[\left(\frac{\varrho'}{\varrho}+\frac{f'}{f}\right)_h\frac{1}{4g_0'(h)} + \int_0^h \frac{1}{16g_0'}\left(\frac{\varrho'}{\varrho}+\frac{f'}{f}\right)^2 dx_2\right](t-g_0(h))^{3/2}\\
&+\frac{(t-3g_0(h))^{3/2}}{6g_0'(0)}\left(\frac{\varrho'}{\varrho}+\frac{f'}{f}\right)_0 + \dots\Bigg\}.
\end{aligned}
$$

In expression (2.23) it is understood that only real values of the square roots are permissible so that $K_1(t)$ is zero for $t < g_0(h)$ and from (2.20) $g_0(h) = \int_0^h \left(\frac{\varrho}{f}\right)^{1/2} dX_2$. When ϱ, ν, μ are constants the non-zero first term in (2.33) agrees with corresponding result in [2].

3. A crack in a variable moduli micropolar elastic strip

The micropolar theory of elasticity has been reviewed by ERINGEN [5] where the field equations are given (see [5] for earlier references to this theory). Perhaps this theory is the least complicated of the generalised theories of elasticity which include the effect of couple stresses; a comparison is made in [5] with the theory of couple stress elasticity and certain deficiencies in this latter theory pointed out. In a recent note [3] we have considered the problem of a crack in a micropolar elastic strip with constant moduli by using a path independent integral which serves to calculate the energy flow into the crack tip. For plane strain the following formula for the energy release rate is given in [3]:

$$G = \int_S [W\delta_{j1} - t_{jl}u_{l,1} - m_{j3}\phi_{3,1}]dS_j, \tag{3.1}$$

where i and j take the values 1 and 2, t_{jl} is the stress tensor in the notation of [5], m_{j3} are components of the couple stress tensor and ϕ_3 the component in the X_3 direction of the micropolar rotation vector. W the strain energy function can be written (see [5])

$$2W = \lambda\varepsilon_{kk}\varepsilon_{ll} + (\mu+\varkappa)\varepsilon_{kl}\varepsilon_{kl} + \mu\varepsilon_{kl}\varepsilon_{lk} + \gamma(\phi_{3,1}^2 + \phi_{3,2}^2), \tag{3.2}$$

k, l take the values 1 and 2, λ, μ, $\varkappa$ and γ are material constants which we assume are functions of X_2. The strain measures ε_{kl} are defined as

$$\varepsilon_{kl} = u_{l,k} + \varepsilon_{lk3}\phi_3, \tag{3.3}$$

where ε_{lk3} is the permutation tensor. The equations of equilibrium can be written

$$\frac{\partial t_{jl}}{\partial X_j} = 0 \tag{3.4}$$

and

$$\frac{\partial m_{j3}}{\partial X_j} - \varepsilon_{lk3}t_{kl} = 0, \tag{3.5}$$

i and j going from 1 to 2. To complete the field equations we need the constitutive equations which are for the linear theory (cf. [5])

$$t_{kl} = \lambda\varepsilon_{rr}\delta_{kl} + (\mu+\varkappa)\varepsilon_{kl} + \mu\varepsilon_{lk}, \qquad m_{k3} = \gamma\phi_{3,k}. \tag{3.6}$$

In the Appendix we derive these equations via a minimum principle based on the strain energy and show that the integral G can be derived in terms of a generalisation of the energy momentum tensor used by ESHELBY [6] for classical elasticity. In particular we show that G is a path independent integral even if the material constants vary in the X_2 direction the crack lying on $X_2 = 0$ as in Fig. 1.

We now consider the problem of a semi-infinite crack in a strip as shown in Fig. 1; the sides of the strip are now subjected to the following boundary conditions:

$$u_2 = \pm u_{20}, \quad u_1 = \pm u_{10}, \quad \phi_3 = 0 \quad \text{on} \quad X_2 = \pm h \quad \text{for all } X_1. \tag{3.7}$$

The calculation of the energy release at the crack tip is done most easily by relating the integral of G around a small contour at the crack tip to that evaluated by taking G around a large contour such as $ABCDEF$ of Fig. 1. The two integrals have the same value since the contributions to G from the path along the surface of the crack AH and FG are both zero from the stress free conditions of the crack. Because of the boundary conditions on the sides of the strip the contributions from BC and ED are also zero, so the only non-zero contributions come from the vertical strips CD and AB and EF. We now assume that these are removed to $X_1 = \pm\infty$ and that so far from the crack tip there are no variations in the displacement and microrotation fields in the X_1 direction. With this assumption the equations of equilibrium (3.4) and (3.5) along CD and BE simplify to

$$t_{21,2} = 0, \quad t_{22,2} = 0,$$

and

$$m_{23,2}+t_{12}-t_{21} = 0. \tag{3.8}$$

Further the strain measures (3.3) simplify to

$$\varepsilon_{11} = 0, \quad \varepsilon_{22} = u_{2,2}, \quad \varepsilon_{21} = u_{1,2}+\phi_3, \quad \varepsilon_{12} = -\phi_3. \tag{3.9}$$

From (3.8) t_{21} and t_{22} must be constant on the vertical strips so we write

$$t_{21} = B_\pm \text{ and } \quad t_{22} = C_\pm \quad \text{on} \quad X_1 = \pm\infty, \tag{3.10}$$

where $B_\pm$ and $C_\pm$ are constants.

Equations (3.6) together with (3.8), (3.9) and (3.10) give the equations

$$\begin{gathered}\frac{d}{dX_2}(\gamma\phi_{3,2})-\varkappa\left\{\frac{2\mu+\varkappa}{\mu+\varkappa}\right\}\phi_3 = \frac{\varkappa B_\pm}{(\mu+\varkappa)},\\ (\mu+\varkappa)u_{1,2} = -\varkappa\phi_3+B_\pm, \quad (\lambda+2\mu+\varkappa)u_{2,2} = C_\pm.\end{gathered} \tag{3.11}$$

The equations where the constants have + subscripts hold along CD and those with minus subscripts hold along BA and FE. The boundary conditions on the crack surface, which is stress free, are

$$t_{21} = 0 = t_{22} \quad \text{and} \quad m_{23} = 0 \quad \text{on} \quad X_2 = 0, \quad X_1 < 0 \tag{3.12}$$

which gives immediately $B_- = 0 = C_-$ from (3.10). A suitable solution to $(3.11)_1$ satisfying the boundary condition (3.7) is then $\phi_3 = 0$ and solutions of $(3.11)_2$ and $(3.11)_3$ subject to (3.7) are $u_1 = \pm u_{10}$ for $X_2 \gtrless 0$ and $u_2 = \pm u_{20}$ for $X_2 \gtrless 0$ when $X_1 = -\infty$. The solution of $(3.11)_1$ on the strip CD when only the boundary conditions (3.7) hold is not so simple when μ and $\varkappa$ both depend on X_2. We thus consider γ as a constant small parameter and look for solutions of (3.11) with small γ. This can be done in a similar way to that outlined for Eq. (2.11). As can be seen from Eq. (3.11) the equation for $u_{2,2}$ and hence t_{22} is independent of ϕ_3 and the contribution of u_{20} to the energy release rate will have a similar form to that derived in [1] for the inhomogeneous elastic strip, so in the following we put $u_{20} = 0$ and consider only the effect of applying a displacement $u_1 = \pm u_{10}$ on $X_2 = \pm h$. After some algebra one obtains the result to order $\gamma^{1/2}$

$$G = \frac{u_{10}^2}{2\int\limits_0^h \frac{dX_2}{(2\mu+\varkappa)}}\left\{1+\frac{\gamma^{1/2}}{2\int\limits_0^h \frac{dX_2}{(2\mu+\varkappa)}}\left[\frac{1}{(2\mu+\varkappa)}\left(\frac{\varkappa}{(\mu+\varkappa)(2\mu+\varkappa)}\right)^{1/2}\right]_h\right\}. \tag{3.13}$$

This agrees with the result for an inhomogeneous elastic strip which could be obtained by the method of [1] when $\gamma = 0$. Also when μ and $\varkappa$ are constants, (3.13) agrees to order

$\gamma^{1/2}$ with Eq. (4.2) of [3] which gives the corresponding result for an homogeneous micropolar strip. Note that the term in square-brackets in (3.13) is evaluated at $X_2 = h$ and we have assumed in the derivation of (3.13) that μ and $\varkappa$ are both even functions of X_2. The same approach should, of course, work for situations where this last assumption does not hold.

4. Concluding remarks

In this paper we have shown how certain problems involving media with variable elastic moduli can be reduced to the solution of ordinary differential equations such as (2.11) or (3.11) and thus that explicit solutions can be obtained for general variations in moduli for either small times (Sec. 2) or small variations from classical elasticity (Sec. 3). It should be noted in Sec. 3 that the parameter γ would usually be considered to be small in situations where the micropolar theory is considered a possible model (cf. [5]). It should also be noted that solutions for large times (p small) in Sec. 2 can also be obtained by perturbation methods and solutions for intermediate times could presumably be obtained efficiently by numerical solution of (2.11), whereas a direct attack on the problem, other than by using the approach of Sec. 2, would be much more complex.

Appendix

The energy-momentum tensor in micropolar elasticity

The results derived in this appendix will not be restricted to plane strain the relevant two-dimensional equations are given in Sec. 3. The argument below parallels that given by ESHELBY [6] for the homogeneous elastic case. Suppose there exists a strain-energy function

$$W = W(u_{i,k}, \phi_J, \phi_{J,s}), \tag{A.1}$$

where ϕ_J is the microrotation vector and a comma denotes partial differentiation with respect to X (a similar argument to that given below could be given for large strains; here we consider only infinitesimal strains). We assume that W has the property that

$$t_{lk} = \frac{\partial W}{\partial u_{k,l}}, \quad m_{lk} = \frac{\partial W}{\partial \phi_{k,l}} \quad \text{and} \quad \frac{\partial W}{\partial \phi_N} = \varepsilon_{lkN} t_{kl}. \tag{A.2}$$

The Euler equations for minimising the functional $\int W dv$ are then

$$\frac{\partial}{\partial X_j}\left(\frac{\partial W}{\partial u_{i,j}}\right) - \frac{\partial W}{\partial u_i} = 0 \quad \text{and} \quad \frac{\partial}{\partial X_j}\left(\frac{\partial W}{\partial \phi_{i,j}}\right) - \frac{\partial W}{\partial \phi_i} = 0, \tag{A.3}$$

where i, j, k, l take values from 1 to 3. Equations (A.2) together with (A.3) give the equations of equilibrium

$$\frac{\partial t_{jl}}{\partial X_j} = 0, \qquad \frac{\partial m_{jl}}{\partial X_j} - \varepsilon_{lkl} t_{kl} = 0. \tag{A.4}$$

The two-dimensional counterparts of Eq. (A.4) are given in (3.4) and (3.5).

If we now define an energy-momentum tensor

$$P_{jl} = W\delta_{jl} - \frac{\partial W}{\partial u_{i,j}} u_{i,l} - \frac{\partial W}{\partial \phi_{i,j}} \phi_{i,l}, \tag{A.5}$$

where i, j, l go from 1 to 3. It is not difficult to show by using (A.3) and some manipulation that

$$\frac{\partial P_{jl}}{\partial X_j} = \left(\frac{\partial W}{\partial X_l}\right)_{\exp}, \tag{A.6}$$

where we define

$$\left(\frac{\partial W}{\partial X_l}\right)_{\exp} = \frac{\partial W(u_{i,k}, \phi_j, \phi_{j,S}, X_m)}{\partial X_l}\Bigg|_{\substack{u_{i,k}, \phi_j, \phi_{j,S}, X_m \text{ constant}, \\ m \neq l}} \tag{A.7}$$

where we are now allowing W to depend explicitly on X_m.

In terms of the energy-momentum tensor the crack extension force or energy release rate can be written

$$G = \int_S P_{j1}\, dS_j, \tag{A.8}$$

where S is a surface enclosing the crack tip; in the plane strain case this surface is that of a cylinder with generators parallel to the X_3 axis so that the integral in (3.1) is effectively a line integral in the (X_1, X_2) plane. In (A.8) P_{j1} is used since the crack extension is assumed to take place in the X_1 direction. Applying the divergence theorem to (A.8) and using (A.6) we note that the integral in (A.8) will be path independent provided W does not depend explicitly on X_1.

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DEPARTMENT OF MATHEMATICS,
IMPERIAL COLLEGE, LONDON, S.W.7

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