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An analysis of elastic-plastic creep buckling of axi-symmetric shells by the finite element method (*)

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THE OBJECTIVE of the present paper is to show the application of the finite element method to the creep buckling analysis of axi-symmetric thin shells of revolution with and without plasticity effect. The numerical examples performed in this paper include the time dependent snap-through creep buckling phenomena of shallow spherical shells under time independent external pressure.

Celem pracy jest pokazanie zastosowania metody elementów skończonych do analizy wyboczenia pełzającego osiowo symetrycznych cienkich powłok obrotowych z uwzględnieniem i bez uwzględnienia efektów plastycznych. Przykłady numeryczne zamieszczone w tej pracy zawierają zjawiska przeskoku przy wyboczeniu pełzającym mało wyniosłych powłok sferycznych poddanych działaniu ciśnienia zewnętrznego niezależnego od czasu.

Целью работы является представление применения метода конечных элементов для анализа ползучего продольного изгиба осесимметричных, тонких, вращательных оболочек с учетом и без учета пластических эффектов. Численные примеры, помещенные в этой работе, содержат явления скачкообразного перехода при ползучем продольном изгибе пологих сферических оболочек, подвергнутых действию внешнего давления, независящего от времени.

1. Introduction

THE IMPORTANCE of creep buckling analyses is well known in designing high temperature structural components such as those used in chemical and reactor plants. The excellent reviews in this area are available in references [1 and 2]. As can be seen from these references, the creep buckling analyses of shells have been exclusively performed on cylindrical shells and few works have been done on the creep buckling of shells other than cylindrical.

On the other hand, the rapid development of the finite element method has made it feasible to undertake various classes of non-linear structural analyses [3, 4].

In this paper, the incremental formulation of elastic-plastic large creep deformation analysis of thin shells of revolution and its application to creep buckling analysis of shallow spherical shell in the pre- and post-buckling state are presented within the framework of the finite element displacement method. The method utilized here is the incremental procedure based on the Lagrangian technique in which all the displacements, stresses and strains are referred to in an initial state and nodal coordinates are kept fixed during the entire calculation [5].

Both material and geometrical non-linearities are taken into account in the incremental

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equations and the former consists of the time independent plasticity, i.e. von Mises yield criterion and the associated flow rule, and the time-dependent creep which is assumed constant during the small time increment. The creep buckling time of the present problem is defined as the time at which the stiffness of the system equation is found to be singular, and the snap-through buckling analysis [6] is then performed until the stable condition appears again, after which the usual analysis of creep large deformation is continued step by step.

2. Theory

An axi-symmetric shell is represented as a series of conical frusta, as shown in Fig. 1 Each conical frustum is assumed to correspond to an element.

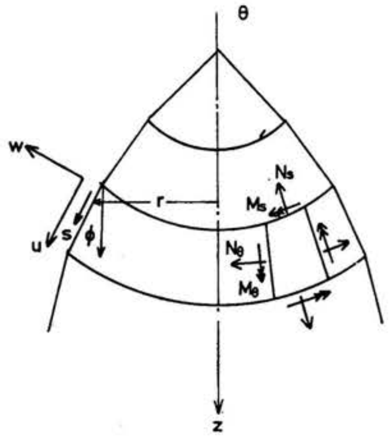


FIG. 1. Axi-symmetric shell represented as a series of conical frusta.

2.1. Stress-strain relation

The relation between stress vector $\boldsymbol{\sigma}$ and elastic strain vector $\boldsymbol{\epsilon}^e$ is given as the following expression in incremental form:

$$\Delta\boldsymbol{\sigma} = \mathbf{D}_1^e \Delta\boldsymbol{\epsilon}^e, \tag{2.1}$$

where the prefix Δ denotes an increment, $\Delta\boldsymbol{\sigma}$ and $\Delta\boldsymbol{\epsilon}^e$ are the following vectors, respectively:

$$\Delta\boldsymbol{\sigma} = \begin{Bmatrix} \Delta\sigma_s \\ \Delta\sigma_\theta \end{Bmatrix}, \tag{2.2}$$

$$\Delta\boldsymbol{\epsilon}^e = \begin{Bmatrix} \Delta\varepsilon_s \\ \Delta\varepsilon_\theta \end{Bmatrix}, \tag{2.3}$$

in which subscripts s and θ represent meridian and circumferential components, respectively. For isotropic material, $\mathbf{D}_1^e$ becomes

$$\mathbf{D}_1^e = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}, \tag{2.4}$$

where E and ν are the Young's modulus and Poisson's ratio, respectively.

For material undergoing creep and plasticity, the incremental elastic strain vector may be expressed in the form:

$$\Delta \boldsymbol{\epsilon}^e = \Delta \boldsymbol{\epsilon} - \Delta \boldsymbol{\epsilon}^p - \Delta \boldsymbol{\epsilon}^c, \tag{2.5}$$

where $\Delta \boldsymbol{\epsilon}$ is the incremental total strain vector, $\Delta \boldsymbol{\epsilon}^p$ is the incremental plastic strain vector and $\Delta \boldsymbol{\epsilon}^c$ is the incremental creep strain vector.

For creep strain, Mises-Mises' creep theory is assumed to apply. Then, $\Delta \boldsymbol{\epsilon}^c$ is written as follows [7]

$$\Delta \boldsymbol{\epsilon}^c = \mathbf{Z} \Delta \bar{\varepsilon}^c = \mathbf{Z} \dot{\bar{\varepsilon}}^c \Delta t, \tag{2.6}$$

in which $\Delta \bar{\varepsilon}^c$ is the incremental equivalent creep strain, $\dot{\bar{\varepsilon}}^c$ is the equivalent creep strain rate characterized by uniaxial creep test and Δt is the time increment. On the other hand, $\mathbf{Z}$ in the Eq. (2.6) is represented as follows by the equivalent stress $\bar{\sigma}$ and the second deviatoric stress invariant J:

$$\mathbf{Z} = \frac{3}{2\bar{\sigma}} \frac{\partial J}{\partial \boldsymbol{\sigma}}. \tag{2.7}$$

Here, $\bar{\sigma}$ and J are defined by

$$\bar{\sigma}^2 = \sigma_s^2 + \sigma_\theta^2 - \sigma_s \sigma_\theta, \quad J = \frac{\bar{\sigma}^2}{3}. \tag{2.8}$$

From the Eqs. $(2.8)_1$ and $(2.8)_2$, the Eq. (2.7) becomes

$$\mathbf{Z} = \frac{3}{2\bar{\sigma}} \begin{Bmatrix} \sigma_s' \\ \sigma_\theta' \end{Bmatrix}, \tag{2.9}$$

in which σ_s' and σ_θ' are deviatoric stress components defined by

$$\sigma_s' = \sigma_s - \frac{\sigma_s + \sigma_\theta}{3}, \quad \sigma_\theta' = \sigma_\theta - \frac{\sigma_s + \sigma_\theta}{3}. \tag{2.10}$$

In order to calculate the incremental plastic strain components, the von Mises yield criterion and the associated flow rule are utilized. From the theory of plastic potential, the incremental plastic strain components must be normal to the yield surface, which gives the equation:

$$\Delta \boldsymbol{\epsilon}^p = \frac{\partial F}{\partial \boldsymbol{\sigma}} \lambda, \tag{2.11}$$

where F is the yield function defined by

$$F = \bar{\sigma}^2 = 3J \tag{2.12}$$

and λ represents the normality parameter. From the Eqs. (2.7) and (2.12) the incremental equivalent stress can be calculated as

$$\Delta \bar{\sigma} = \mathbf{Z}^T \Delta \boldsymbol{\sigma}, \tag{2.13}$$

where the superscript T represents the transpose of matrix. The incremental plastic work ΔW_p can be written either in the form:

$$\Delta W_p = \boldsymbol{\sigma}^T \Delta \boldsymbol{\epsilon}^p, \tag{2.14}$$

or

$$\Delta W_p = \bar{\sigma}\Delta\bar{\varepsilon}^p, \tag{2.15}$$

in which $\Delta\bar{\varepsilon}^p$ is the incremental equivalent plastic strain. Introducing the Eq. (2.11) into the Eq. (2.14) and using the Eq. (2.15) yield

$$\boldsymbol{\sigma}^T \frac{\partial F}{\partial \boldsymbol{\sigma}} \lambda = \bar{\sigma}\Delta\bar{\varepsilon}^p. \tag{2.16}$$

From the Eqs. (2.12) and (2.16), we have

$$\lambda = \frac{1}{2\bar{\sigma}}\Delta\bar{\varepsilon}^p. \tag{2.17}$$

In view of the Eqs. (2.7), (2.11), (2.12) and (2.17), the incremental plastic strain vector may be written as

$$\Delta\boldsymbol{\epsilon}^p = \mathbf{Z}\Delta\bar{\varepsilon}^p. \tag{2.18}$$

Next, introducing the Eq. (2.5) into the Eq. (2.1) yields

$$\Delta\boldsymbol{\sigma} = \mathbf{D}_1^e(\Delta\boldsymbol{\epsilon} - \Delta\boldsymbol{\epsilon}^p) - \mathbf{D}_1^e\Delta\boldsymbol{\epsilon}^c. \tag{2.19}$$

Substituting the Eq. (2.19) into the Eq. (2.13) and assuming the relation:

$$\Delta\bar{\sigma} = E_p\Delta\bar{\varepsilon}^p \tag{2.20}$$

to be valid where E_p represents the uniaxial hardening modulus in stress-strain curve, the incremental equivalent plastic strain can be calculated as

$$\Delta\bar{\varepsilon}^p = \frac{1}{H}(\mathbf{Z}^T\mathbf{D}_1^e\Delta\boldsymbol{\epsilon}^e - \mathbf{Z}^T\mathbf{D}_1^e\Delta\boldsymbol{\epsilon}^c), \tag{2.21}$$

where the Eq. (2.18) is utilized and

$$H = E_p + \mathbf{Z}^T\mathbf{D}_1^e\mathbf{Z}. \tag{2.22}$$

In view of the Eqs. (2.6), (2.18), (2.19) and (2.21), the incremental stress-strain relation is given by

$$\Delta\boldsymbol{\sigma} = (\mathbf{D}_1^e + \mathbf{D}_1^p)\Delta\boldsymbol{\epsilon} - (\mathbf{D}_1^e + \mathbf{D}_1^p)\mathbf{Z}\dot{\bar{\varepsilon}}^c\Delta t \tag{2.23}$$

in which

$$\mathbf{D}_1^p = -\mathbf{D}_1^e\mathbf{Z}\mathbf{Z}^T\mathbf{D}_1^e/H. \tag{2.24}$$

Using the Eqs. (2.4) and (2.9), H and $\mathbf{D}_1^p$ are given as follows:

$$\begin{aligned} H &= E_p + \frac{9}{4\bar{\sigma}^2}(\sigma_s' S_1 + \sigma_\theta' S_2), \\ \mathbf{D}_1^p &= \frac{-1}{S}\begin{bmatrix} S_1^2 & S_1 S_2 \\ S_1 S_2 & S_2^2 \end{bmatrix}, \end{aligned} \tag{2.25}$$

where

$$S_1 = \frac{E}{1-\nu^2}(\sigma_s' + \nu\sigma_\theta'), \quad S_2 = \frac{E}{1-\nu^2}(\nu\sigma_s' + \sigma_\theta') \tag{2.26}$$

and

$$S = \frac{4}{9} H \bar{\sigma}^2. \tag{2.27}$$

Assuming the Kirchhoff-Love hypothesis, $\Delta \boldsymbol{\epsilon}$ is expressed by

$$\Delta \boldsymbol{\epsilon} = \begin{Bmatrix} \Delta e_s - \bar{z}\Delta\chi_s \\ \Delta e_\theta - \bar{z}\Delta\chi_\theta \end{Bmatrix}, \tag{2.28}$$

in which e_s, e_θ and χ_s, χ_θ indicate strains and curvature changes of the middle surface, respectively, and $\bar{z}$ is the distance from the middle surface of the shell.

Substituting the Eqs. (2.4), (2.9), (2.27) and (2.28) into the Eq. (2.23), the incremental stress-strain relation is written as

$$\Delta \boldsymbol{\sigma} = (\mathbf{D}_1^e + \mathbf{D}_1^p)\Delta\boldsymbol{\epsilon} - (\mathbf{D}_1^e + \mathbf{D}_1^p)\mathbf{Z}\dot{\bar{\varepsilon}}^c \Delta t = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}\begin{Bmatrix} \Delta e_s - \bar{z}\Delta\chi_s \\ \Delta e_\theta - \bar{z}\Delta\chi_\theta \end{Bmatrix}$$
$$- \frac{1}{S}\begin{bmatrix} S_1^2 & S_1 S_2 \\ S_1 S_2 & S_2^2 \end{bmatrix}\begin{Bmatrix} \Delta e_s - \bar{z}\Delta\chi_s \\ \Delta e_\theta - \bar{z}\Delta\chi_\theta \end{Bmatrix} - \frac{3\dot{\bar{\varepsilon}}^c \Delta t}{2\bar{\sigma}}\left(\begin{Bmatrix} S_1 \\ S_2 \end{Bmatrix} - \frac{1}{S}\begin{Bmatrix} S_1(S_1\sigma_s' + S_2\sigma_\theta') \\ S_2(S_1\sigma_s' + S_2\sigma_\theta') \end{Bmatrix}\right). \tag{2.29}$$

Next, let N_i and M_i, $i = S, \theta$ represent membrane and moment resultants, respectively, as shown in Fig. 1. The incremental forms of these resultants are assumed to be defined by

$$\begin{aligned} \Delta N_s &= \int_{-h/2}^{h/2} \Delta\sigma_s d\bar{z}, & \Delta N_\theta &= \int_{-h/2}^{h/2} \Delta\sigma_\theta d\bar{z}, \\ \Delta M_s &= \int_{-h/2}^{h/2} \Delta\sigma_s \bar{z} d\bar{z}, & \Delta M_\theta &= \int_{-h/2}^{h/2} \Delta\sigma_\theta \bar{z} d\bar{z}, \end{aligned} \tag{2.30}$$

where h is shell thickness.

Introducing the Eq. (2.29) into the Eq. (2.30), the generalized stress-strain relation is given as follows:

$$\Delta \mathbf{N} = \begin{Bmatrix} \Delta N_s \\ \Delta N_\theta \\ -\Delta M_s \\ -\Delta M_\theta \end{Bmatrix} = (\mathbf{D}^e + \mathbf{D}^p)\Delta\mathbf{e} - (\Delta\mathbf{C}_e + \Delta\mathbf{C}^p), \tag{2.31}$$

in which

$$\Delta \mathbf{e} = \begin{Bmatrix} \Delta e_s \\ \Delta e_\theta \\ \Delta\chi_s \\ \Delta\chi_\theta \end{Bmatrix},$$

$$\mathbf{D}^e = \frac{Eh}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 & 0 \\ & 1 & 0 & 0 \\ \text{SYM.} & & h^2/12 & h^2\nu/12 \\ & & & h^2/12 \end{bmatrix}, \tag{2.32}$$

$$\mathbf{D}^p = \int_{-h/2}^{h/2} \frac{1}{S}\begin{bmatrix} S_1^2 & S_1 S_2 & -\bar{z}S_1^2 & -\bar{z}S_1 S_2 \\ & S_2^2 & -\bar{z}S_1 S_2 & -\bar{z}S_2^2 \\ & & \bar{z}^2 S_1^2 & \bar{z}^2 S_1 S_2 \\ \text{SYM.} & & & \bar{z}^2 S_2^2 \end{bmatrix} d\bar{z},$$

$$\Delta \mathbf{C}^e = \frac{3\Delta t}{2} \int_{-h/2}^{h/2} \begin{Bmatrix} S_1 \\ S_2 \\ -S_1 \bar{z} \\ -S_2 \bar{z} \end{Bmatrix} \frac{\dot{\bar{\varepsilon}}^c}{\bar{\sigma}} d\bar{z},$$

and

$$\Delta \mathbf{C}^p = -\frac{3\Delta t}{2} \int_{-h/2}^{h/2} \begin{Bmatrix} S_1 \\ S_2 \\ -S_1 \bar{z} \\ -S_2 \bar{z} \end{Bmatrix} \frac{S_1 \sigma_s' + S_2 \sigma_\theta'}{S\bar{\sigma}} \dot{\bar{\varepsilon}}^c d\bar{z}. \tag{2.33}$$

2.2. Displacement

Figure 2 shows an axi-symmetric shell element having two nodes i and j, in which s is the meridian distance. The nodal displacement is prescribed by $\bar{u}$, $\bar{w}$ and β, as shown in Fig. 2. Therefore, the displacement of node i can be defined by three components as

$$\mathbf{d}_i = \begin{Bmatrix} \bar{u}_i \\ \bar{w}_i \\ \bar{\beta}_i \end{Bmatrix}. \tag{2.34}$$

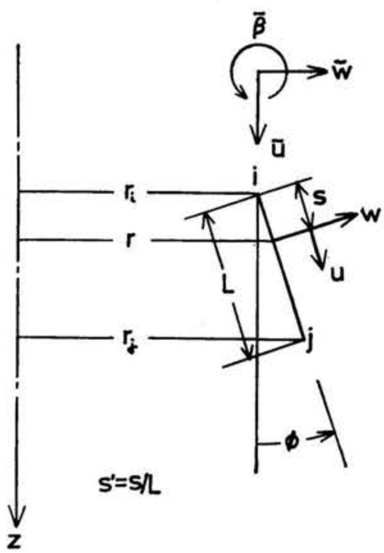


FIG. 2. An element of an axi--symmetric shell.

Thus, the displacement vector $\mathbf{d}$ of the element with two nodes i and j is given by

$$\mathbf{d} = \begin{Bmatrix} \mathbf{d}_i \\ \mathbf{d}_j \end{Bmatrix}. \tag{2.35}$$

The displacement $\mathbf{v}$ within an element is assumed to be interpolated by the nodal displacement $\mathbf{d}$. The shape function which relates these two vectors is determined so as to satisfy the condition of continuity of u, w and dw/ds between neighbouring elements. Then the in-plane displacement u can be taken varying linearly with s and the transverse displacement w as a cubic in s. The relation between $\mathbf{v}$ and $\mathbf{d}$ is expressed with $\mathbf{0}$ being null matrix in the form:

(2.36) $$\mathbf{v} = \begin{Bmatrix} u \\ w \end{Bmatrix} = \mathbf{N}'_d \begin{bmatrix} \boldsymbol{\lambda} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda} \end{bmatrix} \mathbf{d} \equiv \mathbf{N}_d \mathbf{d},$$

where $\boldsymbol{\lambda}$ is the matrix of transformation from $(\bar{u}, \bar{w}, \bar{\beta})$ to $(u, w, dw/ds)$ in Fig. 2 and given by

(2.37) $$\boldsymbol{\lambda} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in which ϕ is the angle between element and axis of rotation. In the Eq. (2.36) $\mathbf{N}'_d$ can be written in the following form:

(2.38) $$\mathbf{N}'_d = \begin{bmatrix} 1-s' & 0 & 0 & s' & 0 & 0 \\ 0 & 1-3s'^2+2s'^3 & L(s'-2s'^2+s'^3) & 0 & 3s'^2-2s'^3 & L(-s'^2+s'^3) \end{bmatrix},$$

where $s' = s/L$, and s' varies from 0 to 1, since L is the length of the element.

In incremental form, the Eq. (2.36) may be transformed into

(2.39) $$\Delta\mathbf{v} = \mathbf{N}_d \Delta\mathbf{d}.$$

2.3. Strain-diplacement relation

The strain-displacement relation is given as follows, with the effect of geometrical non-linearity taken into account:

(2.40) $$e_s = \frac{du}{ds} + \frac{1}{2}\left(\frac{dw}{ds}\right)^2, \qquad \chi_s = \frac{d^2 w}{ds^2},$$
$$e_\theta = (u\sin\phi + w\cos\phi)/r, \quad \chi_\theta = \frac{\sin\phi}{r}\frac{dw}{ds},$$

in which r is the distance between the axis of rotation and the middle surface of the shell. The incremental form of the strain-displacement relation can be expressed as

(2.41) $$\Delta e_s = \frac{d\Delta u}{ds} + \frac{dw}{ds}\frac{d\Delta w}{ds} + \frac{1}{2}\left(\frac{d\Delta w}{ds}\right)^2, \quad \Delta\chi_s = \frac{d^2\Delta w}{ds^2},$$
$$\Delta e_\theta = (\Delta u\sin\phi + \Delta w\cos\phi)/r, \quad \Delta\chi_\theta = \frac{\sin\phi}{r}\frac{d\Delta w}{ds}.$$

Since the third term of the right-hand side in the expression of Δe_s is an infinitesimally small quantity of second order, it is omitted here in the incremental expression except in the derivation of the geometric stiffness matrix. The Eq. (2.41) is divided into a linear part and a non-linear one with respect to incremental value as follows:

(2.42) $$\Delta\mathbf{e} = \Delta\mathbf{e}_L + \Delta\mathbf{e}_{NL},$$

where

$$(2.43)\qquad \Delta \mathbf{e}_L = \begin{bmatrix} \frac{d\Delta u}{ds} + \frac{dw}{ds}\frac{d\Delta w}{ds} \\ (\Delta u \sin\phi + \Delta w \cos\phi)/r \\ \frac{d^2 \Delta w}{ds^2} \\ \frac{\sin\phi}{r}\frac{d\Delta w}{ds} \end{bmatrix}, \quad \Delta \mathbf{e}_{NL} = \begin{bmatrix} \frac{1}{2}\left(\frac{d\Delta w}{ds}\right)^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Using the increment of the nodal displacement $\Delta\mathbf{d}$, $\Delta\mathbf{e}_L$ may be written as follows:

$$(2.44)\qquad \Delta \mathbf{e}_L = (\mathbf{B}_1' + \mathbf{B}_2')\begin{bmatrix} \boldsymbol{\lambda} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda} \end{bmatrix}\Delta\mathbf{d} \equiv (\mathbf{B}_1 + \mathbf{B}_2)\Delta\mathbf{d} \equiv \mathbf{B}\Delta\mathbf{d},$$

where $\mathbf{B}_1'$ or $\mathbf{B}_1$ is independent of the change of shell geometry and is given by

$$(2.45)\qquad \mathbf{B}_1' = \left[\begin{matrix} -1/L & 0 & 0 \\ (1-s')\sin\phi/r & (1-3s'^2+2s'^3)\cos\phi/r & L(s'-2s'^2+s'^3)\cos\phi/r \\ 0 & -(6-12s')/L^2 & -(4-6s')/L \\ 0 & -(6s'-6s'^2)\sin\phi/rL & -(-1+4s'-3s'^2)\sin\phi/r \end{matrix}\right.$$

$$\left.\begin{matrix} 1/L & 0 & 0 \\ s'\sin\phi/r & (3s'^2-2s'^3)\cos\phi/r & L(-s'^2+s'^3)\cos\phi/r \\ 0 & -(12s'-6)/L^2 & -(2-6s')/L \\ 0 & -(-6s'+6s'^2)\sin\phi/rL & -(2s'-3s'^2)\sin\phi/r \end{matrix}\right],$$

whereas $\mathbf{B}_2'$ or $\mathbf{B}_2$ depends on the change of shell geometry and is given by

$$(2.46)\qquad \mathbf{B}_2' = \begin{bmatrix} \mathbf{GdG} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

In the Eq. (2.46), matrix $\mathbf{G}$ is defined by

$$(2.47)\qquad \mathbf{G} = \mathbf{C}\begin{bmatrix} \boldsymbol{\lambda} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda} \end{bmatrix},$$

in which

$$(2.48)\qquad \mathbf{C} = [0 \quad (-6s'+6s'^2)/L \quad 1-4s'+3s'^2 \quad 0 \quad (6s'-6s'^2)/L \quad -2s'+3s'^2].$$

Using the Eq. (2.47), $\Delta\mathbf{e}_{NL}$ is written in the following form:

$$(2.49)\qquad \Delta \mathbf{e}_{NL} = \frac{1}{2}(\mathbf{G}\Delta\mathbf{d})^2 \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}.$$

2.4. Principle of virtual work

Provided the boundary condition are homogeneous, the principle of virtual work is given as follows [8]:

$$\int_A \delta(\Delta \mathbf{e})^T(\mathbf{N}+\Delta \mathbf{N})dA - \int_A \delta(\Delta \mathbf{v})^T(\mathbf{F}+\Delta \mathbf{F})dA = 0, \tag{2.50}$$

in which $\mathbf{F}$ is distributed load per unit area of the middle surface, A is area of the element and δ denotes a virtual variation. Here, dA is written as follows:

$$dA = 2\pi r ds = 2\pi r L ds'. \tag{2.51}$$

Using the Eq. (2.51), the Eq. (2.50) can be transformed into

$$\int_0^1 \delta \Delta \mathbf{e}^T(\mathbf{N}+\Delta \mathbf{N})\, rLds' - \int_0^1 \delta(\Delta \mathbf{v})^T(\mathbf{F}+\Delta \mathbf{F}) rLds' = 0. \tag{2.52}$$

Introducing the Eqs. (2.31), (2.39), (2.42), (2.44) and (2.49) into the Eq. (2.52), neglecting higher-order terms with respect to incremental values and assuming that equilibrium is completely satisfied at the preceeding step, the final form of incremental finite element equation of equilibrium is found to be

$$(\mathbf{k}^e+\mathbf{k}^p+\mathbf{k}^g)\Delta \mathbf{d} = \Delta \mathbf{F}_A + \Delta \mathbf{F}_c^e + \Delta \mathbf{F}_c^p, \tag{2.53}$$

where

$$\mathbf{k}^e = \int_0^1 \mathbf{B}^T \mathbf{D}^e \mathbf{B} rLds', \quad \mathbf{k}^p = \int_0^1 \mathbf{B}^T \mathbf{D}^p \mathbf{B} rLds',$$

$$\mathbf{k}^g = \int_0^1 \mathbf{G}^T N_s \mathbf{G} rLds', \tag{2.54}$$

$$\Delta \mathbf{F}_A = \int_0^1 \mathbf{N}_d^T \Delta \mathbf{F} rLds', \quad \Delta \mathbf{F}_c^e = \int_0^1 \mathbf{B}^T \Delta \mathbf{C}^e rLds',$$

and

$$\Delta \mathbf{F}_c^p = \int_0^1 \mathbf{B}^T \Delta \mathbf{C}^p rLds'. \tag{2.55}$$

Matrices $\mathbf{k}^e$, $\mathbf{k}^p$ and $\mathbf{k}^g$ are the elastic stiffness matrix, the plastic stiffness matrix and the geometric stiffness matrix, respectively. Vectors $\Delta \mathbf{F}_c^e$ and $\Delta \mathbf{F}_c^p$ are increments of elastic and plastic creep load vectors, respectively.

Superposing the Eq. (2.53) for the complete structure, the equation of equilibrium of the total system is given as follows:

$$\sum_{i=1}^{N} (\mathbf{k}^e+\mathbf{k}^p+\mathbf{k}^g)\, \Delta \mathbf{d} = \sum_{i=1}^{N} (\Delta \mathbf{F}_A + \Delta \mathbf{F}_c^e + \Delta \mathbf{F}_c^p), \tag{2.56}$$

in which N is the number of elements.

3. Method of calculation

The process of calculation is described briefly in what follows in the case of the creep buckling of a shallow spherical shell undergoing elastic-plastic and creep deformation.

At time $t = 0$

(i) Let us assume $\Delta \mathbf{F}_c$ be zero in the Eq. (2.56) and calculate the elastic-plastic solution with large deformation effect by the usual incremental method, in which the load is increased statically until the desired quantity is reached.

At $t > 0$

(ii) Modify $\mathbf{k}^e$ and $\mathbf{k}^g$ with respect to the new geometry and N_s at the preceeding step. If the material is found yielding, calculate $\mathbf{k}^p$.

(iii) Calculate $\Delta \mathbf{C}^e$. If the element is found yielding, calculate $\Delta \mathbf{C}^p$ together with $\Delta \mathbf{C}^e$.

(iv) Carry out the processes of (ii) and (iii) for all elements and construct the equation of equilibrium of the total system (2.56).

(v) Solve the Eq. (2.56) to obtain the displacement increment $\Delta \mathbf{d}$ and calculate $\Delta \boldsymbol{\sigma}$ and $\Delta \mathbf{N}$ from $\Delta \mathbf{d}$ thus determined. Then, add $\Delta \mathbf{d}$, $\Delta \boldsymbol{\sigma}$ and $\Delta \mathbf{N}$ to the values of the preceeding step to obtain $\mathbf{d}$, $\boldsymbol{\sigma}$ and $\mathbf{N}$.

(vi) Calculate $\bar{\sigma}$ to find the yielding of any region of the structure.

(vii) Judge the stability of the equilibrium state obtained from the Eq. (2.56). If the equilibrium state is stable, proceed to the next time step and return to (ii). If the equilibrium state is not stable, the snap-through analysis [6] is carried out at that momentum until the original load recovers, in which the load-deflection relation shows the well known "*S*" shaped curve and critical time Tc is defined by this moment.

(viii) Proceed to the next time step and return to (ii).

To obtain $\Delta \mathbf{C}^e$ and $\Delta \mathbf{C}^p$ in the Eqs. $(2.32)_4$ and (2.33), respectively, each element is divided into 20 layers and average values of σ_s', σ_θ' and $\bar{\sigma}$ at each layer of the preceding step are utilized in the computation. The Gauss quadrature formula is employed to calculate each stiffness matrix and load vector in the Eq. $(2.54)_1$ through the Eq. (2.55).

To determine the stability of the equilibrium state, the positive definiteness of the stiffness matrix of the system may be adopted as an important information. In this paper, each diagonal element of the upper triangular matrix obtained in applying the forward process of the Gauss elimination method to the pertinent matrix is traced, since the matrix is known to be positive definite when all of these elements are positive numerals [9].

The time increment Δt is determined in the calculation such that the increment of the displacement does not exceed the prescribed reference value. In the present case, the deflection of the central point of a spherical shell is taken as the representative displacement and 1/800 of the shell thickness is adopted as the prescribed reference value.

4. Results and discussion

An elastic-plastic creep buckling analysis of a clamped shallow spherical shell under uniform pressure P, Fig. 3 is presented, where the dimensions of the shell are as follows:

Radius of curvature $R = 786.25\,\text{mm}$,

Base-plane edge radius $r = 125\,\text{mm}$,

Shell height $W_0 = 10\,\text{mm}$,
Shell thickness $h = 10\,\text{mm}$.

The material constants and creep law are assumed as

Young's modulus $E = 7000\,\text{kg/mm}^2$,
Poisson's ratio $\nu = 0.345$,
Yield stress $\sigma_y = 11\,\text{kg/mm}^2$,
Hardening modulus $E_p = E/200$
Creep strain $\varepsilon^c = A\sigma^n t^m (\sigma\colon \text{kg/mm}^2,\ t\colon hr)$,

where

$A = 4.64 \times 10^{-8}$, $(\text{kg/mm}^2)^{-n}\,(hr)^{-m}$,
$n = 4.43$, $m = 0.273$.

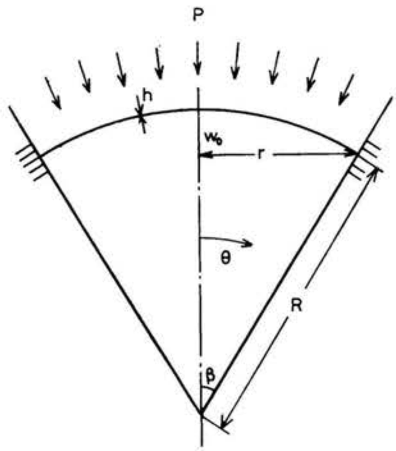


FIG. 3. Shallow spherical shell with clamped edge under uniform external pressure.

The strain hardening rule is employed for the creep calculation and the shell is idealized by 25 conical shell elements. The external pressure P of $26.5\,\text{kg/cm}^2$ is applied to the shell, which is selected so as to be smaller than the upper buckling pressure of $28.6\,\text{kg/cm}^2$ obtained by static analysis.

Figures 4 and 5 display the central deflection of the shell *vs* time and the meridional

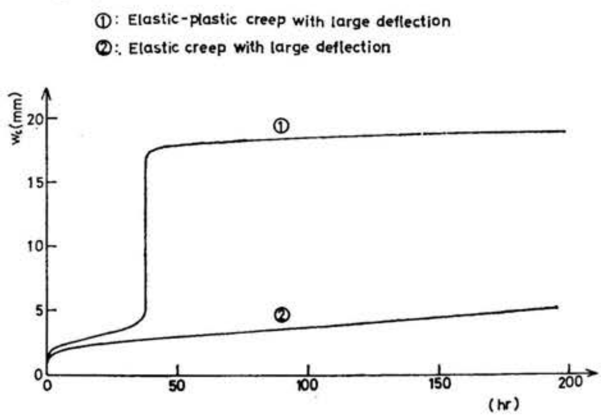


FIG. 4. Variation of central deflection W_C with time.

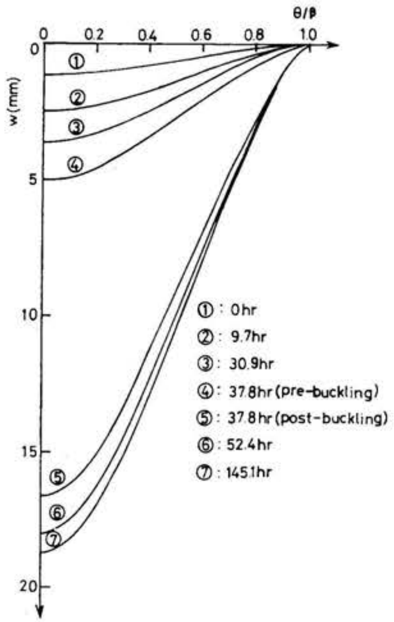
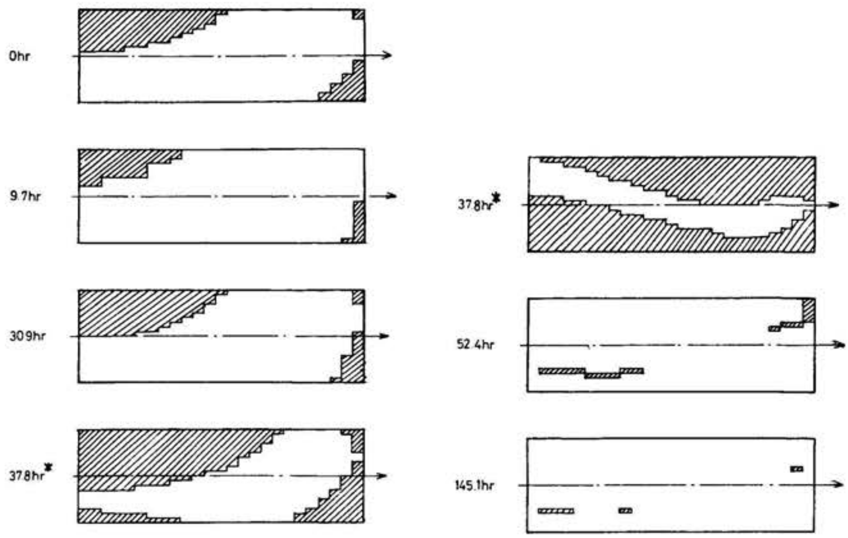


FIG. 5. Meridian distributions of deflection W.



* pre - buckling state

*post - buckling state

FIG. 6. Variation of developed plastic region with time.

distributions of the deflection W, respectively. It is found from the former that the instability does not occur if plastic deformation is not taken into consideration in the theory as far as the time elapsed of 200 hours, whereas the critical time of 37.8 hours is observed in the case of elastic-plastic creep with large deflection, at which the equilibrium becomes unstable and the snap-through buckling is found. After this, the shell deforms again stably by creep with the elapsed time. The process of deformation during the snap-through buckling can be observed in Fig. 5. It is worthy of note that the shell deforms over the shell height at the post-buckling state which is apparently the stable configuration.

Figure 6 illustrates the changes of plastic regions with time. It may be observed that the plastic region increases with respect to time before snap-through buckling, whereas it decreases after this critical time. From Fig. 6, it can be concluded that the reduction of the stiffness due to the increasing plastic regions plays an important part in snap-through buckling.

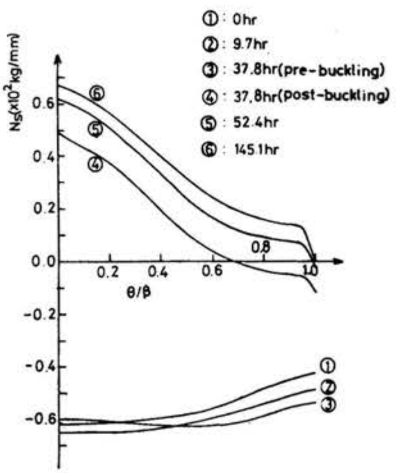


FIG. 7. Meridian distributions of stress resultant Ns.

Finally, Fig. 7 shows the meridian distributions of the membrane stress resultant Ns, from which Ns is observed to change from compression to tension suddenly at the critical time.

It can be concluded from the limited example presented here that the plastic deformation has an important effect on the creep buckling phenomenon of the shallow spherical shell.

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