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Spatially cognitive media (1). II. One-dimensional theory (2)

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UNDER certain hypotheses a general constitutive equation for media with *spatial cognitivity* is cast into a form suitable for the solution of (referential-description) problems posed for one-dimensional continua of this type. Moreover, this deduction-mode automatically yields an expression for the initial-stress distribution within such one-dimensional bodies. Two explicit examples are given; one of these illustrates the extent of an "edge effect" as a function of a *spatial influence parameter* $\sigma \in (0, 1]$.

Czyniąc niektóre hipotezy wyprowadzono ogólne równanie konstytutywne dla ośrodków z przestrzenną świadomością w formie wygodnej do rozwiązywania zagadnień, postawionych dla tego typu jednowymiarowego kontinuum. Stosowany przy tym sposób dedukcji automatycznie spełnia wyrażenie dla rozkładu naprężeń początkowych dla wspomnianych wyżej ciał jednowymiarowych. Przytoczono dwa proste przykłady; jeden z nich ilustruje miarę "efektu krawędziowego" jako funkcji przestrzennego parametru wpływu $\sigma \in (0, 1]$.

Делая некоторые гипотезы выведено общее определяющее уравнение для сред с пространственной познавательностью в форме пригодной для решения задач поставленных для этого типа одномерного континуум. Применяемый при этом способ дедукции автоматически удовлетворяет выражению для распределения начальных напряжений для упомянутых выше одномерных тел. Приведены два простых примера, один из них иллюстрирует меру „краевого эффекта'', как функции пространственного параметра влияния $\sigma \in (0, 1]$.

1. Introduction

A GENERAL constitutive equation for homogeneous isotropic materials with spatial cognitivity is [1](5):

$$\mathbf{T} = \underset{\mathbf{z}\in B_t}{\mathfrak{F}} (\mathbf{x}-\mathbf{z}), \tag{1.1}$$

where the spatial description is employed and where $\underset{\mathbf{z}\in B_t}{\mathfrak{F}}$ is an isotropic rank-two, generally nonsymmetric, tensor valued functional on B_t, the image in the physical space $\mathscr{E}$

(1) Some aspects of the development presented here form part of a 1966 Ph.D. Thesis submitted by one of us (Y.S.P.) to the School of Engineering and Applied Science, University of California, Los Angeles (UCLA). The present formulation, however, is solely that of the senior author.

(2) Cf., [1] for the three-dimensional constitutive theory, where the motivation for this work and further references to the literature are given. In particular, EDELEN [2], ERINGEN & EDELEN [3], and ROGULA [4] have also considered non-local effects.

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(5) Unless otherwise stated, we adhere to the terminology and scheme-of-notation adopted in [1]

of the body at time t. Forms of (1.1) possessing greater explicitness can be obtained upon invoking additional hypotheses. This is done in [1] for the three-dimensional case.

Almost independently of [1], in this article we explore the consequences which, upon introducing additional hypotheses, can be deduced from (1.1) when the body is one-dimensional. This means that *all* images B_t of the body $\mathscr{B}$ occupy connected sub-intervals of a one-dimensional Euclidean point space. Thus, the deformations of the body are maps which take one point on the real line to another. The stress-vector and -tensor are represented by the same real number, its sign specifying the direction of the stress-vector.

Denote two open connected subsets, I_0 and I_t, of the real line $\mathscr{R}$ as follows:

$$I_0 = (L_2, L_1),\ L_2 < L_1, \quad \text{and} \quad I_t = (l_2, l_1),\ l_2 < l_1. \tag{1.2}$$

Then, for our present purpose we adopt the notation [6]:

$B_0 = I_0$, the image of $\mathscr{B}$ in its reference configuration; say: χ_0,

$B_t = I_t$, the image of $\mathscr{B}$ in its configuration at time t, χ_t,

χ : $I_0 \times \mathscr{R} \to \mathscr{R}$, the motion $\mathscr{B}$ — here, a one-parameter (time: $t \in \mathscr{R}$) family of configurations ($\equiv$ homeomorphisms: $\mathscr{R} \to \mathscr{R}$) of $\mathscr{B}$ relative to B_0:

$$x = \chi(X, t) \equiv \chi_t(X) = X + k(X, t),\ x \in I_t,\ X \in I_0,\ k(X, 0) = 0, \tag{1.3}$$

and

$T_{x,t}$ = the stress at the point occupying the place x in B_t, at time t.

The intent of the notation employed in (1.1) is that of indicating that in this investigation we refrain from considering media possessing *both* time-memory and spatial-cognitivity. In fact, we deliberately avoid consideration of the additional complexities introduced by the former and by possible memory-cognitivity interactions. However, we do not exclude the possibility of posing dynamical problems for one-dimensional bodies composed of media with spatial cognitivity. Though this possibility is encompassed by our formulation, it is not exploited in this presentation.

Since it is stated in the spatial description, (1.1) cannot be employed as it stands. First it, or suitable approximations thereto, must be cast in terms of referential variables. This is one of the objectives of the present investigation.

In the course of attaining this objective, after defining certain function spaces, a mathematical actualization[7] of the spatial one-dimensional form of a Postulate of Fading Spatial Cognitivity [1, § 5] is given in Sect. 2. The content of this section permits the present exposition to be almost independent of that given in [1]. Approximations to (1.1), valid in spatial and referential variables, are derived in Sect. 3. Before applications of the first order approximation can be undertaken, it is first cast in a dimensionally homogeneous form (Sect. 4.1) and some of the implications stemming from CAUCHY'S First and Second Laws [5, § 15] are presented in Sect. 4.2. The existence of the solution of the integral equation for $k(X, t)$ (referential description) is established in Sect. 4.3. In Sect. 5 there is stated a theorem, establishing sufficient conditions under which a one-dimensional body composed of a spatially cognitive medium will obey HOOKE'S Law. Finally, two examples of applications of the first order approximation for the response functional

[6] For simplicity, we sometimes refer to B_0 and B_t as one-dimensional bodies.

[7] Others may be possible by relaxing the restrictions on the function spaces here employed.

are given in Sect. 6. One of these illustrates the appearance of an "edge effect" and the significance of the role played by the spatial influence parameter in determining the extent of this effect.

2. Fading spatial cognitivity

In its one-dimensional *qualitative* form we state *a*:

POSTULATE 2.1. *FADING SPATIAL COGNITIVITY. The states of deformation of points occupying distant places in the medium relative to the place x at which the stress is to be determined have less influence on this stress than the deformation states of points occupying places near to x.*

To achieve *a* mathematical rendering of this postulate, the following function spaces are defined (Cf., [1, § 5]):

$\mathfrak{c}_w^*$: the space of all real-valued functions $\mathfrak{h}$ on $\mathscr{R}$, which, when restricted to I_t for each fixed $t \in \mathscr{R}$, possess a norm:

$$\|\mathfrak{h}\|_{w,t}^2 \equiv \int_{I_t} [\mathfrak{h}(z)w(z)]^2 dz \tag{2.1}$$

induced by the inner product:

$$\langle \mathfrak{h}_1, \mathfrak{h}_2 \rangle_{w,t} \equiv \int_{I_t} \mathfrak{h}_1(z)\mathfrak{h}_2(z)w^2(z)dz, \tag{2.2}$$

where w is a weight function, and

$\mathfrak{c}^{**}$: the space of all functionals $\mathfrak{f}: \mathfrak{c}_w^* \to \mathscr{R}$ which, with absolute value taken on the right, possess a finite norm:

$$\|\mathfrak{f}\| \equiv \operatorname*{Sup}_{\mathfrak{h}\in\mathfrak{c}_w^*} \left| \mathfrak{f}\left(\frac{\mathfrak{h}}{\|\mathfrak{h}\|}\right) \right| < \infty. \tag{2.3}$$

Upon completion [6, § 4], the spaces $\mathfrak{c}_w^*$ and $\mathfrak{c}^{**}$ become real HILBERT and BANACH spaces respectively, which, since no ambiguity need arise, we also designate by $\mathfrak{c}_w^*$ and $\mathfrak{c}^{**}$.

As a means for quantitatively specifying the influence of distant places upon the stress $T_{x,t}$, introduce *a*:

DEFINITION 2.1. ***Spatial influence function***. *A scalar-valued function* $\mathfrak{h}$ *defined on the field*

$$v_{(\cdot)}(\cdot) \equiv v(\cdot\,,\,\cdot): \mathscr{R}\times\mathscr{R} \to \mathscr{R}: (x, z) \to v_x(z) \equiv x-z$$

such that for each fixed $x \in \mathscr{R}$:

$$\begin{aligned} &\operatorname{Lim} \mathfrak{h}(v_x(z)) = 0 \quad \textit{as} \quad |v_x(z)| \to \infty, \textit{ and} \\ &\mathfrak{h}(v_x(z)) \neq 0 \quad \textit{for any finite} \quad |v_x(z)| \geqslant 0. \end{aligned} \tag{2.4}$$

In some instances it is convenient to introduce a *spatial influence parameter* $\sigma \in (0, 1]$ to characterize the *rate of decay* of the function $\mathfrak{h}$ by redefining v_x to be $\bar{v}_x(\sigma, z) \equiv \dfrac{1}{\sigma}(x-z)$, with (2.4) also holding for each $\sigma \in (0, 1]$.

Based on the preceding definitions and reasoning as in [1], a mathematical rendering of the preceding qualitatively-stated postulate may be expressed as:

POSTULATE 2.2. FADING SPATIAL COGNITIVITY (Mathematical rendering). The constitutive equation defining the medium has the form:

$$T_{x,t} = \underset{z\in I_t}{\mathfrak{F}}(x-z) = \mathfrak{f}(\mathfrak{h}\circ v_{x;\sigma}), \quad x, z \in I_t, \quad \sigma \in (0, 1], \tag{2.5}$$

where the constitutive functional $\mathfrak{f}$ *is an element in* $\mathfrak{c}^{**}$, *and the spatial influence function* $\mathfrak{h}\circ v_{x,\sigma}$ *is an element of* $\mathfrak{c}_w^*$ *satisfying* (2.4), *where* $v_{x;\sigma}(z) \equiv \bar{v}_x(\sigma, z)$.

This postulate, since it partially identifies $\mathfrak{h}$, serves as a guide for constructing constitutive equations for the class of media under consideration.

3. An approximation to the response functional

The conditions $\mathfrak{f}: \mathfrak{c}_w^* \to \mathscr{R}$, $\mathfrak{f} \in \mathfrak{c}^{**}$, of the preceding postulate are now invoked so as to serve their intended purpose of enabling us to apply the theory of FRÉCHET differentials (Cf., [1] for citations to the literature) and RIESZ's theorem. Thus, the first order approximation to $(2.5)_2$ is the first term in the FRECHET power series expansion of $\mathfrak{f}$ about the null element $\theta \in \mathfrak{c}_w^*$; hence, it takes the form:

$$T_{x,t} = \delta\mathfrak{f}(\theta; \mathfrak{h}\circ v_{x;\sigma}) + O(\|\mathfrak{h}\circ v_{x;\sigma}\|_w), \tag{3.1}$$

where $\delta\mathfrak{f}$ is, for each fixed $x \in I_t$, a linear functional on the space $\mathfrak{c}_w^*$, and where [by (2.4)] $\delta^\circ\mathfrak{f}(\theta, \mathfrak{h}) = \mathfrak{f}(\theta) = 0$.

Supposing $h\circ v_{x;\sigma}$ to be differentiable with respect to its argument; by (1.3), the first term of its Taylor series expansion is:

$$\begin{aligned}(\mathfrak{h}\circ v_{x;\sigma})(z) &= \mathfrak{h}\left(\frac{X-Z}{\sigma} + \frac{1}{\sigma}[k(X,t)-k(Z,t)]\right)\\ &= \mathfrak{h}\left(\frac{X-Z}{\sigma}\right) + \frac{1}{\sigma}[k(X,t)-k(Z,t)]\mathfrak{h}'\left(\frac{X-Z}{\sigma}\right) + O\left\{\frac{1}{\sigma}[k(X,t)-k(Z,t)]\right\}^2\\ &\equiv (\mathfrak{h}\circ v_{X;\sigma})(Z) + (Y_{X,t;\sigma}\circ v_X)(Z) + O\left\{\frac{1}{\sigma}[k(X,t)-k(Z,t)]\right\}^2,\end{aligned} \tag{3.2}$$

where $\mathfrak{h}'(W)$ is the first derivative of $\mathfrak{h}$ evaluated at W. Under the assumption that $k(\cdot, t)$ is differentiable to any required order, the Taylor series expansion for the second term in (3.2) is:

$$(Y_{X,t;\sigma}\circ v_X)(Z) = \left\{\frac{1}{\sigma}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m!}[k^{(m)}(X,t)](X-Z)^m\right\}\mathfrak{h}'\left(\frac{X-Z}{\sigma}\right). \tag{3.3}$$

Thus, by (3.2) and on not exhibiting the error term, (3.1) becomes:

$$T_{x,t} = \delta\mathfrak{f}(\theta; \mathfrak{h}\circ v_{X;\sigma} + Y_{X,t;\sigma}\circ v_X). \tag{3.4}$$

Hence, on requiring that $Y_{X,t;\sigma}\circ v_X$ be an element of $\mathscr{C}_W^*$ — the referential counterpart of $\mathfrak{c}_w^*$ — for each $\sigma \in (0, 1]$, $X \in I_0$, and each $t \in (-\infty, \infty)$, $\delta\mathfrak{f}$ is transformed into a linear functional on $\mathscr{C}_W^*$. By its linearity property, (3.4) can be written as:

$$T_{x,t} = \delta\mathfrak{f}(\theta; \mathfrak{h}\circ v_{X;\sigma}) + \delta\mathfrak{f}(\theta; Y_{X,t;\sigma}\circ v_X). \tag{3.5}$$

By the RIESZ representation theorem [6, § 16] for a continuous linear functional on a real HILBERT space, (3.5) can be expressed as:

$$(3.6)\quad T_{x,t} = \int_{I_0} g(X-Z)\mathfrak{h}\left(\frac{X-Z}{\sigma}\right)W^2(X-Z)dZ + \int_{I_0} g(X-Z)Y_{X,t;\sigma}(X-Z)W^2(X-Z)dZ,$$

where

$$T_{x,t} = \tilde{T}_{x,t}(X, t).$$

This is the form of the first order approximation to (2.5) which will be employed in the sequel.

The accuracy of the approximations (3.1) & (3.2), and (3.6), increases as, respectively, $||\mathfrak{h}||_w$ (or $||\mathfrak{h}||_W$, as appropriate) and

$$\frac{1}{\sigma}|k(X, t)-k(Z, t)| \to 0 \quad \text{for} \quad \textit{each } X \in I_0, \textit{ each } t \in (-\infty, \infty), \textit{ each}$$
$$\sigma \in (0, 1], \text{ and } \textit{all } Z \in I_0.$$

4. Investigation of the first order approximation

For convenience we restrict the subsequent development to the case where the spatial influence is homogeneous; i.e., $W = 1^\dagger$, the unit function on I_0.

Upon setting $k(X, t) = k(Z, t) = L$, a rigid-body translation by L, for all $X, Z \in I_0$ and each $t \in (-\infty, \infty)$, the second term on the right-hand side of (3.5) vanishes and the first term represents an initial (or, pre-) stress([8]) which is not due to the deformation from I_0 to I_t. The "pure stress", $T_{p;x,}$ is then given by:

$$(4.1)\qquad T_{p;x,t} = T_{x,t} - T_0 = \int_{I_0} g(X-Z)\left\{\frac{1}{\sigma}[k(X, t)-k(Z, t)]\right\}\frac{d\mathfrak{h}}{dS}\bigg|_{S=\frac{1}{\sigma}(X-Z)} dZ,$$

which, as it should be, is insensitive to a rigid-body translation.

4.1. Dimensional homogeneity

As it stands (4.1) is not dimensionally homogeneous. However, recollecting that at our disposal we have a characteristic-length Λ and a -time τ, which are not explicitly exhibited in (2.7), we may easily render (4.1) dimensionally homogeneous by setting:

$$(4.2)\quad X = \Lambda\bar{X}, \quad Z = \Lambda\bar{Z}, \quad t = \tau\bar{t}, \quad L_{1,2} = \Lambda\bar{L}_{1,2}, \quad k(X, t) \equiv \Lambda\bar{k}(\bar{X}, t) \equiv \tilde{k}(\bar{X}, t),$$

$$(4.3)\qquad \mathfrak{h}\left(\frac{X-Z}{\sigma}\right) \equiv \bar{\mathfrak{h}}\left(\frac{\bar{X}-\bar{Z}}{\sigma}\right),$$

([8]) This result (cf., also, [1]) is worthy of emphasis. On the one hand, because it shows that our method of derivation *automatically* yields an expression for the initial stress distribution within a (one-dimensional) body. On the other hand, because it supports the assertion that *nearly* all bodies possess a non-zero initial-stress state.

and

$$g(X-Z) \equiv \frac{\tilde{E}}{\Lambda}\bar{g}(\bar{X}-\bar{Z}), \tag{4.4}$$

where $\tilde{E}$ is a modulus with dimension "stress". Hence, by (4.2) to (4.4), (4.1) transforms into:

$$\bar{T}_{p;x,t} = \tilde{\bar{T}}_{p;x,t}(\bar{X},\bar{t}) \equiv \tilde{\bar{T}}_{pt}(\bar{X}) \equiv \frac{T_{p;x,t}}{\tilde{E}} = \int\limits_{\bar{I}_0} \frac{1}{\sigma}[\bar{k}(\bar{X},\bar{t})-\bar{k}(\bar{Z},\bar{t})]\frac{d\bar{\mathfrak{h}}}{d\bar{S}}\bigg|_{\bar{S}=\frac{1}{\sigma}(\bar{X}-\bar{Z})} d\bar{Z}, \tag{4.5}$$

where, for simplicity, we have set $\bar{g}(\bar{X}-\bar{Z}) \equiv 1$ for all $\bar{X}, \bar{Z} \in \bar{I}_0$. This is the case whose consequences we shall develop further in the sequel.

4.2. The equation of motion

Respectively, the one-dimensional forms of CAUCHY'S first and second laws, expressed in terms of referential variables — $T(x) = \bar{T}(\chi_t(X)) = (\bar{T}\circ\chi_t)(X) \equiv T_t(X)$ — are:

$$\frac{dT_t}{dX}\cdot\frac{d\chi_t^{-1}}{dx}\bigg|_{x=\chi_t(X)} + \varrho b = \varrho\ddot{x}, \tag{4.6}$$

and, for each $t \in (-\infty, \infty)$ and all $X_0 \in I_0$,

$$T_t^+(X_0) \equiv T_t(X)|_{X=X_0^+} = T_t(X)|_{X=X_0^-} \equiv T_t^-(X_0), \tag{4.7}$$

where, respectively, b and $\ddot{x} = \partial^2\chi/\partial t^2$ are the referential forms of the body-force and acceleration fields acting on B_t; and $T_t^+(X_0)$ $[T_t^-(X_0)]$ is the stress acting at $x_0 = \chi_t(X_0)$ at time t in the positive [negative] direction of x.

Let

$$G\left(\frac{\bar{X}-\bar{Z}}{\sigma}\right) = \frac{d\bar{\mathfrak{h}}}{d\bar{S}}\bigg|_{\bar{S}=\frac{1}{\sigma}(\bar{X}-\bar{Z})} \equiv \bar{\mathfrak{h}}'\left(\frac{\bar{X}-\bar{Z}}{\sigma}\right) \quad \text{and} \quad \bar{k}'(\bar{X},\bar{t}) = \frac{\partial\bar{k}}{\partial\bar{X}}(\bar{X},\bar{t}). \tag{4.8}$$

Then, by (4.5),

$$\frac{d\tilde{\bar{T}}_{pt}}{d\bar{X}} = \int\limits_{\bar{I}_0}\left\{\frac{1}{\sigma^2}[\bar{k}(\bar{X},\bar{t})-\bar{k}(\bar{Z},\bar{t})]G'\left(\frac{\bar{X}-\bar{Z}}{\sigma}\right)+\frac{1}{\sigma}\bar{k}'(\bar{X},\bar{t})G\left(\frac{\bar{X}-\bar{Z}}{\sigma}\right)\right\}d\bar{Z}. \tag{4.9}$$

If in its dimensionless form (4.6) is to hold for *all* $\bar{X} \in \bar{I}_0$ and *all* $\bar{t} \in \mathscr{I}_t \equiv \mathscr{R} \equiv (-\infty, \infty)$, then [provided that the body force $b\circ\mathscr{X}_t$ is continuous on $\bar{I}_0 \times \mathscr{I}_t$ and the motion is once (twice) differentiable with respect to $\bar{X} \in \bar{I}_0$ $(t \in \mathscr{I}_t)$] the general spatial influence function must be a twice differentiable function of $\bar{S} \equiv \frac{1}{\sigma}(\bar{X}-\bar{Z})$.

For the case where for each $t \in \mathscr{I}_t$ and each $X \in I_0$, $(b\circ\mathscr{X}_t)(X) \equiv 0$ and $\ddot{\mathscr{X}}_t(X) \equiv 0$, on employing dimensionless variables and the notation of (4.5), (4.6) reduces to:

$$\frac{\tilde{E}}{\Lambda^2}\frac{d\tilde{\bar{T}}_{pt}}{d\bar{X}}\cdot\frac{d\bar{\mathscr{X}}_t^{-1}}{d\bar{x}}\bigg|_{\bar{x}=\bar{\mathscr{X}}(\bar{X},\bar{t})} = 0. \tag{4.10}$$

From (4.10) we can at once read off: *If for all* $\bar{X} \in \bar{I}_0$, $t \in \mathscr{I}_t$, $(d\bar{\mathscr{X}}_t^{-1}/d\bar{x})|_{\bar{x}=\bar{\mathscr{X}}(\bar{X},\bar{t})} \neq 0$, *then* $d\tilde{\bar{T}}_{pt}/d\bar{X} = 0$ *on* $\bar{I}_0$. Therefore, if the body is in equilibrium in the absence of a body-force- and an acceleration-field, then the "pure" stress state within it is a constant stress-field.

4.3. Existence of solutions

Cleaving to the static case, rewrite (4.5) as follows:

$$f(\bar{X}) = \bar{k}(\bar{X}) - \int_{\bar{I}_0} \mathscr{L}\left((\bar{Z};\bar{X},\frac{1}{\sigma}\right)\bar{k}(\bar{Z})d\bar{Z}, \tag{4.11}$$

where

$$f(\bar{X}) = \frac{\bar{T}_p}{p(\bar{X})}, \quad p(\bar{X}) = \bar{\mathfrak{h}}\left(\frac{\bar{X}-\bar{L}_2}{\sigma}\right) - \bar{\mathfrak{h}}\left(\frac{\bar{X}-\bar{L}_1}{\sigma}\right) \tag{4.12}$$

and

$$\mathscr{L}\left(\bar{Z};\bar{X},\frac{1}{\sigma}\right) = \frac{1}{\sigma p(\bar{X})} G\left(\frac{\bar{X}-\bar{Z}}{\sigma}\right).$$

Equation (4.11) is a linear integral equation of the FREDHOLM type. Let $\bar{\mathscr{L}}^2$ be the set of all square integrable real-valued functions on $\bar{I}_0$, equipped with the corresponding induced inner product. There exists a solution $\bar{k}(\bar{X}) \in \bar{\mathscr{L}}^2$ of (4.11) if and only if its adjoint homogeneous equation possesses only the trivial solution [6]. This is true for (4.11); hence, it has a solution.

Since the homogeneous counterpart of (4.11) has as its only solution $\bar{k}(\bar{X}) =$ constant on $\bar{I}_0$, the solution of (4.11) is unique up to a constant; i.e., a rigid-body motion.

5. Hooke's law

As has been shown in [1], Hookean materials are a special case of all materials with spatial cognitivity. For the one-dimensional static case, based on (4.5), we state:

THEOREM 5.1. *If the general spatial influence function* $\bar{\mathfrak{h}}$ *is such that:*

$$\lim_{\sigma\to 0} \bar{\mathfrak{h}}\left(\frac{1}{\sigma}(\bar{X}-\bar{Z}\right) = -\delta(\bar{X}-\bar{Z}), \tag{5.1}$$

where δ *is the* DIRAC *δ-function, then* $T_p = \tilde{E}e$, *where* $e = (d\bar{k}/d\bar{X})$ *and* $\tilde{E}$ *is a constant.*

In summary, as a guide for the construction of spatial influence functions $\bar{\mathfrak{h}}$, we have:

$$1. \quad \lim_{\left|\frac{1}{\sigma}(\bar{X}-\bar{Z})\right|\to\infty} \bar{\mathfrak{h}}\left(\frac{1}{\sigma}(\bar{X}-\bar{Z})\right) = 0, \quad \sigma \neq 0,$$

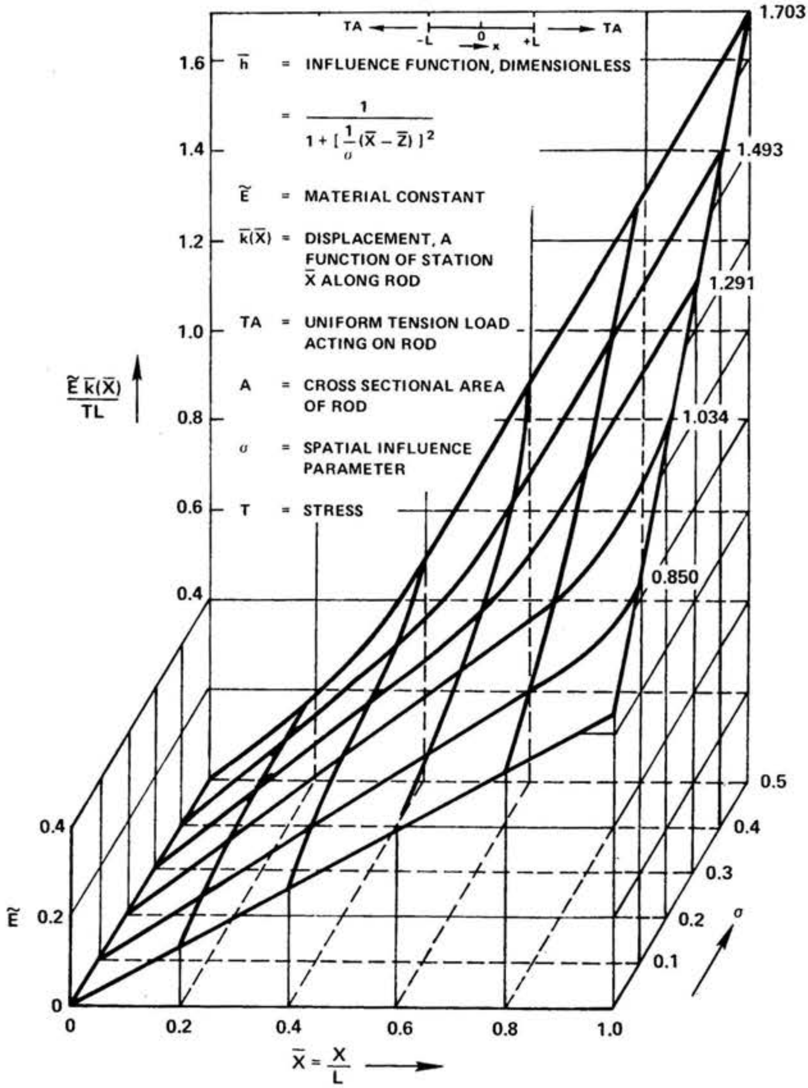
$$2. \quad \lim_{\sigma\to 0} \bar{\mathfrak{h}}\left(\frac{1}{\sigma}(\bar{X}-\bar{Z})\right) = -\delta(\bar{X}-\bar{Z}) \quad \text{for all} \quad \bar{X},\bar{Z} \in \bar{I}_0, \tag{5.2}$$

and

3. $\bar{\mathfrak{h}}$ must be at least a twice differentiable function with respect to $\bar{S} \equiv \frac{1}{\sigma}(\bar{X}-\bar{Z})$.

6. Examples

Two simple examples are presented in this section. The one-dimensional body under consideration is assumed to be in its static equilibrium state under the action of a *null* body-force field — then, as shown above, the stress-field will be a constant field on the body. The solution for one of these two examples exhibits the *boundary layer effect* mentioned in [1] and demonstrates the significance of the influence parameter σ for the medium. In the case of Example I the detailed calculations were performed on an IBM 7094 digital computer. A graph of the results is given in Fig. 1.



Dimensionless Displacement vs Dimensionless Stations for Various Values of the Spatial Influence Parameter σ

FIG. 1.

6.1. Example I

The spatial influence function $\bar{\mathfrak{h}}$ is chosen to be:

$$\bar{\mathfrak{h}}\left(\frac{1}{\sigma}(\bar{X}-\bar{Z})\right) \equiv -\frac{1}{\pi\sigma}\cdot\frac{1}{1+\left[\frac{1}{\sigma}(\bar{X}-\bar{Z})\right]^2}4. \tag{6.1}$$

This function satisfies all of the criteria (5.2). If the half-length L of the body is taken to be the characteristic length of the medium, then $(4.12)_{1,3}$ become:

$$\begin{aligned} p(\bar{X}) &= \frac{1}{\pi\sigma}\left[\frac{1}{1+[(\bar{X}-1)/\sigma]^2}-\frac{1}{1+[(\bar{X}+1)/\sigma]^2}\right],\\ \mathcal{L}\left(\bar{Z};\bar{X},\frac{1}{\sigma}\right) &= \frac{2}{\pi\sigma^3}\cdot\frac{1}{p(\bar{X})}\cdot\frac{\bar{X}-\bar{Z}}{\{1+[(\bar{X}-\bar{Z})/\sigma]^2\}^2}. \end{aligned} \tag{6.2}$$

It can be verified from (4.5) that as $\sigma \to 0$ in this case, $\bar{k}(\bar{X}) = \lambda\bar{X}$, where λ is a constant, will yield a constant stress. Therefore, for this medium, as $\sigma \to 0$, the deformation which will produce a constant stress distribution within the body is:

$$\chi(\bar{X}) = (1+\lambda)\bar{X}, \tag{6.3}$$

the simple-extension case. The corresponding stress is given by

$$T = \tilde{E}\lambda,$$

where $\tilde{E}$ may be identified with the usual YOUNG's modulus.

6.2. Example II

Choose

$$\bar{\mathfrak{h}}\left(\frac{1}{\sigma}(\bar{X}-\bar{Z})\right) = -\frac{1}{2\sqrt{\pi}\,\sigma}\exp\left\{-\left[\frac{1}{2\sigma}(\bar{X}-\bar{Z})\right]^2\right\}. \tag{6.4}$$

This spatial influence function satisfies conditions (5.2). Taking $\Lambda = L$, the half-length of the body, (4.5) reduces to:

$$T_p = -\frac{\tilde{E}}{4\sqrt{\pi}\,\sigma^3}\int_{-1}^{1}(\bar{X}-\bar{Z})e^{-[(\bar{X}-\bar{Z})/2\sigma]^2}\{\bar{k}(\bar{X})-\bar{k}(\bar{Z})\}\,d\bar{Z}. \tag{6.5}$$

It is easy to verify that, in the limit $\sigma \to 0$, (6.5) will yield a constant stress when $\bar{k}(\bar{X}) = \lambda\bar{X}$, where λ is a constant.

7. Concluding remarks

From the results exhibited in Fig. 1 it is evident that an "edge effect" occurs for small σ. As $\sigma \to 1$ this effect spreads into the body, as would be intuitively expected. Thus, it seems that we are justified in asserting that the spatial influence parameter σ does serve to partially characterize such an effect.

Naturally, explicit examples of two- and three-dimensional problems remain to be constructed and solved. Of particular interest would be boundary-value problems whose solution could be subjected to experimental verification.

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