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A general theory for floating ice plates

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THE NON-LINEAR basic equations which govern the motion of floating ice plates are developed from the three-dimensional theory of elasticity by means of a general expansion procedure. The result is a hierarchy of two-dimensional approximate equations. This theory is valid for non-uniform temperature distribution across the thickness of the ice and takes the effect of thermal stresses into account. A special simplified "generalized Reissner-von Kármán" theory is derived, which reduces to the classical plate theory when the usual simplified assumptions are made. For a significant temperature profile, the plate constants are determined and it is shown that for fresh water ice the influence of the temperature variation across the depth is negligible. For sea ice the influence of the temperature distribution is substantial. Explicit calculations, however, will be presented elsewhere.

Nieliniowe równania podstawowe rządzące ruchem pływających płyt lodowych zostały wyprowadzone z trójwymiarowej teorii sprężystości poprzez zastosowanie ogólnej procedury rozwinięcia. W rezultacie otrzymano przybliżone równania teorii dwuwymiarowej. Teoria ta słuszna jest dla niejednorodnego rozkładu temperatury wzdłuż grubości bryły lodowej oraz uwzględnia wpływ naprężeń termicznych. Wyprowadzona została szczególna uproszczona "uogólniona teoria Reissnera- von Kármána", która, po dokonaniu zwykle przyjmowanych założeń upraszczających, sprowadza się do klasycznej teorii płyt. Stałe występujące w problemie płyty określono dla znacznego zakresu temperatury. Wykazano również, że dla lodu powstałego ze świeżej wody wpływ zmiany temperatury wzdłuż grubości płyty jest pomijany. Dla lodu z wody morskiej wpływ rozkładu temperatury jest istotny. Konkretne obliczenia numeryczne będą przedmiotem innej publikacji.

Нелинейные основные уравнения описывающие движение плавающих ледяних плит выведены из трехмерной теории упругости путем применения общей процедуры разложения. В результате получены приближенные уравнения двумерной теории. Эта теория справедлива для неоднородного распределения температуры вдоль толщины ледяного тела, а также учитывает влияние термических напряжений. Выведена частная упрощенная "обобщенная теория Рейсснера-Кармана", которая, после проведения обычно принимаемых упрощающих предположений, сводится к классической теории плит. Постоянные выступающие в задаче плиты определены для значительного интервала температуры. Показано тоже, что для льда возникшего из пресной воды влиянием изменения температуры вдоль толщины плиты можно пренебречь. Для льда из морской воды влияние распределения температуры существенно. Конкретные численные расчеты будут предметом другой публикации.

Notation

IN THIS article symbolic and Cartesian tensor notation is used. Accordingly, Latin and Greek indices assume the values 1, 2, 3 and 1, 2, respectively. Einstein's summation convention is used according to which summation is understood over doubly repeated indices. Commas indicate differentiation with respect to a space variable, while dots denote total (material) time derivatives.

Subsequently a list of symbols is given.

C_{ij} right Cauchy-Green deformation tensor,

$C_{ij}^{(m)}$ right Cauchy-Green deformation tensor of order m,

$\mathscr{C}_{jikl}$ first-order elastic constants of three-dimensional anisotropic linear elasticity,

$\mathfrak{C}^{(m)}_{ijkl}$ first-order elastic constants of order m of two-dimensional anisotropic finite linear elasticity,
$\mathscr{D}^{(0)}, \mathscr{D}^{(1)}$ zeroth and first-order plate rigidities,
$\mathfrak{D}^{(0)}, \mathfrak{D}^{(1)}$ zeroth and first-order flexural rigidities,
E modulus of elasticity,
E_{ij} elongation tensor or Lagrangian strain tensor,
$E^{(m)}_{ij}$ Lagrangian strain tensor of order m,
$\tilde{E}_{ij} = u_{(i,j)} = \frac{1}{2}(u_{i,j}+u_{j,i})$,
F stress function,
F_{ij} deformation gradient,
f_i body force per unit mass,
$F^{(m)}_i$ body force of order m (per unit area),
h thickness of the plate,
$\mathfrak{H}$ constant in "macroscopic plane stress" situation,
$I^{(p)}$ moment of inertia of order p,
J det F_{ij}, Jacobian determinant of the motion $\chi(\cdot)$,
L reference length,
$\mathscr{M}, m$ mass densities,
$M^{(p)}$ averaged shear modulus of order p in isotropic finite linear elastic plates,
M_x, M_y bending moments,
M_{xy} twisting moments,
N_i external normal vector in the reference configuration,
$\mathscr{N}^{(p)}, \mathfrak{N}^{(p)}$ "generalized Poisson's ratios",
Q_i heat flux (Lagrangian),
Q_x, Q_y shear forces,
$\tilde{R}_{ij} = u_{[i,j]}$
$\mathscr{S}$ surface area, static moment,
$S^{(p)}_k$ surface load of order p,
t_i, t^*_i stress vector,
T_{ij} first Piola-Kirchhoff stress tensor,
u velocity in the x_1-direction,
u_i displacement vector,
$\ddot{U}^{(m)}_i$ acceleration vector of order m,
v velocity in the y-direction,
v_i velocity vector,
V volume,
w velocity component in z-direction,
X, Y particle label,
α_{ij} strain at constant stresses,
Γ averaged shear modulus,
δ_{ij} Kronecker delta, unit tensor, $\delta_{ij} = 1$, if $i = j$ and $\delta_{ij} = 0$, if $i \neq j$,
$\eta = u^{(0)}_3$ zeroth order displacement,
η entropy,
$\varphi = -u^{(1)}_1$ first-order displacement in the x-direction,
$\emptyset$ empty set,
$\psi = -u^{(1)}_2$ first-order displacement in the y-direction,
λ, μ Lamé's constants,
$\Lambda^{(p)}, M^{(p)}$ averaged Lamé's constants of order p,
ϑ absolute temperature,

$\Theta^{(m)}$, $\theta^{(m)}$ temperature resultants of order m,
Σ_{ij} second Piola-Kirchhoff stress tensor,
ω frequency,
ω_{ij}, $\omega\delta_{ij}$ thermal expansion coefficient.

1. Statement and motivation of the problem

In the past, the analyses of floating ice plates subjected to static and dynamic loads were based on the theory of thin homogeneous elastic plates, although in actual floating ice plates the material constants may vary strongly with depth. The reason for the variation of the material constants is chiefly due to a non-uniform distribution of the temperature with depth, but also due to a non-homogeneity induced by the freezing process.

With regard to the elastic behaviour, *A*. Assur has suggested that the elastic plate theory may be applied in floating ice plates provided the plate constant (Young's modulus) is replaced by a quantity which is averaged over the depth of the plate [1]. This conjecture has been substantiated by Kerr and Palmer [2], who on the basis of linear elasticity theory show that the usual plate theory emerges when the pertinent equations are averaged over the thickness of the plate. Kerr and Palmer's result, however, is based on the tacit assumption that Poisson's ratio does not vary with depth, an assumption which is certainly not correct for sea ice. Moreover, they do not take thermodynamic arguments into account. These, so we believe, are nontheless important, because the temperature varies from one material point to another.

A reinvestigation of the entire matter seems to be necessary, because, apart from the above reasons, various effects have not been studied in the past. First, it is not clear *a priori* that the non-uniformity of the temperature distribution can simply be interpreted as a nonhomogeneity of the material constants. In fact this is certainly not so in the theory of viscoelasticity. Second, dependent upon the local air temperature the corresponding temperature distribution in the plate will change and so do the local material constants and the averaged constants as well. Third, thermal stresses are induced through the temperature variation. These thermal stresses, in general, will cause deformations in the plane of the plate. A proper derivation thus should include thermal stresses. Fourth, the situation is not too rare where displacements are of the order of, or larger than, the thickness of the ice plate. It therefore also seems to be justified that the von Kármán plate theory is investigated with regard to its validity in the situation of floating ice plates.

So far, only the behaviour of the plates has been considered. An equally important part of the description of floating ice is the one of its underlying water, because it determines the boundary forces at the interface plate-water. While this interaction is generally only taken into account by assuming that the water pressure is proportional to the plate deflection, a moment thought shows that this is only correct in the static situation. Dynamic processes are much more complex. In particular, there are various physical situations for which a different set of differential equations applies. Generally, lake and sea ice require a different treatment of the fluid equations than ice on rivers, or ice interacting with tidal motion. We shall not list the equations for the fluid. They are contained in [3].

With regard to various physical aspects we do not assume that the ice plate is in an isothermal state, because the water-ice interface is at freezing temperature, while its upper surface depends on the air temperature and may vary quite considerably. The time scale of these thermal processes is much larger than the one of the wave motion, so that it seems justified to neglect true thermoelastic effects and to simply assume that the plate material constants are inhomogeneous. All material coefficients depend upon a parameter — the temperature — which in turn is a function of position.

Apart from this complication in the physical description of the ice plate other effects should also be included. Due to the thermal stress effects, prestress either as tension or pressure, may occur and a proper description should also take these effects into account. Moreover, the freezing process is such that various degrees of anisotropy are induced in the actual ice plate, which in general thus should not be treated as an isotropic material.

The technique to derive the governing equations of two-dimensional elasticity, which we shall use is one which has become increasingly fashionable in recent years. The plate equations shall be derived by a "smearing procedure" of the governing equations of three-dimensional elasticity. The method goes back to CAUCHY [4], but has increasingly been applied recently. MINDLIN was the first to derive the two-dimensional plate equations by a Cauchy series expansion from the equations of three-dimensional elasticity. MINDLIN and MEDICK [5] derived a linear theory of plates using expansion procedures. GOLDENVEIZER [6] discusses the possible aspect of such asymptotic or iterative methods. Similar methods are used by GREEN, LAWS and NAGHDI [7] and by DÖKMECI [8, 9], DÖKMECI and HUTTER [10], NIGUL [11], KALININ [12], KOITER [13] and WIDERA [14, 15] in other contexts of general plate and shell theories.

Finally, we mention two recent accounts. The first by KAUL [16] deals with thermal oscillations, the second by GREEN and NAGHDI [17] is concerned with the derivation of general theories of shells and rods. While the intentions of Green and Naghdi's memoir is different, Kaul's approach appears to be very similar to ours. He, however, assumes that the material constants do not vary with temperature, an assumption not made by us.

The major objective in the present paper is a consistent derivation of equations *similar* to those of von Kármán's plate equations. The conventional Kirchhoff-Love hypothesis, that is the assumption that directors perpendicular to the middle surface remain perpendicular under deformation, is abrogated. The theory is developed by means of the Hamiltonian principle and a separation of variables technique is used by which the three-dimensional field equations are converted into two-dimensional ones. A sequence of approximate equations which include the effects of transverse shear and normal strains, acceleration and rotatory inertia is thus consistently constructed. The governing equations consist of the macroscopic equations of motion together with the relevant boundary conditions, the constitutive equations, the strain displacement relations and the strain energy expression.

Specifically the paper is arranged as follows:

In Sec. 2 the kinematic variables are introduced, while in Sec. 3 the strain displacement relations are derived. In Sec. 4, after presenting the local balance equations, the Hamiltonian principle is derived, which will be used to derive the plate equations of motion. With the definitions of the load and stress resultants of Sec. 5 and the presentation of the constitutive equations in Sec. 6 we then possess the apparatus to derive the plate equations of motion in Sec. 7. These equations are then simplified in Sec. 8 where a *generalized* Reissner-von Kármán plate theory is derived. Sec. 9 finally deals with the numerical determination of the temperature dependent plate constants.

2. Kinematic variables

Consider an open regular region $\mathscr{D}$ with boundary $\partial\mathscr{D}$ in three-dimensional Euclidean space $\mathscr{E}^3$. Let $\partial\mathscr{D} = \partial\mathscr{D}_U \cup \partial\mathscr{D}_L \cup \partial\mathscr{D}_E$ where $\partial\mathscr{D}_U$, $\partial\mathscr{D}_L$ and $\partial\mathscr{D}_E$ are, respectively, the upper and lower faces and the edge boundary face. Clearly,

$$\mathscr{D} \cup \partial\mathscr{D} = \overline{\mathscr{D}}, \quad \partial\mathscr{D}_U \cap \partial\mathscr{D}_E = \emptyset,$$
$$\partial\mathscr{D}_U \cap \partial\mathscr{D}_L = \emptyset, \quad \partial\mathscr{D}_L \cap \partial\mathscr{D}_E = \emptyset,$$

where $\overline{\mathscr{D}}$ is the closure of $\mathscr{D}$. The edge boundary surface is taken as a cylindrical surface perpendicular to the flat middle surface of the undeformed plate. The plate in its reference

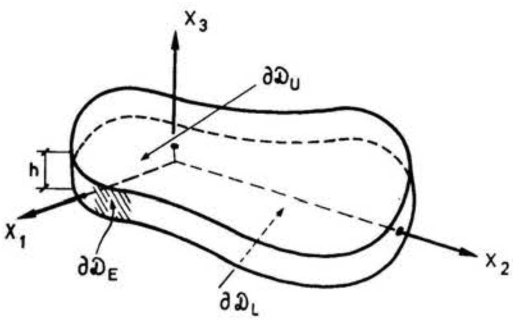

FIG. 1.

configuration is referred to a right-handed (convected) coordinate system x_i of which the first two axes lie in the undeformed reference surface. The positive direction of the third coordinate x_3 is taken to be upward and $x_3 = 0$ coincides with a plane between the upper and lower faces, the position of which will be determined in the course of calculations.

We assume the thickness h of the plate to be much smaller than any of its dimensions in the (x_1, x_2)-plane. Denoting the smallest value of the length dimension in the horizontal reference plane by L we thus have $h/L \ll 1$. This assumption allows us to treat the plate as a two-dimensional model of a three-dimensional deformable body. Moreover, it implies that stress and displacement fields do not vary violently across the thickness of the plate.

In the subsequent calculations we shall choose the Lagrangian description for the motion of the plate. All coordinates are consequently taken in the reference frame.

On the above basis the displacement components of a generic point in $\overline{\mathscr{D}}$ can be represented as

$$(2.1) \qquad u_i(x_k, t) = \sum_{m=0}^{\infty} P_m(x_3) u_i^{(m)}(x_\alpha, t), \quad i, k = 1, 2, 3, \quad \alpha = 1, 2$$

with

(2.2) $$P_0(x_3) = 1, \quad P_1(x_3) = x_3.$$

Here, the vector functions $u_i^{(m)}$ are unknown *a priori* and independent functions defined on $\overline{\mathscr{D}}$. Moreover, it is assumed that the functions $u_i^{(m)}$ exist, are single valued and of class C^2 at least.

In the final analysis only the two functions P_0 and P_1 will be used. They will be considered to be taken from the set $P_m(x_3) = x_3^m$. If the displacement vector u_i is analytic in $\overline{\mathscr{D}}$ with respect to the coordinate x_3 then (2.1) can be interpreted as a Taylor series expansion of u_i about $x_3 = 0$ which is uniformly convergent in $\overline{\mathscr{D}}$. However, in our case $u_i^{(m)}$ are independent. In fact, P_m could be any other convenient functions such as Legendre polynomials.

By virtue of the representation (2.2) the Kirchhoff-Love hypothesis is eliminated, i.e. directors perpendicular to the middle surface need not remain perpendicular to the deformed middle surface in the course of the motion.

3. Strain displacement relations

The right Cauchy-Green deformation tensor, denoted by C_{ij} is expressed in terms of the displacement components [18] by

(3.1) $$C_{ij} = u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} + \delta_{ij}$$

and

(3.2) $$C_{ij} = 2\tilde{E}_{ij} + [\tilde{E}_{ki}\tilde{R}_{kj} + \tilde{E}_{kj}\tilde{R}_{ki} + \tilde{E}_{ik}\tilde{E}_{jk} + \tilde{R}_{ki}\tilde{R}_{kj}] + \delta_{ij}$$

with

(3.3) $$\tilde{E}_{ij} = 1/2(u_{i,j} + u_{j,i}) = u_{(i,j)}, \quad \tilde{R}_{ij} = 1/2(u_{i,j} - u_{j,i}) = u_{[i,j]}.$$

Following TRUESDELL and TOUPIN [18] (footnote, p. 306), we do not neglect in (3.2) the first two terms in the bracket as was done by NOVOZHILOV [19].

The series expansions in all displacement components as assumed in (2.1) imply that$_\alpha$ the Cauchy-Green deformation tensor be of the following form:

(3.4) $$C_{ij} = \delta_{ij} + \sum_{m=0}^{\infty} x_3^m C_{ij}^{(m)}(x_\alpha, t),$$

where

(3.5) $$C_{ij}^{(m)} = u_{i,\alpha}^{(m)}\delta_{j\alpha} + u_{j,\alpha}^{(m)}\delta_{i\alpha} + (u_i^{(m+1)}\delta_{3j} + u_j^{(m+1)}\delta_{3i})(m+1)$$
$$+ \sum_{p=0}^{\infty} [u_{k,\alpha}^{(m-p)}u_{k,\beta}^{(p)}\delta_{i\alpha}\delta_{\beta j} + (p+1)u_{k,\alpha}^{(m-p)}u_k^{(p+1)}\delta_{i3}\delta_j$$
$$+ (p+1)u_{k,\alpha}^{(m-p)}u_k^{(p+1)}\delta_{i\alpha}\delta_{j3} + (p+1)(m-p+1)u_k^{(m-p+1)}u_k^{(p+1)}\delta_{i3}\delta_{j3}]$$

is a measure of the strain of order (m). Equivalently, $u_i^{(m)}$ is the displacement field of order (m).

The representation (3.1) is exact. A simpler approximate theory emerges when the Cauchy-Green deformation tensor is approximated by

$$C_{ij} = \delta_{ij}+u_{i,j}+u_{j,i}+u_{3\alpha}u_{3,\beta}\delta_{i\alpha}\delta_{j\beta}. \tag{3.6}$$

The representation (3.4) still remains correct, but now

$$C_{ij}^{(m)} = u_{i,\alpha}^{(m)}\delta_{j\alpha}+u_{j,\alpha}^{(m)}\delta_{i\alpha}+(m+1)u_i^{(m+1)}\delta_{3j}+(m+1)u_j^{(m+1)}\delta_{i3}+\sum_{p=0}^{m}u_{3,\alpha}^{(m-p)}u_{3,\beta}^{(p)}\delta_{i\alpha}\delta_{j\beta}. \tag{3.7}$$

In thin plates with moderately large displacements it is appropriate to retain only first-order terms and of the non-linearities of (3.7) only the zeroth order terms; then

$$\begin{aligned} &\frac{1}{2}C_{\alpha\beta}^{(0)} = E_{\alpha\beta}^{(0)} = u_{(\alpha,\beta)}^{(0)}+\frac{1}{2}\frac{\partial\eta}{\partial x_\alpha}\frac{\partial\eta}{\partial x_\beta};\ \frac{1}{2}C_{11}^{(1)} = E_{11}^{(1)} = -\frac{\partial\varphi}{\partial x_1},\\ &\frac{1}{2}C_{13}^{(0)} = E_{13}^{(0)} = \frac{1}{2}\left(\frac{\partial\eta}{\partial x_1}-\varphi\right),\quad \frac{1}{2}C_{22}^{(1)} = E_{22}^{(1)} = -\frac{\partial\psi}{\partial x_2},\\ &\frac{1}{2}C_{23}^{(0)} = E_{23}^{(0)} = \frac{1}{2}\left(\frac{\partial\eta}{\partial x_2}-\psi\right),\quad \frac{1}{2}C_{12}^{(1)} = E_{12}^{(1)} = \frac{1}{2}\left(\frac{\partial\psi}{\partial x_1}+\frac{\partial\varphi}{\partial x_2}\right), \end{aligned} \tag{3.8}$$

where

$$\varphi = -u_1^{(1)},\quad \psi = -u_2^{(1)},\quad \eta = u_3^{(0)},\quad C_{33}^{(0)} = u_3^{(1)}. \tag{3.9}$$

We shall call this approximation the "von Kármán approximation", because it corresponds to the one occuring in his plate theory [20].

4. The equations of balance

The dynamic equations of balance of linear momentum and energy may essentially be stated in two different forms, dependent upon whether they are referred to the reference or present configuration. Here, where we have chosen the reference configuration they assume the following form:

balance of linear momentum

$$\varrho_0\ddot{u}_i = T_{ik,k}+\varrho_0 f_i \tag{4.1}$$

balance of energy

$$\varrho_0\dot{\varepsilon} = T_{ik}\dot{F}_{ik}-Q_{k,k}+\varrho_0 r. \tag{4.2}$$

Apart from these equations one also has the balance of mass $\varrho_0 = \varrho J$ and the balance of moment of momentum $F_{ik}T_{jk} = F_{jk}T_{ik}$. ϱ_0 denotes the density in the reference configuration, while ϱ is the density in the present configuration. f_i and T_{ij} are the body force and Piola-Kirchhoff stress tensor, respectively, Q_k is the heat flux vector, r the energy supply and ε the internal energy density. Moreover,

$$F_{ik} = u_{i,k}+\delta_{ik},\quad J = \det(F_{ik}) \tag{4.3}$$

and

$$T_{ik} = F_{il}\Sigma_{kl} \tag{4.4}$$

where Σ_{ik} is the (symmetric) second Piola-Kirchhoff stress tensor adopting the definition of TRUESDELL and NOLL [21].

We shall henceforth neglect heat sources and restrict our considerations to processes for which $T_{ik}\dot{F}_{ik}$ is negligible. With appropriate boundary conditions, the energy equation then separates from the momentum equation. Moreover, the balance law of moment of momentum is satisfied identically once the constitutive equations are formulated. The only remaining field equation is then (4.1), because the balance law of mass can be considered to be an equation for ϱ.

Let t_i^* and u_i^* be the prescribed values of the stress and displacement vectors on the boundary surface. More precisely, let $\partial\mathscr{D}_u$ and $\partial\mathscr{D}_\sigma$ be disjoint sets of the boundary surface such that $\partial\mathscr{D} = \partial\mathscr{D}_u \cup \partial\mathscr{D}_\sigma$; then the boundary conditions can be written in the form

$$u_k^* - u_k = 0; \quad (x_1, x_2) \in \partial\mathscr{D}_u; \quad t_k^* - t_k = 0; \quad (x_1, x_2) \in \partial\mathscr{D}_\sigma \tag{4.5}$$

with

$$t_k = T_{kl}N_l, \tag{4.6}$$

where N_l is the normal vector in the reference configuration.

We proceed to formulate the variational principle. To this end, let t_1 and t_2 be two arbitrary but fixed times such that $t_2 > t_1$ and let δ indicate variation. Then it follows that

$$\delta\mathfrak{J} = \delta\mathfrak{J}_1 + \delta\mathfrak{J}_2 + \delta\mathfrak{J}_3 = 0,$$

where

$$\delta\mathfrak{J} = \int_{t_1}^{t_2} dt \int_{\mathscr{D}} [T_{ik,k} - \varrho_0(\ddot{u}_i - f_i)]\,\delta u_i\,dv + \int_{\partial\mathscr{D}_u} (u_k^* - u_k)\,\delta t_k\,dA + \int_{\partial\mathscr{D}_\sigma} (t_k^* - t_k)\,\delta u_k\,dA \tag{4.7}$$

is equivalent to the local equations (4.1) and (4.5). In fact, because the variations δt_k and δu_i are arbitrary, they can be varied independently, implying that the coefficients of δu_i and δt_k must vanish separately over the body $\mathscr{D}$ and the boundary $\partial\mathscr{D}_\sigma$ and $\partial\mathscr{D}_u$, respectively. Choosing $\delta t_k = 0$ and δu_k with compact support in $\bar{\mathscr{D}}$ implies

$$\delta\mathfrak{J}_1 = 0. \tag{4.8}$$

Similarly, one can show

$$\delta\mathfrak{J}_2 = 0 \quad \text{and} \quad \delta\mathfrak{J}_3 = 0. \tag{4.9}$$

The variational integrals will be used in the following sections to derive the macroscopic equations. We emphasize that its applicability is limited to the case when thermal effects are known *a priori* or when their change under the processes under investigation is insignificant. In particular, it cannot be used for the derivation of plate equations in thermoelasticity and other more complex theories.

5. Load and stress resultants

In order to facilitate notation in the subsequent analysis it is advantageous to introduce the following shorthand notation.

We define a *body force resultant* of order m by

$$F_i^{(m)} = \int_h \varrho_0 f_i x_3^m ds, \tag{5.1}$$

a *stress resultant*

$$T_{ij}^{(m)} = \int_h x_3^m \Sigma_{ij} ds \tag{5.2}$$

surface loads of order m

$$\begin{aligned} S_k^{(m)} &= [x_3^m \Sigma_{3k}]_1^u = x_3^m \Sigma_{3k}|_{\text{upper}} - x_3^m \Sigma_{3k}|_{\text{lower}}, \\ f_k^{*(m)} &= x_3^m t_k^*|_{\text{upper}} + x_3^m t_k^*|_{\text{lower}}. \end{aligned} \tag{5.3}$$

In the above relations ds is the line element and integration is over the height of the undeformed plate.

Note that the definitions (5.1) to (5.3) are more or less arbitrary. In particular, one could also define $F_i^{(m)}$, etc. by integrating over the height of the deformed body or one could define $T_{ij}^{(m)}$ in terms of T_{ij} rather than Σ_{ij}. This arbitrariness, however does not correspond to a non-uniqueness in the emerging theory. But it means that the averaged quantities $T_{ij}^{(m)}$, $S_i^{(m)}$ should by no means be given explicit physical meaning. Uniqueness is not required except for physically measurable quantities such as displacements, velocity and the like. A formulation must be unique only with respect to these definitely observable quantities. On the other hand, that (5.2) and (5.3) are the most convenient choices will be seen in Sec. 6.

Similarly in (5.1) to (5.3) we define an *acceleration resultant* by

$$\ddot{U}_i^{(m)} = \sum_{p=0}^{\infty} I^{(m+p)} \ddot{u}_i^{(p)}, \tag{5.4}$$

where the *m-th moment of inertia* is given by

$$I^{(m)} = \int_h x_3^m ds. \tag{5.5}$$

Moreover, the prescribed *stress resultant* of order m is given by

$$t_k^{*(m)} = \int_h x_3^m t_k^* ds. \tag{5.6}$$

Later, we shall also need two *temperature resultants* of order m which are given by $\theta^{(m)}$ and $\Theta^{(m)}$:

$$\theta^{(m)} = \int_h x_3^m \vartheta ds, \quad \vartheta = \sum_{n=0}^{\infty} x_3^n \Theta^{(n)}. \tag{5.7}$$

It seems appropriate to recall here that we do not assume that $I^{(1)} = 0$, which would imply that the plane $x_3 = 0$ would lie midway between the upper and lower surfaces.

Note further, that it is through $t_k^{*(m)}$ that an interaction with the sublaying water or the wind load is achieved. Denoting the Cauchy stress tensor in the neighbouring fluid by σ_{ij} it is readily shown that

$$t_j^* = +JF_{ik}\sigma_{ij}N_k. \tag{5.8}$$

6. Constitutive equations. Elastic material

Following the usual lines of argument, introducing the Clausius-Duhem inequality

$$\varrho_0\dot{\eta}-\left(\frac{Q_k}{\vartheta}\right)_{,k} \geqslant \varrho_0 \frac{r}{\vartheta} \tag{6.1}$$

it can be shown that

$$\Sigma_{ij} = \frac{\partial \mathfrak{P}}{\partial E_{ij}}, \quad \eta = -\frac{\partial \mathfrak{P}}{\partial \vartheta}. \tag{6.2}$$

Thus, according to (5.2) we have

$$T_{ij}^{(m)} = \int_h x_3^{(m)} \frac{\partial \mathfrak{P}}{\partial E_{ij}} ds. \tag{6.3}$$

Let $\vartheta' = \vartheta - \vartheta_0$, where ϑ_0 is a reference temperature. The polynomial representation

$$\mathfrak{P} = \frac{1}{2}\mathscr{C}_{ijkl}(E_{ij}-\vartheta'\omega_{ij})(E_{kl}-\vartheta'\omega_{kl}) \tag{6.4}$$

neglects real thermoelasticity effects but it produces the constitutive law

$$\Sigma_{ij} = \mathscr{C}_{ijkl}(\vartheta)(E_{kl}-\vartheta'\omega_{kl}), \tag{6.5}$$

which is *linear* in the elongation tensor E_{ij} and accounts for thermal stresses. Note, however, that E_{ij} is still the Lagrangian strain of finite elasticity. In the above equations $\mathscr{C}_{ijkl}$ and ω_{ij} are the isothermal first-order elasticity coefficients and thermal expansion coefficients, respectively. They satisfy the symmetry relations

$$\mathscr{C}_{ijkl} = \mathscr{C}_{jikl} = \mathscr{C}_{klij}, \quad \omega_{ij} = \omega_{ji} \tag{6.6}$$

of hyperelasticity. In the case of isotropic material they reduce to

$$\mathscr{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \quad \omega_{ij} = \omega\delta_{ij}, \tag{6.7}$$

where ω is the coefficient of linear thermal expansion and λ and μ are Lame's constants. One usually assumes

$$\mu > 0, \quad 3\lambda+2\mu > 0. \tag{6.8}$$

With $\mathfrak{P}(E_{ij}, \vartheta)$ being an unspecified non-linear function of the elongation tensor and the temperature [see (6.3)], not much simplification can be achieved when calculating the stress resultants. When $\mathfrak{P}(\cdot)$ is a polynomial, then a reduction is possible. In particular, the macroscopic constitutive equations in the linear form are obtained by substituting (6.5) into (5.2):

$$T_{ij}^{(m)} = \int_h x_3^m \mathscr{C}_{ijkl}(\vartheta')(E_{kl}-\vartheta'\omega_{kl})ds. \tag{6.9}$$

In floating ice plates it is not justified to assume that the elastic constants be temperature independent. Thus

$$(6.10)\qquad T_{ij}^{(m)} = \sum_{p=0}^{\infty} \mathfrak{C}_{ijkl}^{(m+p)}[E_{kl}^{(p)} - \omega_{kl}\Theta^{(p)}],$$

with

$$(6.11)^{(1)}\qquad \mathfrak{C}_{ijkl}^{(m)} = \int_h x_3^m \mathscr{C}_{ijkl}(\vartheta)\,ds.$$

In the isotropic case

$$(6.12)\qquad T_{ij}^{(m)} = \sum_{p=0}^{\infty} \{\Lambda^{(m+p)}(E_{kk}^{(p)} - 3\omega\Theta^{(p)})\,\delta_{ij} + 2M^{(m+p)}(E_{ij}^{(p)} - \omega\Theta^{(p)}\delta_{ij})\}$$

with

$$(6.13)\qquad \Lambda^{(m)} = \int_h x_3^m \lambda(\vartheta)\,ds, \qquad M^{(m)} = \int_h x_3^m \mu(\vartheta)\,ds.$$

It is this law we shall use in the linearized theory of deflection of ice plates. Finally, from (6.10) it follows that

$$(6.14)\qquad T_{ij}^{(m)} = \frac{1}{2}\left[\frac{\partial\overline{\mathfrak{P}}}{\partial E_{ij}^{(m)}} + \frac{\partial\overline{\mathfrak{P}}}{\partial E_{ji}^{(m)}}\right],$$

where

$$(6.15)\qquad \overline{\mathfrak{P}} = \frac{1}{2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \mathfrak{C}_{ijkl}^{(p+q)}[E_{ij}^{(p)} - \omega_{ij}\Theta^{(p)}]\,[E_{kl}^{(q)} - \omega_{kl}\Theta^{(q)}].$$

A different representation of $T_{ij}^{(m)}$ is obtained when $(5.7)_1$ is used instead of $(5.7)_2$. (6.12) then becomes

$$(6.16)\qquad T_{ij}^{(m)} = \sum_{p=0}^{\infty} \mathfrak{C}_{ijkl}^{(m+p)} E_{kl}^{(p)} - \alpha_{ij}^{(m)}.$$

where

$$(6.17)\qquad \alpha_{ij}^{(m)} = \int_h x_3^m \mathscr{C}_{ijkl}(\vartheta)\,\omega_{kl}(\vartheta)\,\vartheta\,ds.$$

This representation lacks the symmetry properties enjoyed by (6.13) but may be advantageous in special approximations.

7. The plate equations of motion

We now proceed to develop the non-linear field equations of plates in terms of the displacement components. In this Section our starting equation is the variational principle (4.7).

(1) We have only made a ϑ-dependence of $\mathscr{C}_{ijkl}$ explicit. A correct notation would be $\mathscr{C}_{ijkl}(x_3, \vartheta(x_3))$.

To begin with, consider the contribution from the volume integral

$$\delta\mathfrak{I}_1 = \int\limits_{t_1}^{t_2} dt \Big[\int\limits_{\mathscr{D}} [T_{ij,j} - \varrho_0(\ddot{u}_i - f_i)]\,\delta u_i\Big] dv \tag{7.1}$$

$$= \int\limits_{t_1}^{t_2} dt \Big[\int\limits_{\mathscr{S}} dA \int\limits_{h} [T_{ij,j} - \varrho_0(\ddot{u}_i - F_i)]\,\delta u_i\Big]\, ds.$$

Substituting for $u_i b$ and correspondingly for δu_i the representation (2.1), using (4.3), (4.4) and (5.1) and (5.2), we obtain

$$\delta\mathfrak{I}_1 = \int\limits_{t_1}^{t_2} dt \int\limits_{\mathscr{S}} dA \Big\{\sum_{m=0}^{\infty} [-\varrho_0\, \ddot{U}_i^{(m)} + F_i^{(m)} + P_i^{(m)} + N_i^{(m)}\Big\}\,\delta u_i^{(m)}, \tag{7.2}$$

where

$$N_i^{(m)} = T_{\beta i,\beta}^{(m)} - mT_{3i}^{(m-1)} + \sum_{p=0}^{\infty} \{T_{\beta\alpha}^{(m+p)} u_{i,\alpha\beta}^{(p)} + pT_{\beta 3}^{(m+p-1)} u_{i,\beta}^{(p)} + T_{\beta,\alpha}^{(m+p)} u_{i,\beta}^{(p)} \tag{7.3}$$

$$+ pT_{\beta 3,\beta}^{(m+p-1)} u_i^{(p)} - mT_{3\alpha}^{(m+p-1)} u_{i,\alpha}^{(p)} - mpT_{33}^{(m+p-2)} u_i^{(p)}\}$$

and

$$P_i^{(m)} = \sum_{p=0}^{\infty} \{S_i^{(m)} + [S_\alpha^{(m+p)} u_{i,\alpha}^{(m)} + pS_3^{(m+p-1)} u_i^{(p)}]\}. \tag{7.4}$$

Note that it is $P_i^{(m)}$ which contains any possible interaction with water and/or wind load.

Further approximation is achieved by neglecting special terms. If only zeroth and first-order terms are retained and of the non-linearities only products with $u_3^{(0)}$, then (7.3) and (7.4) are approximated by

$$\begin{aligned} N_i^{(0)} &= T_{\beta i,\beta}^{(0)} + (T_{\beta\alpha}^{(0)} u_{3,\alpha\beta}^{(0)} + T_{\alpha\beta,\alpha}^{(0)} u_{3,\beta}^{(0)})\,\delta_{i3}, \\ N_i^{(1)} &= T_{\beta i,\beta}^{(1)}, \\ P_i^{(0)} &= S_i^{(0)} + S_\alpha^{(0)} u_{3,\alpha}^{(0)}\,\delta_{i3}, \\ P_i^{(1)} &= S_i^{(1)}, \end{aligned} \tag{7.5}$$

or

$$\begin{aligned} N_\gamma^{(0)} &= T_{\beta\gamma,\beta}^{(0)}, \\ N_3^{(0)} &= T_{\beta 3,\beta}^{(0)} + (T_{\beta\alpha}^{(0)} u_{3,\alpha\beta}^{(0)} + T_{\alpha\beta,\alpha}^{(0)} u_{3,\beta}^{(0)}), \\ N_i^{(1)} &= T_{\beta i,\beta}^{(1)}, \\ P_\gamma^{(0)} &= S_\gamma^{(0)}, \\ P_3^{(0)} &= S_3^{(0)} + S_\alpha^{(0)} u_{3,\alpha}^{(0)}, \\ P_i^{(1)} &= S_i^{(1)}. \end{aligned} \tag{7.6}$$

The surface integrals in (3.20) are

$$\delta\mathfrak{I}_2 = \int\limits_{t_1}^{t_2} dt \int\limits_{\partial\mathscr{D}_d} (u_k^* - u_k)\,\delta t_k\, dA \tag{7.7}$$

and

$$\delta\mathfrak{J}_3 = \int_{t_1}^{t_2} dt\Big\{\int_{\partial\mathcal{D}_\sigma} (t_k^* - t_k)\,\delta u_k dA + \int_{\partial\mathcal{D}_t} (t_k^* - t_k)\,\delta u_k dA\Big\}. \tag{7.8}$$

Here, $\partial\mathcal{D}_d$ is that part of the surface where the displacement is prescribed. $\partial\mathcal{D}_\sigma$ is the surface portion on $\partial\mathcal{D}_L$ or $\partial\mathcal{D}_U$, respectively, where the traction is prescribed, while $\partial\mathcal{D}_t$ is that portion of $\partial\mathcal{D}_E$ where the stress is prescribed.

Using (4.3), (4.4), (5.2) and (5.3), performing the integrations in (7.7) and (7.8), we obtain

$$\delta\mathfrak{J}_2 = \int_{t_1}^{t_2} dt \int_{\partial\mathcal{D}_d} \sum_{m=0}^{\infty} x_3^m (u_k^{*(m)} - u_k^{(m)})\,\delta t_k dA, \tag{7.9}$$

$$\delta\mathfrak{J}_3^{(1)} = \int_{t_1}^{t_2} dt \sum_{m=0}^{\infty} \int_{\partial\mathcal{D}_\sigma} \Big\{f_k^{*(m)} - S_k^{(m)} - \sum_{p=0}^{\infty} [S_\alpha^{(m+p)} u_{k,\alpha}^{(p)} + pS_3^{(m+p-1)} u_k^{(p)}]\,\delta u_k^{(m)}\Big\} dA, \tag{7.10}$$

$$\delta\mathfrak{J}_3^{(2)} = \int_{t_1}^{t_2} dt \sum_{m=0}^{\infty} \oint_{\mathscr{C}_t} \Big\{t_k^{*(m)} - T_{\alpha k}^{(m)} N_\alpha - \sum_{p=0}^{\infty} [T_{\alpha\beta}^{(m+p)} N_\alpha u_{k,\beta}^{(p)} + pT_{\alpha 3}^{(m+p-1)} N_\alpha u_k^{(p)}]\delta u_k^{(m)}\Big\} ds.$$

With (7.1), (7.9) and (7.10) we are now in position to apply the Hamilton principle. On setting separately each variation of the functionals $\mathfrak{J}_k$ equal to zero, *viz.*

$$\delta\mathfrak{J}_1 = \delta\mathfrak{J}_2 = \delta\mathfrak{J}_3^{(1)} = \delta\mathfrak{J}_3^{(2)} = 0 \tag{7.11}$$

for arbitrary variations of the displacement and traction vector $\delta u_i^{(m)}$ and $\delta t_k^{(m)}$, the following hierarchy of boundary value problems is obtained;

$$\begin{aligned}
&\varrho_0 \dot{U}_i^{(m)} = F_i^{(m)} + N_i^{(m)} + P_i^{(m)}, \qquad (x_1, x_2) \in \mathscr{S},\\
&u_k^{*(m)} - u_k^{(m)} = 0, \qquad (x_1, x_2) \in \partial\mathcal{D}_d,\\
&[F_k^{*(m)} = S_k^{(m)} + \sum_{p=0}^{\infty} [S_\alpha^{(m+p)} u_{k,\alpha}^{(p)} + pS_3^{(m+p-1)} u_k^{(p)}], \qquad (x_1, x_2) \in \partial\mathcal{D}_\sigma,\\
&t_k^{*(m)} = T_{\alpha k}^{(m)} N_\alpha + \sum_{n=0}^{\infty} [T_{\alpha\beta}^{(m+n)} N_\alpha u_{k,\beta}^{(n)} + nT_{\alpha 3}^{(m+n-1)} N_\alpha u_k^{(n)}], \qquad (x_1, x_2) \in \partial\mathcal{D}_t.
\end{aligned} \tag{7.12}$$

These equations will henceforth be called the macroscopic equations of motion and boundary conditions of order m.

So far, a fully non-linear plate theory has been established. It consists of a one parameter family of differential equations and boundary conditions (7.12). The stress resultants $T_{ij}^{(m)}$ are given in terms of the elongation tensor E_{ij} in (6.10) or (6.16), the latter being defined in Sec. 3 in terms of the strains. Note that for a well-set initial value problem the above equations must be complemented by initial conditions. Let u_0 and $\dot{u}_0$ be the displacement and velocity field prescribed at time t_0. Then, by a Taylor series expansion

$$\begin{aligned}
u_0 &= \sum_{m=0}^{\infty} x_3^m \frac{\partial^m u_0}{m!\,\partial x_3^m}\bigg|_{x_3=0} = \sum_{m=0}^{\infty} x_3^m u_0^{(m)},\\
\dot{u}_0 &= \sum_{m=}^{\infty} x_3^m \frac{\partial^m \dot{u}_0}{m!\,\partial x_3^m}\bigg|_{x_3=0} = \sum_{m=0}^{\infty} x_3^m \dot{u}_0^{(m)},
\end{aligned} \tag{7.13}$$

one obtains the initial values for the $u_0^{(m)}$ and $\dot{u}_0^{(m)}$.

Clearly, the set of equations introduced above is complex. In fact it forms an infinity of equations and in this form cannot be used for practical analysis. We must search for a consistent reduction of the equations by truncating the series. Thus the plate theory of order M is defined by

$$u_i(x_k, t) = \sum_{m=0}^{M} P_m(x_3) u_i^{(m)}(x_\alpha, t) \tag{7.14}$$

together with the condition

$$u_k^{(m)} = 0, \quad \text{for all } m > M. \tag{7.15}$$

Accordingly, only those quantities in (7.14) are considered the order of which is not greater than M. This results in a finite set of non-linear partial differential equations and corresponding boundary conditions.

8. Further simplifications

The general theory which characterizes the non-linear behaviour of plates has been formulated in the preceding Sections. This theory is still very complex and calls for further simplifications. Such various simplifications are possible, depending upon the degree of further neglections. Here we derive a "generalized Reissner-von Kármán theory".

To begin with recall that the macroscopic constants $\Lambda^{(i)}$ and $M^{(i)}$ for plates isotropic in the (x_1, x_2)-plane are known, once the temperature variation across the plate thickness is prescribed. Note further that we have not fixed the (x_1, x_2)-plane yet. This will be done now with the condition

$$\Lambda^{(1)} = \int_h x_3 \lambda(\vartheta) ds = 0. \tag{8.1}$$

We assume henceforth that the coordinates (x_1, x_2, x_3) are chosen accordingly. To be precise, however, one must mention that the normalization (8.1) is only useful provided the temperature distribution is such that the surface $x_3 = 0$ is flat. This implies that λ does not vary within the (x_1, x_2)-plane.

Satisfying (8.1) does not, in general, imply

$$M^{(1)} = \int_h x_3 \mu(\vartheta) ds = 0, \tag{8.2}$$

but if it does, then $\mu(\vartheta) = K\lambda(\vartheta)$ and this in turn implies that Poisson's ratio must be independent of the temperature. We shall not assume it in the sequel, because we are interested in its significance.

The theory presented here can be considered to be an order "1" theory. Accordingly we shall keep 0(1)-terms, while neglecting all higher-order terms. With regard to the strain displacement relations we shall keep the non-linear terms which correspond to the von Kármán assumption. The theory presented here may therefore be called a physically linear but geometrically partially non-linear theory. We also assume isotropy(2), so that the stress

(2) More precisely we assume that the macroscopic equations are isotropic. This only requires the equations of three-dimensional elasticity to be orthotropic such that material properties in the plane of the plate and perpendicular to it are different.

strain relations (6.12) can be used. These formulas are used to calculate the zeroth-order stress resultants:

$$
\begin{aligned}
&T_{13}^{(0)} = Q_x = 2M^{(0)}E_{13}^{(0)}+2M^{(1)}E_{13}^{(1)},\\
&T_{23}^{(0)} = Q_y = 2M^{(0)}E_{23}^{(0)}+2M^{(1)}E_{23}^{(1)},\\
(8.3)\quad &T_{11}^{(0)} = N_x = \Lambda^{(0)}E_{kk}^{(0)}+2M^{(0)}E_{11}^{(0)}-\omega(3\Lambda^{(0)}+2M^{(0)})\Theta^{(0)}+2M^{(1)}(E_{11}^{(1)}-\omega\Theta^{(1)}),\\
&T_{22}^{(0)} = N_y = \Lambda^{(0)}E_{kk}^{(0)}+2M^{(0)}E_{11}^{(0)}-\omega(3\Lambda^{(0)}+2M^{(0)})\Theta^{(0)}+2M^{(1)}(E_{22}^{(1)}-\omega\Theta^{(1)}),\\
&T_{12}^{(0)} = N_{xy} = 2M^{(0)}E_{12}^{(0)}+2M^{(1)}E_{12}^{(1)},
\end{aligned}
$$

which we shall assume to be nonzero. The resultant $T_{33}^{(0)}$, however, is assumed to be zero. Thus,

$$
(8.4)\quad T_{33}^{(0)} = \Lambda^{(0)}E_{kk}^{(0)}+2M^{(0)}E_{33}^{(0)}-(3\omega\Lambda^{(0)}+2M^{(0)})\Theta^{(0)}+2M^{(1)}(E_{33}^{(1)}-\omega\Theta^{(1)}) = 0.
$$

As far as first-order stress resultants are concerned it seems appropriate to assume that $T_{11}^{(1)}$, $T_{22}^{(1)}$ and $T_{12}^{(1)}$ are nonzero, while all other first-order stress resultants do vanish, *viz*:

$$
\begin{aligned}
&T_{11}^{(1)} = M_x = 2M^{(1)}E_{11}^{(0)}+\Lambda^{(2)}E_{kk}^{(1)}+2M^{(2)}E_{11}^{(1)}\\
&\qquad\qquad -[2M^{(1)}\omega\Theta^{(0)}+3\Lambda^{(2)}\omega\Theta^{(1)}+2M^{(2)}\omega\Theta^{(1)}],\\
(8.5)\quad &T_{22}^{(1)} = M_y = 2M^{(1)}E_{22}^{(0)}+\Lambda^{(2)}E_{kk}^{(1)}+2M^{(2)}E_{22}^{(1)}\\
&\qquad\qquad -[2M^{(1)}\omega\Theta^{(0)}+3\Lambda^{(2)}\omega\Theta^{(1)}+2M^{(2)}\omega\Theta^{(1)}],\\
&T_{12}^{(1)} = -M_{xy} = 2M^{(1)}E_{12}^{(0)}+2M^{(2)}E_{12}^{(1)}
\end{aligned}
$$

and

$$
\begin{aligned}
&T_{13}^{(1)} = 2M^{(1)}E_{13}^{(0)}+2M^{(2)}E_{13}^{(1)} = 0,\\
(8.6)\quad &T_{23}^{(1)} = 2M^{(1)}E_{23}^{(0)}+2M^{(2)}E_{23}^{(1)} = 0,\\
&T_{33}^{(1)} = \Lambda^{(2)}(E_{kk}^{(1)}-3\omega\Theta^{(1)})+2M^{(2)}(E_{33}^{(1)}-\omega\Theta^{(1)})+2M^{(1)}(E_{33}-\omega\Theta^{(0)}) = 0.
\end{aligned}
$$

It follows from (8.6) that some zeroth and first-order strains are not independent. In particular (8.6) imply that

$$
(8.7)\qquad E_{13}^{(1)} = \frac{M^{(1)}}{M^{(2)}}E_{13}^{(0)},\qquad E_{23}^{(1)} = \frac{M^{(1)}}{M^{(2)}}E_{23}^{(0)},
$$

while (8.4) and $(8.6)_3$ lead to

$$
(8.8)\qquad
\begin{aligned}
E_{33}^{(0)} &= A^{(0)}E_{\alpha\alpha}^{(0)}+B^{(0)}E_{\alpha\alpha}^{(1)}+C^{(0)},\\
E_{33}^{(1)} &= A^{(1)}E_{\alpha\alpha}^{0))}+B^{(1)}E_{\alpha\alpha}^{(1)}+C^{(1)},
\end{aligned}
$$

with

$$
(8.9)\qquad A^{(0)} = -\Lambda^{(0)}(\Lambda^{(2)}+2M^{(2)})/\Delta,\qquad B^{(0)} = 2\Lambda^{(2)}M^{(1)}/\Delta,
$$

$$
\begin{aligned}
(8.10)\quad C^{(0)} = \omega[&(\Lambda^{(2)}+2M^{(2)})(3\Lambda^{(0)}\Theta^{(0)}+2M^{(0)}\Theta^{(0)}+2M^{(1)}\Theta^{(1)}\\
&-2M^{(1)}(3\Lambda^{(2)}\Theta^{(1)}+2M^{(2)}\Theta^{(1)}+2M^{(1)}\Theta^{(0)})]/\Delta,
\end{aligned}
$$

and

$$
\begin{aligned}
&A^{(1)} = 2\Lambda^{(0)}M^{(1)}/\Delta,\qquad B^{(1)} = -\Lambda^{(2)}(\Lambda^{(0)}+2M^{(0)})/\Delta,\\
&C^{(1)} = \omega[(\Lambda^{(0)}+2M^{(0)})(3\Lambda^{(2)}\Theta^{(1)}+2M^{(2)}\Theta^{(1)}+2M^{(1)}\Theta^{(0)})\\
&\qquad\qquad -2M^{(1)}(3\Lambda^{(0)}\Theta^{(0)}+2M^{(0)}\Theta^{(0)}+2M^{(1)}\Theta^{(1)}]/\Delta,
\end{aligned}
$$

where

$$\Delta = [\Lambda^{(0)}+2M^{(0)}][\Lambda^{(2)}+2M^{(2)}]-4(M^{(1)})^2.$$

Note that $B^{(0)}$ and $A^{(1)}$ vanish when $M^{(1)} = 0$. Substituting (8.7) and (8.8) into (8.3) we obtain

(8.11) $$Q_x = \Gamma E_{13}^{(0)}, \quad Q_y = \Gamma E_{23}^{(0)},$$

with

(8.12) $$\Gamma = \left(2M^{(0)}+\frac{2(M^{(1)})^2}{M^{(2)}}\right),$$

and

(8.13)
$$\begin{aligned} N_x &= \mathscr{D}^{(0)}(E_{11}^{(0)}+\mathscr{N}^{(0)}E_{22}^{(0)})+\mathscr{D}^{(1)}(E_{11}^{(1)}+\mathscr{N}^{(1)}E_{22}^{(1)})+\mathscr{T},\\ N_y &= \mathscr{D}^{(0)}(\mathscr{N}^{(0)}E_{11}^{(0)}+E_{22}^{(0)})+\mathscr{D}^{(1)}(\mathscr{N}^{(1)}E_{11}^{(1)}+E_{22}^{(1)})+\mathscr{T},\\ N_{xy} &= 2M^{(0)}E_{12}^{(0)}+2M^{(1)}E_{12}^{(1)}, \end{aligned}$$

where

(8.14)
$$\begin{aligned} \mathscr{D}^{(0)} &= [\Lambda^{(0)}(A^{(0)}+1)+2M^{(0)}], \quad \mathscr{N}^{(0)} = \frac{\Lambda^{(0)}}{\mathscr{D}^{(0)}}(A^{(0)}+1),\\ \mathscr{D}^{(1)} &= [\Lambda^{(0)}B^{(0)}+2M^{(1)}], \quad \mathscr{N}^{(1)} = \frac{1}{\mathscr{D}^{(1)}}(\Lambda^{(0)}B^{(0)}),\\ \mathscr{T} &= [\Lambda^{(0)}C^{(0)}-\omega(3\Lambda^{(0)}+2M^{(0)})\Theta^{(0)}-2\omega M^{(1)}\Theta^{(1)}]. \end{aligned}$$

$M^{(1)} = 0$ implies $\mathscr{D}^{(1)} = 0$ and $\mathscr{D}^{(1)}\mathscr{N}^{(1)} = 0$, in which case the relations (8.13) simplify considerably.

Similarly, one obtains from (8.5) the following relations:

(8.15)
$$\begin{aligned} M_x &= \mathfrak{D}^{(0)}(E_{11}^{(0)}+\mathfrak{N}^{(0)}E_{22}^{(0)})+\mathfrak{D}^{(1)}(E_{11}^{(1)}+\mathfrak{N}^{(1)}E_{22}^{(1)})+\mathfrak{T},\\ M_y &= \mathfrak{D}^{(0)}(\mathfrak{N}^{(0)}E_{11}^{(0)}+E_{22}^{(0)})+\mathfrak{D}^{(1)}(\mathfrak{N}^{(1)}E_{11}^{(1)}+E_{22}^{(1)})+\mathfrak{T}.\\ M_{xy} &= -[2M^{(1)}E_{12}^{(0)}+2M^{(2)}E_{12}^{(1)}], \end{aligned}$$

where

(8.16)
$$\begin{aligned} \mathfrak{D}^{(0)} &= (\Lambda^{(2)}A^{(1)}+2M^{(1)}), \quad \mathfrak{N}^{(0)} = \frac{1}{\mathfrak{D}^{(0)}}(\Lambda^{(2)}A^{(1)}),\\ \mathfrak{D}^{(1)} &= (\Lambda^{(2)}(B^{(1)}+1)+2M^{(2)}), \quad \mathfrak{N}^{(1)} = \frac{1}{\mathfrak{D}^{(1)}}(\Lambda^{(2)}(B^{(1)}+1)),\\ \mathfrak{T} &= -\{[2M^{(1)}\Theta^{(0)}+3\Lambda^{(2)}\Theta^{(1)}+2M^{(2)}\Theta^{(1)}]-\Lambda^{(2)}C^{(1)}\}. \end{aligned}$$

Observe again that in the case $M^{(1)} = 0$, then $\mathfrak{D}^{(0)} = 0$ and $\mathfrak{D}^{(0)}\mathfrak{N}^{(0)} = 0$. In this case there is a true separation of (8.15) from (8.13).

With the definitions

(8.17) $$\varrho_0 I^{(0)} = \mathscr{M}, \quad \mathscr{S} = \varrho_0 I^{(1)}, \quad \mathscr{I} = \varrho_0 I^{(2)},$$

the equations of motion can be obtained in the form

$$\mathscr{M}\ddot{u}-\mathscr{S}\ddot{\varphi} = \frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}+T_x^{(0)},$$

$$\mathscr{M}\ddot{v}-\mathscr{S}\ddot{\psi} = \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}+T_y^{(0)},$$

(8.18) $$\mathfrak{M}\ddot{\eta}+\mathfrak{S}\ddot{u}_3^{(1)} = \frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q+N_x\frac{\partial^2\eta}{\partial x^2}+2N_{xy}\frac{\partial^2\eta}{\partial x\,\partial y}+N_y\frac{\partial^2\eta}{\partial y^2}+\left(\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}\right)\frac{\partial\eta}{\partial x}+\left(\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}\right)\frac{\partial\eta}{\partial y},$$

$$\mathfrak{S}\ddot{u}-\mathscr{I}\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial M_x}{\partial x}-\frac{\partial M_{xy}}{\partial y}-Q_x+T_x^{(1)},$$

$$\mathfrak{S}\ddot{v}-\mathscr{I}\frac{\partial^2\psi}{\partial t^2} = -\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y+T_y^{(1)},$$

where we have defined

(8.19) $$T_x^{(i)} \equiv F_x^{(i)}+P_x^{(i)}, \quad T_y^{(i)} \equiv F_y^{(i)}+P_x^{(i)}, \quad q \equiv F_z^{(i)}+P_z^{(i)}$$

and

(8.20) $$u \equiv u_1^{(0)}, \quad v \equiv u_2^{(0)}.$$

Note that the unknown $u_3^{(1)}$ in the equations (8.18) can be expressed in terms of u, v, φ, ψ and η. To this end, observe that [see (3.9)] $C_{33}^{(0)} = 2E_{33}^{(0)} = u_3^{(1)}$. On the other hand, the Eq. (8.8) must hold. Thus

(8.21) $$\ddot{u}_3^{(1)} = \frac{\partial^2}{\partial t^2}\left\{A^{(0)}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial\eta}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial\eta}{\partial y}\right)^2\right]-B^{(0)}\left[\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial y}\right]+C^{(0)}+C^{(1)}\right\}.$$

With this set of equations a definite set has been obtained describing the dynamic response of ice plates to instationary motions. The equations of motion are (8.18); they form a set of 5 equations for the unknowns u, v, φ, ψ, and η. The stress-strain relations for the macroscopic equations are (8.11), (8.13) and (8.15), while the strains are given in (3.8).

Often one is concerned with a theory where the following simplifying assumptions are made:

(8.22)
(i) $\mathscr{S} = 0,$
(ii) $M^{(1)} = 0,$
(iii) $T_x^{(0)} = T_y^{(0)} = T_x^{(1)} = T_y^{(1)} = 0.$

This theory could justly be called a *generalized Reissner-von Kármán* theory. Assumption (ii) implies that $\mathfrak{D}^{(0)} = \mathscr{D}^{(1)} = \mathfrak{D}^{(0)}\mathfrak{N}^{(0)} = \mathscr{D}^{(1)}\mathscr{N}^{(1)} = 0$. (iii), on the other hand, implies that there are no horizontal surface forces. Drag due to wind and water motion is thus neglected.

For flexural motion further simplification is achieved by neglecting horizontal accelerations $\ddot{u}$ and $\ddot{v}$. In this event the equilibrium equations $(8.18)_{1,2}$ for the membrane forces are satisfied identically by introducing the stress function:

(8.23) $$N_x = \frac{\partial^2 F}{\partial y^2}, \quad N_y = \frac{\partial^2 F}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2 F}{\partial x\,\partial y}.$$

Using the stress-strain relations it then can easily be shown that the equilibrium equations for the membrane forces reduce to

$$\frac{\partial^4 F}{\partial x^4}+\mathfrak{H}\frac{\partial^4 F}{\partial x^2 \partial y^2}+\frac{\partial^4 F}{\partial y^4}=\mathfrak{E}\left[\left(\frac{\partial^2 \eta}{\partial x \partial y}\right)^2-\frac{\partial^2 \eta}{\partial x^2}\frac{\partial^2 \eta}{\partial y^2}\right], \tag{8.24}$$

with

$$\mathfrak{H}=\frac{\mathscr{D}^{(0)}(1-(\mathscr{N}^{(0)})^2}{2M^{(0)}}-2\mathscr{N}^{(0)}, \quad \mathfrak{E}=\mathscr{D}^{(0)}(1-(\mathscr{N}^{(0)})^2, \tag{8.25}$$

where use has been made of the fact that temperature varies only in the direction perpendicular to the plate reference plane. Note that an equation similar to (8.24) also occurs in the von Kármán plate theory with $\mathfrak{H}=2$. Here, we note that this assumption cannot be made.

The equations of motion and constitutive equations on the other hand reduce to

$$\begin{aligned}
\mathscr{M}\ddot{\eta} &= \frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q+\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 \eta}{\partial x^2}+2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 \eta}{\partial x \partial y}+\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 \eta}{\partial y^2},\\
-\mathscr{I}\frac{\partial^2 \varphi}{\partial t^2} &= \frac{\partial M_x}{\partial x}-\frac{\partial M_{xy}}{\partial y}-Q_x,\\
-\mathscr{I}\frac{\partial^2 \psi}{\partial t^2} &= \frac{\partial M_y}{\partial y}-\frac{\partial M_{xy}}{\partial x}-Q_y,\\
M_x &= -\mathfrak{D}^{(1)}\left[\frac{\partial \varphi}{\partial x}+\mathfrak{N}\frac{\partial \psi}{\partial y}\right]+\mathfrak{T}, \quad Q_x=\frac{1}{2}\Gamma\left(\frac{\partial \eta}{\partial x}-\varphi\right),\\
M_y &= -\mathfrak{D}^{(1)}\left[\frac{\partial \varphi}{\partial y}+\mathfrak{N}\frac{\partial \psi}{\partial x}\right]+\mathfrak{T}, \quad Q_y=\frac{1}{2}\Gamma\left(\frac{\partial \eta}{\partial y}-\psi\right),\\
M_{xy} &= -M^{(2)}\left(\frac{\partial \varphi}{\partial x}+\frac{\partial \psi}{\partial y}\right).
\end{aligned} \tag{8.27}$$

Moreover, neglecting all non-linear terms we obtain the linear bending theory accounting for shear deformation. Similar, but not entirely the same equations, have been derived by REISSNER [22] and, by a different method of derivation, also by GREEN [23] (see also [24]).

9. Numerical values for the plate constants

The preceding calculations made use of various constants which are different from the familiar ones in ordinary plate theory. To make this theory amenable to explicit calculations we shall list the pertinent constants in this section. Because temperature variation is essential in this theory, we first need some information with regard to the temperature variation of the elastic constants, such as for Young's modulus, the shear modulus, Poisson's ratio and Lamé's constants, respectively. For polycrystalline ice with randomly oriented crystals these constants have been calculated by various authors from the corresponding single crystal properties. RÖTHLISBERGER [25] summarizes the work of JONA and

SCHERRER [26], BASS [27] *et al*, BROCKAMP and QUERFURTH [28] and BENNET [29]. After some reinterpretations, these results can be summarized as follows:

$$
\left.\begin{aligned}
\frac{E(\vartheta)}{E(0)} &= 1-0.00146\,\vartheta, \qquad E(0) = 9.21\times 10^{10}\ [\text{dynes/cm}^2],\\
\nu(\vartheta) &= 0.314,\\
\frac{\mu(\vartheta)}{\mu(0)} &= 1-0.00146\,\vartheta, \qquad \mu(0) = 3.5\times 10^{10}\ [\text{dynes/cm}^2],\\
\frac{\lambda(\vartheta)}{\lambda(0)} &= 1-0.00146\,\vartheta, \qquad \lambda(0) = 5.95\times 10^{10}\ [\text{dynes/cm}^2],
\end{aligned}\right\}\ 0 \geqslant \vartheta \geqslant -30°\text{C}). \tag{9.1}
$$

In these formulas the temperature is given in centigrades.

In order to obtain numerical values for $\Lambda^{(m)}$ and $M^{(m)}$ as given by (6.13) we assume for the sake of simplicity a linear distribution of the temperature across the thickness of the ice with temperatures ϑ^u and ϑ^l at the upper and lower face, respectively. Without loss of generality we may assume $\vartheta^l = 0$, because the water temperature at the lower face is at freezing point. Then (Fig. 2)

$$\Lambda_1 = \int_h (x_3-\delta)\,\lambda(\vartheta)\,ds = 0, \tag{9.2}$$

implies

$$\frac{\delta}{h} = 0.5+0.000122\,\vartheta^u, \tag{9.3}$$

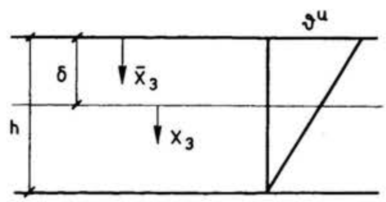

FIG. 2.

where ϑ^u is to be expressed in centigrades. Furthermore, after lengthy manipulations, one deduces

$$
\begin{aligned}
\Lambda^{(0)} &= \lambda(0)h[1-0.000732\,\vartheta^u], \qquad M^{(0)} = \mu(0)h[1-0.000732\,\vartheta^u],\\
\Lambda^{(2)} &= \frac{\lambda(0)h^3}{12}[1-0.000293\,\vartheta^u], \qquad M^{(2)} = \frac{\mu(0)h^3}{12}[1-0.000293\,\vartheta^u],\\
\mathscr{D}^{(0)} &= \frac{2\lambda(0)}{1-\nu}h(1-0.000732\,\vartheta^u), \qquad \mathfrak{D}^{(1)} = \frac{h^3}{12}\frac{2\mu(0)}{1-\nu}(1-0.000293\,\vartheta^u),
\end{aligned} \tag{9.4}
$$

$$\mathscr{N}^{(0)} = \nu; \qquad \mathfrak{N}^{(1)} = \nu, \tag{9.5}$$

$$\Gamma = 2\mu(0)h(1-0.000732\,\vartheta^u). \tag{9.6}$$

With (Fig. 2)

$$\vartheta(x_3) = \left(1-\frac{\delta}{h}\right)\vartheta^u - \frac{x_3}{h}\vartheta^u = \Theta^{(0)}+x_3\Theta^{(1)}, \tag{9.7}$$

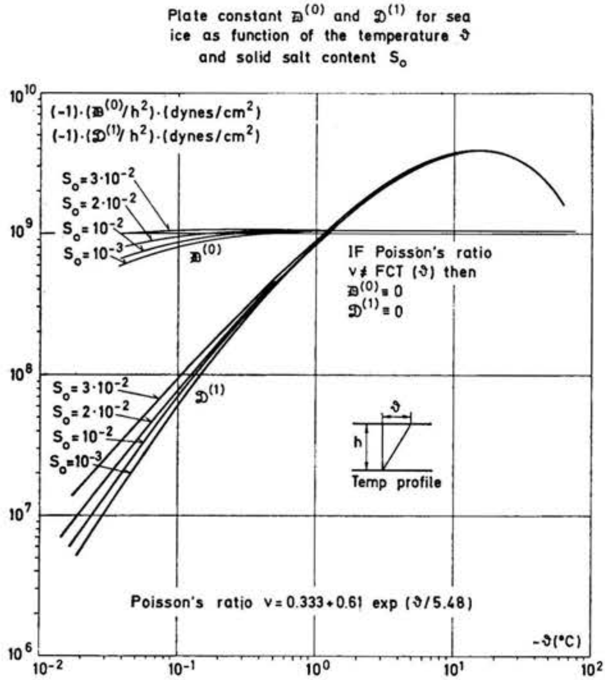

FIG. 3.

where

$$\Theta^{(0)} = \left(1-\frac{\delta}{h}\right)\vartheta^u \approx 0.5\,\vartheta^u, \quad \Theta^{(1)} = -\vartheta^u_{\iota}/h, \tag{9.8}$$

we also obtain

$$\mathscr{T} = -3\omega h \frac{K}{1-\nu}(1-0.000732\,\vartheta^u), \tag{9.9}$$

$$\mathfrak{T} = -\omega h^3 \frac{K}{4(1-\nu)}(1-0.000732\,\vartheta^u), \tag{9.10}$$

where K is the bulk modulus, given by

$$K = \big(3\lambda(0)+2\mu(0)\big)/3. \tag{9.11}$$

The above calculations are performed for fresh water ice and they amply indicate that the influence of the variability of the temperature across the thickness of the ice is negligible. This is not so for sea ice as will be demonstrated now.

The reason for the different behaviour of sea ice must be sought in the presence of salt which not only makes the freezing point of brine temperature dependent, but also determines the amount of brine inclusions according to the phase diagram. Therefore, dependent upon the temperature distribution there is a more or less substantial contribution of brine inclusions in the ice which weakens its strength. Extensive studies have been performed recently. Among those, we mention the illuminating article by ASSUR [30] which among many other things presents the numerical values for the relative volume of brine, n (volume porosity), in "standard sea ice" as a function of the temperature. It then suffices to relate the elastic properties of a solid without porosity to those of the same solid when containing holes. This relation, that is, the dependence of the elastic moduli upon the porosity has extensively been studied from an experimental point of view (see WEEKS and ASSUR [31]). It seems that Young's modulus is linearly related to the porosity but there is insufficient information on the variation of Poisson's ratio with the state of the sea ice.

Rather than relying on such insufficient information our ultimate goal would be to perform a model calculation, by assuming that the voids are of special form. These results are not available now but extensive computer calculations based on results of [31] have proved that there is in fact a substantial influence of the temperature distribution on the plate constants. The calculations are tedious and not pertinent to the matter in hand. It may suffice to substantiate the above conjectures by Fig. 3 where $\mathscr{D}^{(1)}$ and $\mathfrak{D}^{(0)}$ are plotted as functions of the surface temperature ϑ and the solid salt content S_0.

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Added in proof. The linear version of the plate theory of this article has been applied to simple plate bending problems. Results on the deflection of floating plates to strip like loads and the response of such plates to plane waves are reported in [32, 33 and 34]. It is shown in these articles that for sea ice the dependence of the Poisson ratio on temperature (or brine content) may have an influence on the solutions which deviate from the classical solutions only by roughly 3—7%. For *natural* ice this is too small to be of any significance.

Thus, the theory presented in this article has turned out to be too general for ice plates. Nevertheless for *composite* plates and *laboratory* ice where deviations of 3 to 7% can experimentally be detected, because there are fewer imurities present, the above plate theory does have its values.

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