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An analogue for non-linear hereditary viscoelastic membranes

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THE PAPER deals with an analogue which exists between the problem of instantaneous deformation of physically and geometrically non-linear membrane and a similar problem for shallow membrane under internal pressure which is made of the material showing the hereditary type deformation dependent entirely of time. This analogue holds for physical operators of superposition with the same degree of homogeneity. The solution for the time function and the interpretation of the results obtained are presented.

W pracy wykazuje się analogię, jaka istnieje pomiędzy zagadnieniem deformacji natychmiastowej fizykalnie i geometrycznie nieliniowej membrany a podobnym zagadnieniem dla membrany małowyniosłej pod ciśnieniem wewnętrznym z materiału wykazującego deformację, zależną wyłącznie od czasu, typu dziedziczenia. Analogia ta zachodzi dla fizykalnych operatorów superpozycji o identycznym stopniu jednorodności. Podane zostało rozwiązanie dla funkcji czasu oraz interpretacja wyników.

В работе показывается аналогия, существующая между проблемой мгновенной деформации физически и геометрически нелинейной мембраны и аналогичной проблемой для мембраны малого подъема под внешним давлением из материала обладающего деформацией зависящей исключительно от времени, типа наследственности. Эта аналогия имеет место для физических операторов суперпозиции с одинаковой степенью однородности. Даются решение для функции времени, а также интерпретация результатов.

THE KNOWN analogues in the theory of viscoelasticity are of great theoretical and practical significance. They enable us to make use of the existing instantaneous solutions and apply them in presenting time dependent ones. The idea of analogues is founded on the formal similarity of fundamental equations for the analogous problems formulated within the two theories. For example, such an analogue is well-known in the linear theory of viscoelasticity.

As regards the non-linear creep theory, there is an analogue known as the Hoff analogy for problems based on the power (homogeneous) constitutive forms set up for non-linear instantaneous and creep behaviour. However, its validity is limited as the analogue finds application within the geometrically linear deformation theory only.

It has been shown by one of the authors [1, 3] that some analogues founded on power laws may also find application in geometrically non-linear two-dimensional problems.

For example, such analogues have been found for non-linear shallow membranes [3]. Recently, the present authors [4] pointed out an analogue which indicates the broader possibility of application as regards the physical aspects. It is concerned with geometrically non-linear membranes made of non-linear hereditary viscoelastic materials, characterized by a non-stationary deformation process. The analogue has been given in the special case of a cylindrical membrane under internal pressure. This specific problem seems to be of great importance from a practical point of view.

In the present paper the above analogue is generalized and applied to a non-linear shallow membrane which undergoes a hereditary deformation process. The time-dependent solution is founded on the analogous non-linear elastic problem.

1

Let the form of the non-linear constitutive equation for a homogeneous incompressible viscoelastic material be

$$e_{ij}(t) = \mathbf{N}^{(v)} s_{ij}. \tag{1.1}$$

Here, e_{ij} is the strain tensor, s_{ij} the stress deviato rand $\mathbf{N}^{(v)}$ is a non-linear hereditary operator defined within the time interval $T = [t_0, t]$

$$\mathbf{N}^{(v)} s_{ij} = -\int_T H[t, \tau, s(\tau)] s_{ij}(\tau) d\tau. \tag{1.1a}$$

Alternatively, the operator $\mathbf{N}^{(v)}$ can be presented as

$$\mathbf{N}^{(v)} = \mathbf{L}\mathbf{H},$$

where $\mathbf{L}$ is a linear integral time-operator and $\mathbf{H}$ the superpositional homogeneous operator of the $\varkappa$-th degree (of homogeneity).

It means that

$$\mathbf{H}\mu s_{ij} = \mu^{\varkappa} \mathbf{H} s_{ij}, \tag{1.2}$$

where μ is an arbitrary constant.

Further, let us assume that

$$e_{ij} = \mathbf{N}^{(e)} s_{ij}, \tag{1.3}$$

represents the constitutive law describing the instantaneous (elastic) state, where $\mathbf{N}^{(e)}$ denotes a homogeneous superpositional operator(1) of the degree ϱ. Thus,

$$\mathbf{N}^{(e)} \lambda s_{ij} = \lambda^{\varrho} \mathbf{N}^{(e)} s_{ij}, \tag{1.4}$$

λ being an arbitrary non-dimensional constant.

In order to point out the analogue we shall make use of the forms of the Eqs. (1.1) and (1.3). It exists between the analogous problems of shallow non-linear membranes made of two different materials. The former is characterized by an instantaneous reaction, as given by the Eq. (1.4), while the properties of the latter are time-dependent, as shown by the Eq. (1.1).

2

The system of equations for shallow non-linear elastic and a shallow hereditary membrane undergoing large deflections reduces to a system of two analogous differential and integro-differential equations, respectively. These systems are obtained as follows.

(1) Such operators has been treated by Caratheodory.

We assume a set of coordinates $x_\alpha(\alpha = 1, 2)$ which coincide with the lines of the main curvatures of the membrane $k^{(\alpha)}$. Denoting by $(\ )_{,\alpha}$ the differentiation with respect to x_α, we wri te strain components in the membrane surface

$$e_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta}+u_{\beta,\alpha}+w_{,\alpha}w_{,\beta}-2k^{(\alpha)}w\delta_{\alpha\beta}), \tag{2.1}$$

which satisfy the condition of compatibility

$$e_{\alpha\alpha,\beta\beta}+e_{\beta\beta,\alpha\alpha}-2e_{\alpha\beta,\alpha\beta} = \Omega, \quad (\alpha, \beta \text{ — not summed}). \tag{2.2}$$

Here Ω denotes

$$\Omega = (w_{,\alpha\alpha})^2-w_{,\alpha\alpha}w_{,\beta\beta}-(k^{(\alpha)}w_{,\beta\beta}+k^{(\beta)}w_{,\alpha\alpha}), \quad (\alpha, \beta \text{ — not summed}). \tag{2.3}$$

The stresses $\sigma_{\alpha\beta}$ in the membrane surface satisfy the condition of equilibrium

$$\sigma_{\alpha\beta,\alpha} = 0, \tag{2.4}$$

and, therefore, the equilibrium of the deformed membrane is given by the equation

$$\sigma_{\alpha\beta}(w_{,\alpha\beta}+k^{(\alpha)}\delta_{\alpha\beta})+\frac{q}{h} = 0. \tag{2.5}$$

Here q is the constant uniform pressure and h the membrane thickness.

Further, by introducing the stress function Φ defined as

$$\sigma_{\alpha\alpha} = \Phi_{,\beta\beta}, \quad \sigma_{\alpha\beta} = -\Phi_{,\alpha\beta}, \quad \alpha \neq \beta, \tag{2.6}$$

and the constitutive forms of the Eqs. (1.1) and (1.3), we obtain the conditions of the Eqs. (2.2) and (2.5), respectively,

$$\partial_{22}\mathbf{N}^{(k)}\left[\frac{1}{3}(2\Phi^{(k)}_{,22}-\Phi^{(k)}_{,11})\right]+\partial_{11}\mathbf{N}^{(k)}\left[\frac{1}{3}(2\Phi^{(k)}_{,11}-\Phi^{(k)}_{,22})\right]-2\partial_{12}\mathbf{N}^{k}_{.}(-\Phi^{(k)}_{,12}) = \Omega^{(k)}, \tag{2.7}$$

$$\Phi^{(k)}_{,22}(k^{(1)}+w^{(k)}_{,11})+\Phi^{(k)}_{,11}(k^{(2)}+w^{(k)}_{,22})+2(-\Phi^{(k)}_{,12})w^{(k)}_{,12} = -\frac{q}{h}. \tag{2.8}$$

The unknown functions in the above sets of equations are the stress function $\Phi^{(k)}$ and the deflection $w^{(k)}$, where $k = e, v$ we put for elastic and hereditary problems, respectively.

As follows from the form of the Eq. (2.8), its left-hand side must be time-independent as $q/h = \text{const}(t)$. This fact should be taken into account when performing the separation of variables, as shown below. Here

$$\partial_{\alpha\beta} = \partial^2/\partial x_\alpha \partial x_\beta, \quad \Phi_{,\alpha\beta} = \partial^2\Phi/\partial x_\alpha \partial x_\beta, \tag{2.9}$$

the operators $\mathbf{N}^{(e)}$ and $\mathbf{N}^{(v)}$ being given by the Eqs. (1.3) and (1.1), respectively.
If we apply to the Eq. (2.7) the inverse operation $\mathbf{L}^* = \mathbf{L}^{-1}$ we obtain

$$\partial_{22}\mathbf{H}\left[\frac{1}{3}2\Phi^{(v)}_{,22}-\Phi^{(v)}_{,11})\right]+\partial_{11}\mathbf{H}\left[\frac{1}{3}(2\Phi^{(v)}_{,11}-\Phi^{(v)}_{,22})\right]-2\partial_{12}\mathbf{H}(-\Phi^{(v)}_{,12}) = \mathbf{L}^*\Omega^{(v)}. \tag{2.10}$$

In the following we substitute into the Eq. (2.10) the separated fo rmsof resolving functions

$$w^{(v)}(x_\alpha, t) = \tilde{w}(x_\alpha)\varphi(t), \quad \Phi^{(v)}(x_\alpha, t) = \tilde{\Phi}(x_\alpha\psi(t). \tag{2.11}$$

Furthermore, in the Eq. (2.8) we assume

$$k^{(\alpha)} = \tilde{k}^{(\alpha)}\varphi(t). \tag{2.12}$$

This is due to the fact that with such an assumption the left-hand side of the said equation becomes a constant. This is also justified from the physical point of view as stress is proportional to the current curvature. It follows from the assumption made that the Eq. (2.3) is excluded from our considerations as regards analogy.

Further, we put

$$\mathbf{H} = \mathbf{B}\tilde{\mathbf{H}}, \tag{2.13}$$

where **B** is a function of time. Doing so, we find

$$\Omega_{(v)}^{-1}\left\{\partial_{22}\tilde{\mathbf{H}}\left[\frac{1}{3}2\tilde{\Phi}_{,22}-\tilde{\Phi}_{,11})\right]+\partial_{11}\tilde{\mathbf{H}}\left[\frac{1}{3}(2\tilde{\Phi}_{,11}-\tilde{\Phi}_{,22})\right] -2\partial_{12}\tilde{\mathbf{H}}(-\Phi_{,12})\right\} = (\mathbf{B}\psi)^{-1}\mathbf{L}^{*}\varphi^{2}. \tag{2.14}$$

Furthermore, by putting

$$\mathbf{N}^{(e)} = A\tilde{\mathbf{F}}, \tag{2.15}$$

where A is elastic constant, from the Eq. (2.7) we get the form analogous to (2.14)

$$\Omega_{(e)}^{-1}\left\{\partial_{22}\tilde{\mathbf{F}}\left[\frac{1}{3}(2\Phi^{(e)}_{,22}-\Phi^{(e)}_{,11})\right]+\partial_{11}\tilde{\mathbf{F}}\left[\frac{1}{3}(2\Phi^{(e)}_{,11}-\Phi^{(e)}_{,22})\right]-2\tilde{\mathbf{F}}(-\Phi^{(e)}_{,12}) = A^{-1}. \tag{2.16}$$

3

Now, we shall point out the existing analogue between the solutions of the Eq. (2.14) and the Eq. (2.16). By dividing side by side the former through the latter we obtain

$$\frac{R_v}{R_e} = \frac{(\mathbf{B}\psi)^{-1}\mathbf{L}^{*}\varphi^{2}}{A^{-1}}, \tag{3.1}$$

where R_e, R_v denote the left-hand sides of the Eqs. (2.14) and (2.16), respectively. In particular, if the degree of homogeneity for both the physical operators is the same, i.e. $\varkappa = \varrho$, then we can write

$$A\mathbf{L}^{*}\varphi^{2} = \mathbf{B}\psi. \tag{3.2}$$

The result indicates that the time-independent solutions of both the differential equations are identical(²). Thus, the Eq. (3.1) gives the time-dependent solution for the hereditary membrane. Here, the function φ is assumed as reciprocal of ψ [see the Eq. (2.11)]. The above assumption follows from the fact that the stress state in the membrane is proportional to the radii of curvature $R_{\alpha\alpha}$ being considered as time functions. The operator $\mathbf{L}^{*}$ is

(²) It may easily be shown that the identical result is obtained from the boundary conditions for a fixed flat contour of the membrane.

inverse to **L** and, therefore, may assume an integral or differential form in dependence on the kernel of the latter. The symbol **B** in the Eq. (3.2) is a multiplicative time function following from the representation (2.11).

4

In order to specify our considerations let us assume in accordance with the Eqs. (1.1) and (1.2) a particular form of constitutive law which specifies the kernel of the integral operator **L**;

$$\mathbf{N}^{(v)} s_{ij} = \mathbf{L}\mathbf{H} s_{ij} = \mathbf{L}\mathbf{B}\tilde{\mathbf{H}} s_{ij}, \tag{4.1}$$

where

$$\mathbf{L} \dots = \int_T \dots K(t. \tau) d\tau, \tag{4.2}$$

$$\mathbf{B}\psi = \psi^{n-1}\psi = \psi^n, \tag{4.3}$$

$$\tilde{\mathbf{H}} \dots = s^{n-1} \dots, \tag{4.4}$$

and

$$s^2 = \frac{3}{2}(s_{kl} s_{kl}). \tag{4.5}$$

Here K denotes the kernel of the operator **L** which we put into the form

$$K(t, \tau) = K(t-\tau) = -\gamma C_0 e^{-\gamma(t-\tau)}. \tag{4.6}$$

In such a case the inverse operator becomes a differential one, as may by checked by the differentiation of the Eq. (4.2) where the Eq. (4.6) is substituted,

$$\mathbf{L}^* \dots = (C_0\gamma)^{-1}(D+\gamma) \dots, \tag{4.7}$$

D denoting the time derivative.

Thus, from the Eq. (3.2) follows

$$A(C_0\gamma)^{-1}(D+\gamma)\varphi^2 = \varphi^{-n}. \tag{4.8}$$

The solution of the above differential equation reads

$$\varphi(t) = [C_0/A(1-e^{-\gamma/2(n-1)(t-t_0)})]^{1/(n+1)} \tag{4.9}$$

and may be identified with the creep function of the structure. Here C_0, γ, n, are creep constants and A should be treated as an analogue constant. In such a way the solution of the time-dependent problem is given by

$$w^{(v)} = \tilde{w}(x_\alpha)\varphi(t), \quad \Phi^{(v)} = \tilde{\Phi}(x_\alpha)\varphi^{-1}(t). \tag{4.10}$$

Both the values of the Eq. (4.10) disappear at $t = t_0$ as $\tilde{w}$ and $\tilde{\Phi}$ represent the instantaneous solutions.

The results of the analysis are given in Figs. 1, 2 and 3. In Fig. 1 is shown the dependence of the solution (4.9) on time with varying n. It is seen that all curves possess the iden-

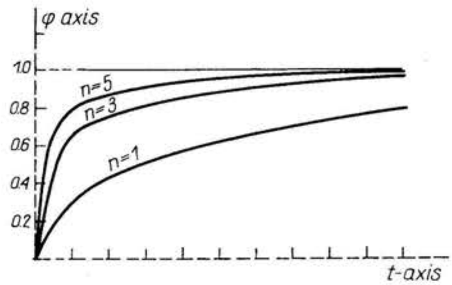

FIG. 1.

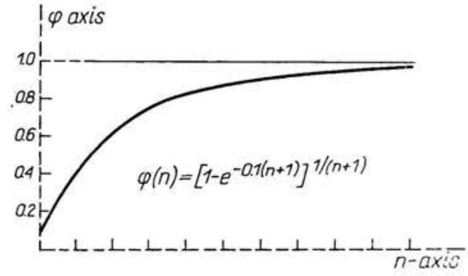

FIG. 2.

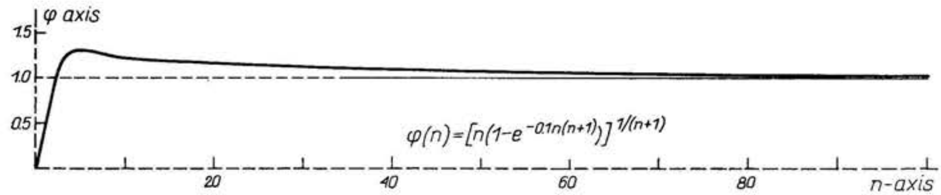

FIG. 3.

tical asymptote for $t \to \infty$. On the other hand, for $n \to \infty$ the solution tends to the same asymptotic value. This is clearly shown in Fig. 2. where as before the analogy coefficient is put $C_0/A = 1$. In Fig. 3 is shown the dependence of the solution on the number n, if all physical parameters are assumed as proportional to this number. The asymptote again assumes the value $\varphi(\infty) = 1$ although the solution approaches it from above.

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