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Symmetric waves in materials with internal state variables (*)

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CONDITIONS for the existence of symmetric waves are given. A non-linear Cattaneo's heat conduction constitutive equation is found and heat flux becomes an internal state variable. It is proved under certain conditions that there are two real symmetric and coupled acceleration waves propagating with finite speeds. The shock waves of small amplitude propagate with a velocity close to the adiabatic sound speed.

Podano warunki istnienia fal symetrycznych. Wyprowadzono nieliniowe równanie przewodnictwa ciepła Cattaneo. Strumień ciepła występuje jako wewnętrzna zmienna stanu. Udowodniono przy niektórych warunkach, że istnieją dwie rzeczywiste symetryczne i sprzężone fale przyśpieszenia, rozprzestrzeniające się ze skończonymi prędkościami. Fale uderzeniowe o małej amplitudzie rozprzestrzeniają się z prędkością bliską adiabatycznej prędkości dźwięku.

Даются условия существования симметричных волн. Выведено нелинейное уравнение теплопроводности Каттанео. Поток тепла выступает как внутренняя переменная состояния. Доказано, при некоторых условиях, что существуют две действительные симметричные и сопряженные волны ускорения, распраcтраняющиеся с конечными скоростями. Ударные волны малой амплитуды распространяются со скоростью близкой адиабатической скорости звука.

1. Introduction

THE ONE-DIMENSIONAL theory is considered. The framework concerning internal state variables is that of COLEMAN and GURTIN [1]. The present work may be considered as a further development of KOSIŃSKI and PERZYNA's paper [2] in which some additional assumptions are made. These assumptions are inspired by CATTANEO's work [10, 11] on hyperbolic heat conduction constitutive assumption (used instead of Fourier parabolic heat conduction constitutive assumption). See also VERNOTTE [16, 17], KALISKI [12], CHESTER [27], BAUMEISTER and HAMILL [18], MEIXNER [28], TAITEL [29] for the linear hyperbolic heat conduction constitutive equation; LYKOV [19], LORD and SCHULMAN [20], POPOV [21], ACHENBACH [22], KALISKI [23], TOKUOKA [26], for coupled linear thermoelasticity with a total hyperbolic system of equations. For the non-linear constitutive equations see SULICIU [13], GURTIN and PIPKIN [14], CHEN [24], KOSIŃSKI and PERZYNA [2], MCCARTHY [32]. BOGY and NAGHDI [33] found that the acceleration waves are not generally symmetric with respect to the direction of propagation, though they become symmetric when the corresponding constitutive equations are linearized (see also CHEN [25]).

It is proved, under the constitutive assumptions (3.1), (3.2), (3.11) and (3.12), that the acceleration waves are symmetric and, if the conditions of Proposition 4.1 and (5.4)–(5.6)

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are satisfied, the accelerations waves are real in the neighbourhood of a totally relaxed state. The obtained acceleration waves are coupled i.e. they carry jump discontinuities of both mathematical and thermal quantities. It is also proved that heat flux does not jump across the shock waves.

When only thermal waves are considered, the second law of thermodynamics together with (5.5) imply here that the acceleration wave velocities are real (and symmetric), while in the framework of I. MÜLER [15] these velocities might only be real, i.e. the second law does not imply that acceleration waves are necessarily real.

2. Preliminaries

A material point X of a one-dimensional body $\mathscr{B}$ is characterized here, by two groups of four functions: F, S, N, Q and $\overline{F}, \overline{S}, \overline{N}, \overline{Q}$. These functions have the following meaning: if at a time t, the one-dimensional strain $\varepsilon = (\partial\chi/\partial X)-1$ (where $x = \chi(X, t)$ gives the motion of the point $X \in \mathscr{B}$), the absolute temperature θ and its gradient $g = \partial\theta/\partial X$ and the internal state variables $\overline{\psi}, \overline{\sigma}, \overline{\eta}$ and $\overline{q}$ are known, then the free energy ψ, the one-dimensional stress σ, the entropy η and the heat flux q are determined by

$$
(2.1)\qquad
\begin{aligned}
\psi &= F(\varepsilon, \theta, g; \overline{\sigma}, \overline{\eta}, \overline{\psi}, \overline{q}),\\
\sigma &= S(\varepsilon, \theta, g; \overline{\sigma}, \overline{\eta}, \overline{\psi}, \overline{q}),\\
\eta &= N(\varepsilon, \theta, g; \overline{\sigma}, \overline{\eta}, \overline{\psi}, \overline{q}),\\
q &= Q(\varepsilon, \theta, g; \overline{\sigma}, \overline{\eta}, \overline{\psi}, \overline{q}),
\end{aligned}
$$

while the time derivatives of the corresponding internal state variables are determined by

$$
(2.2)\qquad
\begin{aligned}
\dot{\overline{\psi}} &= \overline{F}(\varepsilon, \theta, g; \overline{\sigma}, \overline{\eta}, \overline{\psi}, \overline{q}),\\
\dot{\overline{\sigma}} &= \overline{S}(\varepsilon, \theta, g; \overline{\sigma}, \overline{\eta}, \overline{\psi}, \overline{q}),\\
\dot{\overline{\eta}} &= \overline{N}(\varepsilon, \theta, g; \overline{\sigma}, \overline{\eta}, \overline{\psi}, \overline{q}),\\
\dot{\overline{q}} &= \overline{Q}(\varepsilon, \theta, g; \overline{\sigma}, \overline{\eta}, \overline{\psi}, \overline{q}).
\end{aligned}
$$

The set of numbers $s = (\varepsilon, \theta, g; \sigma, \eta, \psi, q)$ will be called a thermodynamic state. A family of real valued functions $s(t) = (\varepsilon(t), \theta(t), g(t); \sigma(t), \eta(t), \psi(t), q(t))$ defined of an interval $[t_i, t_f]$, will be called a thermodynamic process at a material point $X \in \mathscr{B}$, if $s \in \mathrm{R}^1[t_i, t_f]$ and if it satisfies the second law of thermodynamics

$$
(2.3)\qquad -\dot{\psi}+\frac{1}{\varrho_0}\sigma\dot{\varepsilon}-\eta\dot{\theta}-\frac{1}{\varrho_0\theta}qg \geqslant 0,
$$

where ϱ_0 is the mass density in the reference configuration. We say that a real valued function $f \in \mathrm{R}^1[t_i, t_f]$ if f is a regulated function on $[t_i, t_f]$ and has one-sided derivatives which are regulated functions on $[t_i, t_f]$ (see for instance DIEUDONNÉ, Sec. 7.6, [5], SULICIU [6, 7]).

The functions of class R^1 naturally appear in the studies on wave propagations.

If, for the Eq. (2.2), the initial conditions

$$\bar{\sigma}(t_i) = \sigma_i, \quad \bar{\eta}(t_i) = \eta_i, \quad \bar{\psi}(t_i) = \psi_i, \quad \bar{q}(t_i) = q_i \tag{2.4}$$

and the functions $\varepsilon, \theta, g \in \mathsf{R}^1[t_i, t_f]$ are given, and $\bar{F}, \bar{S}, \bar{N}, \bar{Q}$ are good enough functions (for instance Lipschitzian functions with respect to $s^* = (\bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q})$), then the initial value problem (2.2) and (2.4) has a continuous and unique solution $\bar{\sigma}(t)$, $\bar{\eta}(t), \bar{\psi}(t)$, $\bar{q}(t)$ for $t \in [t_i, \omega)$, where $\omega \in (t_i, t_f]$ is a determined number; moreover, for a fixed $t \in [t_i, \omega)$, this solution is continuous with respect to $\varepsilon(\tau)$, $\theta(\tau)$, $g(\tau)$, $\tau \in [t_i, t)$, in the topology of uniform convergence. (For additional properties and discussions on this matter, see SULICIU [6, 7]).

From now on we assume that F, S, N, Q are at least smooth functions and $\bar{F}, \bar{S}, \bar{N}, \bar{Q}$ are at least continuous functions, with the property that any initial value problem for (2.2), with given functions $\varepsilon, \theta, g \in \mathsf{R}^1[t_i, t_f]$, has a unique solution.

The restriction imposed on the functions F, S, N, Q and $\bar{F}, \bar{S}, \bar{N}, \bar{Q}$ by the inequality (2.3), obtained by COLEMAN and GURTIN [1], are

$$S = \varrho_0 \frac{\partial F}{\partial \varepsilon}, \quad N = -\frac{\partial F}{\partial \theta}, \quad \frac{\partial F}{\partial g} = 0, \tag{2.5}$$

$$\frac{\partial F}{\partial \bar{\sigma}}\bar{S} + \frac{\partial F}{\partial \bar{\eta}}\bar{N} + \frac{\partial F}{\partial \bar{\psi}}\bar{F} + \frac{\partial F}{\partial \bar{q}}\bar{Q} + \frac{1}{\varrho_0 \theta} Q g \leqslant 0. \tag{2.6}$$

Next, we wish to define the notion of instantaneous response.

Let $\varepsilon, \theta, g \in \mathsf{R}^1[t_i, t_f]$ and $\sigma_i, \eta_i, \psi_i, q_i$ be given. For $t_0 \in [t_i, \omega)$, we denote

$$\varepsilon_0 = \varepsilon(t_0 - 0), \ldots, q_0 = Q(\varepsilon_0\, \theta_0, g_0; \bar{\sigma}_0, \bar{\eta}_0, \bar{\psi}_0, \bar{q}_0).$$

Then, due to the continuity of the solution $\bar{\sigma}(t)$, $\bar{\psi}(t)$, $\bar{\eta}(t)$, $\bar{q}(t)$ with respect to t, for $t \in [t_i, \omega)$, we have

$$\psi(t_0+0) = F(\varepsilon(t_0+0), \theta(t_0+0), g(t_0+0), \bar{\sigma}_0, \bar{\eta}_0, \bar{\psi}_0, \bar{q}_0)$$

and we get similar results for $\sigma(t_0+0)$, $\eta(t_0+0)$, $q(t_0+0)$.

We call the instantaneous response functions with respect to the given histories $\varepsilon(\tau)$, $\theta(\tau)$, $g(\tau)$, for $\tau \in [t_i, t_0)$ and the given initial conditions $s_i^* = (\sigma_i, \eta_i, \psi_i, q_i)$, the following well defined functions

$$\begin{aligned} \psi &= \psi_I(\varepsilon, \theta) = F(\varepsilon, \theta; s_0^*), \\ \sigma &= \sigma_I(\varepsilon, \theta) = S(\varepsilon, \theta; s_0^*), \\ \eta &= \eta_I(\varepsilon, \theta) = N(\varepsilon, \theta; s_0^*), \\ q &= q_I(\varepsilon, \theta, g) = Q(\varepsilon, \theta, g; s_0^*), \end{aligned} \tag{2.7}$$

where $s_0^* = (\bar{\sigma}_0, \bar{\eta}_0, \bar{\psi}_0, \bar{q}_0)$.

From (2.5) it is obvious that

$$\sigma_I = \varrho_0 \frac{\partial \psi_I}{\partial \varepsilon}, \quad \eta_I = -\frac{\partial \psi_I}{\partial \theta}. \tag{2.8}$$

3. Acceleration waves

Since it is known that the family of all possible acceleration waves is a subfamily of all characteristics of a quasi-linear hyperbolic system of partial differential equations, we shall compute here the characteristics of the corresponding hyperbolic system.

We put down now the following additional constitutive assumptions:

(3.1) $Q, \bar{S}, \bar{F}, \bar{N}$ do not depend on g and $\bar{Q}$ is linear in g, i.e.

$$\bar{Q}(\varepsilon, \theta; g; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}) = -\theta\mu(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q})\frac{\partial\theta}{\partial X} + \theta\lambda(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}). \tag{3.2}$$

The complete system of partial differential equations governing the one-dimensional thermomechanical motion of the body $\mathscr{B}$, can be written as

$$\varrho_0\frac{\partial v}{\partial t} - S_\varepsilon\frac{\partial\varepsilon}{\partial X} - S_\theta\frac{\partial\theta}{\partial X} - S_{\bar{\sigma}}\frac{\partial\bar{\sigma}}{\partial X} - S_{\bar{\eta}}\frac{\partial\bar{\eta}}{\partial X} - S_{\bar{\psi}}\frac{\partial\bar{\psi}}{\partial X} - S_{\bar{q}}\frac{\partial\bar{q}}{\partial X} = 0, \tag{3.3}$$

$$\frac{\partial\varepsilon}{\partial t} - \frac{\partial v}{\partial X} = 0, \tag{3.4}$$

$$\begin{aligned}\varrho_0\theta N_\varepsilon\frac{\partial\varepsilon}{\partial t} + \varrho_0\theta N_\theta\frac{\partial\theta}{\partial t} + Q_\varepsilon\frac{\partial\varepsilon}{\partial X} + [Q_\theta - \varrho_0(F_{\bar{q}} + \theta N_{\bar{q}})\theta\mu]\frac{\partial\theta}{\partial X} + Q_{\bar{\sigma}}\frac{\partial\bar{\sigma}}{\partial X}\\ + Q_{\bar{\eta}}\frac{\partial\bar{\eta}}{\partial X} + Q_{\bar{\psi}}\frac{\partial\bar{\psi}}{\partial X} + Q_{\bar{q}}\frac{\partial\bar{q}}{\partial X} + \varrho_0(F_{\bar{\sigma}} + \theta N_{\bar{\sigma}})\bar{S} + \varrho_0(F_{\bar{\eta}} + \theta N_{\bar{\eta}})\bar{N}\\ + \varrho_0(F_{\bar{\psi}} + \theta N_{\bar{\psi}})\bar{F} + \varrho_0(F_{\bar{q}} + \theta N_{\bar{q}})\theta\lambda = 0,\end{aligned} \tag{3.5}$$

$$\frac{\partial\bar{\psi}}{\partial t} = \bar{F}(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}), \tag{3.6}$$

$$\frac{\partial\bar{\sigma}}{\partial t} = \bar{S}(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}), \tag{3.7}$$

$$\frac{\partial\bar{\eta}}{\partial t} = \bar{N}(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}), \tag{3.8}$$

$$\frac{\partial\bar{q}}{\partial t} + \theta\mu(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q})\frac{\partial\theta}{\partial X} = \theta\lambda(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}), \tag{3.9}$$

where $v = \partial\chi/\partial t$ is the particle velocity.

The characteristic equation for this system will be

$$\begin{aligned}&\varrho_0^2\theta N_\theta c^4 - \varrho_0[Q_\theta - \varrho_0\theta\mu(F_{\bar{q}} + \theta N_{\bar{q}})]c^3 - \varrho_0\theta[S_\varepsilon N_\theta - N_\varepsilon S_\theta + \mu Q_{\bar{q}}]c^2\\ &+ \{S_\varepsilon[Q_\theta - \varrho_0\theta\mu(F_{\bar{q}} + \theta N_{\bar{q}})] + \varrho_0\theta^2 N_\varepsilon\mu S_{\bar{q}} - S_\theta Q_\varepsilon\}c + \theta\mu S_\varepsilon Q_{\bar{q}} - \theta\mu S_{\bar{q}} Q_\varepsilon = 0,\\ &c = \frac{dX}{dt}.\end{aligned} \tag{3.10}$$

This characteristic equation coincides with the Eq. (3.12) of KOSIŃSKI and PERZYNA [2] under the hypotheses considered here.

Of course, a necessary and sufficient condition to obtain symmetric roots from the Eq. (3.10) is to vanish the coefficients of c and c^3. But, for the obtained relations, between the functions F, N, S, Q and μ, it is not easy to find a physical meaning.

If we set

(3.11) $$S_{\bar{q}} = 0, \quad F_{\bar{q}}+\theta N_{\bar{q}} = 0$$

then the symmetry requirement implies

(3.12) $$Q_\theta = 0, \quad Q_\varepsilon = 0$$

and *vice versa*.

Conditions (3.11) involve

(3.13) $$F(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}) = F_1(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi})+\theta F_2(\bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}).$$

Under the conditions (3.11) to (3.12) the characteristic equation (3.10) becomes

(3.14) $$\varrho_0^2 N_\theta c^4 - \varrho_0(S_\varepsilon N_\theta + \varrho_0 N_\varepsilon^2 + \mu Q_{\bar{q}})c^2 + \mu S_\varepsilon Q_{\bar{q}} = 0.$$

From (3.1) and (3.12) we obtain

(3.15) $$q = Q(\bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}),$$

i.e. q becomes itself an internal state variable.

4. Further consequences of the second law of thermodynamics

4.1

First we wish to examine some of the second law consequences in the context of the symmetry assumption.

The hypotheses (3.1), (3.2) and the form (3.13) of the free energy function F used in the inequality (2.6) lead to the following conclusions

(4.1) $$Q = \varrho_0 \theta^3 \mu \frac{\partial F_2}{\partial \bar{q}}$$

and

(4.2) $$\frac{\partial F}{\partial \bar{\sigma}}\bar{S} + \frac{\partial F}{\partial \bar{\eta}}\bar{N} + \frac{\partial F}{\partial \bar{\psi}}\bar{F} + \theta^2 \lambda \frac{\partial F_2}{\partial \bar{q}} \leqslant 0.$$

Since Q and F_2 depend only on the internal state variables, the function μ must be of the form

(4.3) $$\mu(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}) = \frac{\bar{\mu}(\bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q})}{\theta^3}.$$

We may choose

(4.4) $$q = Q(\bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}) = \bar{q},$$

which implies that the function $\bar{\mu}$ and $\partial F_2/\partial \bar{q}$ must be related by

(4.5) $$\bar{q} = \varrho_0 \bar{\mu}(\bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}) \frac{\partial F_2}{\partial \bar{q}}(\bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}).$$

4.2

Another important requirement of the constitutive assumptions (2.1) and (2.2) is that the symmetric roots of the Eq. (3.14) must be real. This requirement together with the second law of thermodynamics will make the above constitutive equations more precise. But the determination of the set of all functions, which will satisfy both conditions, is a very difficult problem to solve.

Since we wish to get a better insight concerning the above requirements, we shall restrict, in that which follows, our generality. To restrict our generality, more than was done by the symmetry requirement, we introduce the notion of a totally relaxed state.

We say that a state $s_i = (\varepsilon_i, \theta_i, g_i = 0; \sigma_i, \eta_i, \psi_i, q_i)$ is totally relaxed if

$$\begin{aligned} \bar{F}(s_i) &= \bar{F}(\varepsilon_i, \theta_i; \bar{\sigma}_i, \bar{\eta}_i, \bar{\psi}_i, \bar{q}_i) = 0, \\ \bar{S}(s_i) &= \bar{S}(\varepsilon_i, \theta_i; \bar{\sigma}_i, \bar{\eta}_i, \bar{\psi}_i, \bar{q}_i) = 0, \\ \bar{N}(s_i) &= \bar{N}(\varepsilon_i, \theta_i; \bar{\sigma}_i, \bar{\eta}_i, \bar{\psi}_i, \bar{q}_i) = 0, \\ \lambda(s_i) &= \lambda(\varepsilon_i, \theta_i; \bar{\sigma}_i, \bar{\eta}_i, \bar{\psi}_i, \bar{q}_i) = 0, \end{aligned} \tag{4.6}$$

where we have chosen $\bar{\sigma}_i = \sigma_i$, etc. We observe that if at $t = t_i$ we are at a totally relaxed state we remain there for all $t > t_i$, since it was supposed that the functions $\bar{F}, \bar{S}, \bar{N}, \mu$ and λ are defined in such a way that any initial value problem for the Eqs. (3.6)–(3.9) has a unique solution.

We can prove the following:

PROPOSITION 4.1. *If the symmetry requirement is satisfied* [*i.e. if* (3.11) *and* (3.12) *hold*] *and if: a)* μ *and* λ *are of class* C^2 *with respect to* $\bar{q}$ *and* $\mu \neq 0$, $\partial\lambda/\partial\bar{q} \neq 0$; *b)* $\bar{F}, \bar{N}, \bar{S}$ *do not depend on* $\bar{q}$ *and c)* $s_i = (\varepsilon_i, \theta_i, g_i = 0; \sigma_i, \eta_i, \psi_i, q_i)$ *is a totally relaxed state, then*:

$$q_i = 0, \tag{4.7}$$

$$\bar{\mu}(s_i)\,\lambda_{\bar{q}}(s_i) \leqslant 0. \tag{4.8}$$

P r o o f. Consider the process generated by

$$\varepsilon(t) = \varepsilon_i, \quad \theta(t) = \theta_i, \quad g(t) = g = const, \quad t \geqslant t_i \tag{4.9}$$

and which starts at $t = t_i$ with

$$\bar{\sigma}(t_i) = \sigma_i, \quad \bar{\eta}(t_i) = \eta_i, \quad \bar{\psi}(t_i) = \psi_i, \quad \bar{q}(t_i) = q_i, \tag{4.10}$$

where $s_i = (\varepsilon_i, \theta_i, g = 0; \sigma_i, \eta_i, \psi_i, q_i)$ is a fixed totally relaxed state. Then according to conditions b) and c) the initial value of problem (3.6)–(3.9) and (4.10) has a unique solution

$$\begin{aligned} &\bar{\psi}(t) = \psi_i, \quad \bar{\sigma}(t) = \sigma_i, \quad \bar{\eta}(t) = \eta_i, \\ &\bar{q}(t) = \tilde{q}(t, g) = \tilde{q}(t, \varepsilon_i, \theta_i, g; \sigma_i, \eta_i, \psi_i, q_i), \quad t \geqslant t_i, \\ &\tilde{q}(t, 0) = q_i. \end{aligned} \tag{4.11}$$

The solution (4.11), the condition c) and (4.5) used in (4.2) lead to the following inequality

$$A(t, g) = \frac{\lambda(\varepsilon_i, \theta_i; \sigma_i, \eta_i, \psi_i, \tilde{q}(t, g))\tilde{q}(t, g)}{\bar{\mu}(\sigma_i, \eta_i, \psi_i, \tilde{q}(t, g))} \leqslant 0. \tag{4.12}$$

We observe that

$$A(t, 0) = 0 \quad \text{for any} \quad t \geqslant t_i, \tag{4.13}$$

i.e. for any fixed t, A as a function of g has a maximum at $g = 0$.

Thus we have

$$\left.\frac{\partial A}{\partial g}\right|_{g=0} = \frac{[\overline{\mu}(s_i)\big(\lambda_{\overline{q}}(s_i)q_i + \lambda(s_i)\big) - \lambda(s_i)q_i\overline{\mu}_{\overline{q}}(s_i)]\dfrac{\partial \tilde{q}}{\partial g}(t, 0)}{(\overline{\mu}(s_i))^2} = 0. \tag{4.14}$$

Since

$$Y(t) = \frac{\partial \tilde{q}}{\partial g}(t, g)$$

is the solution of the initial value problem (see for instance Hartman, Chap. V, § 3 [9])

$$\frac{dY}{dt} + \theta_i\left(g\frac{\partial \mu}{\partial \overline{q}} - \frac{\partial \lambda}{\partial \overline{q}}\right)Y + \theta_i\mu = 0, \quad Y(t_i) = 0,$$

it follows by condition a) of the proposition that

$$\frac{\partial \tilde{q}}{\partial g}(t, 0) \neq 0, \quad t > t_i. \tag{4.15}$$

Now we choose a fixed $t > t_i$ in (4.14) and we find that (4.7) holds. (4.8) follows from $\partial^2 A/\partial g^2 \leqslant 0$ at $g = 0$ in the same way. Q.E.D.

Under the conditions of Propositions 4.1 the second law implies that at a relaxed state the heat flux must vanish. The meaning of the relation (4.8) will be discussed later on.

Consider now the state

$$s_0 = (\varepsilon_0, \theta_0, g_0 = 0; \sigma_0, \eta_0, \psi_0, q_0 = 0)$$

which is totally relaxed. Suppose that there exists a neighbourhood U_0 of the point $(\varepsilon_0, \theta_0)$ in the (ε, θ)-plane, such that the solution of the system (3.6)–(3.9) with the initial conditions

$$\overline{\psi}(t_i) = \psi_0, \quad \overline{\sigma}(t_i) = \sigma_0, \quad \overline{\eta}(t_i) = \eta_0, \quad \overline{q}(t_i) = 0 \tag{4.16}$$

and

$$\varepsilon(t) = \varepsilon = \text{const}, \quad \theta(t) = \theta = \text{const}, \quad t \geqslant t_i \tag{4.17}$$

has a finite and unique limit when $t \to \infty$, i.e.

$$\begin{aligned} &\lim_{t\to\infty}\overline{\psi}(t) = \overline{\psi}_R(\varepsilon, \theta), \quad \lim_{t\to\infty}\overline{\sigma}(t) = \overline{\sigma}_R(\varepsilon, \theta), \\ &\lim_{t\to\infty}\overline{\eta}(t) = \overline{\eta}_R(\varepsilon, \theta), \quad \lim_{t\to\infty}\overline{q}(t) = \overline{q}_R(\varepsilon, \theta), \end{aligned} \tag{4.18}$$

where $s_0^* = (\sigma_0, \eta_0, \psi_0, q_0 = 0)$ is fixed. Of course, in this case t_f is supposed to be infinite.

From (4.18) and the continuity of $\overline{F}$, $\overline{S}$, $\overline{N}$ and λ, we obtain $\lim_{t\to\infty}\dot{\overline{\psi}}(t) = 0, \ldots, \lim_{t\to\infty}\dot{\overline{q}}(t) = 0$, i.e.

$$\begin{aligned} &\overline{F}(\varepsilon, \theta, \overline{\sigma}_R(\varepsilon, \theta), \overline{\eta}_R(\varepsilon, \theta), \overline{\psi}_R(\varepsilon, \theta), \overline{q}_R(\varepsilon, \theta) = 0, \\ &\lambda(\varepsilon, \theta, \overline{\sigma}_R(\varepsilon, \theta), \overline{\eta}_R(\varepsilon, \theta), \overline{\psi}_R(\varepsilon, \theta), \overline{q}_R(\varepsilon, \theta)) = 0. \end{aligned} \tag{4.19}$$

for any $(\varepsilon, \theta) \in U_0$. This means that any state $s = \big(\varepsilon, g = 0; \sigma_R(\varepsilon, \theta), \eta_R(\varepsilon, \theta), \psi_R(\varepsilon, \theta), q_R(\varepsilon, \theta)\big)$, $(\varepsilon, \theta) \in U_0$ (where $\sigma_R(\varepsilon, \theta) = S(\varepsilon, \theta; \bar{\sigma}_R(\varepsilon, \theta), \bar{\eta}_R(\varepsilon, \theta), \bar{\psi}_R(\varepsilon, \theta))$, etc.) is a totally relaxed state. It is obvious that

$$\psi_R(\varepsilon_0, \theta_0) = \psi_0, \text{ etc.} \tag{4.20}$$

Concerning $(4.6)_4$ and the function $\bar{q}_R(\varepsilon, \theta)$, we observe that if the material is isotropic (with respect to the full orthogonal group, which in the one-dimensional case reduces to the elements $1, -1$), then

$$\lambda(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, 0) = 0 \tag{4.21}$$

for any $s = (\varepsilon, \theta, g = 0, \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q} = 0)$, which means that $\bar{q}(t) = 0$ and $\bar{q}_R(\varepsilon, \theta) = 0$. Next, we consider isotropic materials only.

From the inequality (4.2) we obtain

$$\frac{\partial \psi}{\partial t} = \frac{\partial F}{\partial \bar{\sigma}}\bar{S} + \frac{\partial F}{\partial \bar{\eta}}\bar{N} + \frac{\partial F}{\partial \bar{\psi}}\bar{F} \leqslant 0$$

which leads to

$$\psi_{\mathrm{I}}(\varepsilon, \theta, s_0^*) \geqslant \psi(t) \geqslant \psi_R(\varepsilon, \theta), \tag{4.22}$$

where $\psi(t) = F\big(\varepsilon, \theta, \bar{\sigma}(t), \bar{\eta}(t), \bar{\psi}(t)\big)$.

The relations (4.22) tell us that the function

$$\Lambda(\varepsilon, \theta) = \psi_{\mathrm{I}}(\varepsilon, \theta, s_0^*) - \psi_R(\varepsilon, \theta) \tag{4.23}$$

has a minimum at $\varepsilon = \varepsilon_0$, $\theta = \theta_0$. Thus, we can write

$$\begin{aligned} \sigma_0 &= \varrho_0 \frac{\partial \psi_{\mathrm{I}}}{\partial \varepsilon}(\varepsilon_0, \theta_0, s_0^*) = \varrho_0 \frac{\partial \psi_R}{\partial \varepsilon}(\varepsilon_0, \theta_0) = \sigma_R(\varepsilon_0, \theta_0), \\ \eta_0 &= -\frac{\partial \psi_{\mathrm{I}}}{\partial \theta}(\varepsilon_0, \theta_0, s_0^*) = -\frac{\partial \psi_R}{\partial \theta}(\varepsilon_0, \theta_0) = \eta_R(\varepsilon_0, \theta_0), \end{aligned} \tag{4.24}$$

and

$$\frac{\partial^2 \psi_{\mathrm{I}}}{\partial \varepsilon^2}(\varepsilon_0, \theta_0, s_0^*) \geqslant \frac{\partial^2 \psi_R}{\partial \varepsilon^2}(\varepsilon_0, \theta_0), \tag{4.25}$$

$$\frac{\partial^2 \psi_{\mathrm{I}}}{\partial \theta^2}(\varepsilon_0, \theta_0, s_0^*) \geqslant \frac{\partial^2 \psi_R}{\partial \theta^2}(\varepsilon_0, \theta_0), \tag{4.26}$$

$$\begin{aligned} &\left[\frac{\partial \sigma_{\mathrm{I}}(\varepsilon_0, \theta_0, s_0^*)}{\partial \varepsilon} - \frac{\partial \sigma_R(\varepsilon_0, \theta_0)}{\partial \varepsilon}\right]\left[\frac{\partial \eta_R(\varepsilon_0, \theta_0)}{\partial \theta} - \frac{\partial \eta_{\mathrm{I}}(\varepsilon_0, \theta_0, s_0^*)}{\partial \theta}\right] \\ &\qquad - \left[\frac{\partial \sigma_{\mathrm{I}}}{\partial \theta}(\varepsilon_0, \theta_0, s_0^*) - \frac{\partial \sigma_R}{\partial \theta}(\varepsilon_0, \theta_0)\right]^2 \geqslant 0; \end{aligned} \tag{4.27}$$

(4.24) to (4.27) are valid for any totally relaxed state

$$s_0 = (\varepsilon_0, \theta_0, g_0 = 0; \sigma_0, \eta_0, \psi_0, q_0 = 0).$$

Such relations of the form (4.24) and (4.25)–(4.27) are known in the literature. For relations of the form (4.24) see COLEMAN and GURTIN [1] and for relations of the form (4.25)–(4.2) see COLEMAN [3] and COLEMAN and GURTIN [4].

5. On the real acceleration waves assumption

We return now to the problem of the existence of real waves in a material for which the symmetry assumption is verified. In this case, taking into account (2.5), (3.13) and (4.4), the constitutive assumptions (2.1) can be written as

$$(5.1)\qquad \begin{aligned} \psi &= F(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}) = F_1(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}) + \theta F_2(\bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}),\\ \sigma &= S(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}) = \varrho_0 \frac{\partial F_1}{\partial \varepsilon}(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}),\\ \eta &= N(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}) = -\frac{\partial F_1}{\partial \theta}(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}) - F_2(\bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}),\\ q &= \bar{q}. \end{aligned}$$

The internal state variables verify the Eqs. (3.6)–(3.9), where μ has the form (4.3) and $\bar{\mu}$ is related to F_2 by (4.5).

The roots of the Eqs. (3.14) can be written as

$$(5.2)\qquad c_{\mathrm{I}}^2 = \frac{S_\varepsilon N_\theta + \varrho_0 N_\varepsilon^2 + \mu - \sqrt{(S_\varepsilon N_\theta + \varrho_0 N_\varepsilon^2 + \mu)^2 - 4\mu S_\varepsilon N_\theta}}{2\varrho_0 N_\theta},$$

$$(5.3)\qquad c_{\mathrm{II}}^2 = \frac{S_\varepsilon N_\theta + \varrho_0 N_\varepsilon^2 + \mu + \sqrt{(S_\varepsilon N_\theta + \varrho_0 N_\varepsilon^2 + \mu)^2 - 4\mu S_\varepsilon N_\theta}}{2\varrho_0 N_\theta},$$

where $(5.1)_4$ was used.

Suppose that for an isotropic body there exists a relaxed state s_0 and a neighbourhood of it, where the limits (4.18) exist and where the conditions of Proposition 4.1 are satisfied. Suppose we know about the relaxed state that

$$(5.4)\qquad \frac{\partial^2 \psi_R(\varepsilon_0, \theta_0)}{\partial \varepsilon^2} > 0$$

and

$$(5.5)\qquad \frac{\partial \lambda}{\partial \bar{q}}(s_0) < 0.$$

If we know about the instantaneous response that

$$(5.6)\qquad \frac{\partial N}{\partial \theta}(s_0) > 0;$$

then the second law of thermodynamics implies [see formulas (4.25) and (4.8)]

$$(5.7)\qquad \frac{\partial S(s_0)}{\partial \varepsilon} > 0$$

and

$$(5.8)\qquad \mu(s_0) > 0,$$

i.e. the acceleration waves are real ($c_{\mathrm{I}}^2 > 0$, $c_{\mathrm{II}}^2 > 0$). If we know (5.4) and (5.5) and we assume real acceleration waves, then the second law implies (5.6)

The inequality (5.5) is a justified requirement, since in the linear case $\lambda = -\varkappa\bar{q}$, $\varkappa = \text{const} > 0$.

In what follows we shall assume

$$\lambda(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}) = -\varkappa(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q})\bar{q}. \tag{5.9}$$

The Eq. (3.9) can be written as

$$\dot{\bar{q}} = -\frac{\bar{\mu}(\bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q})}{\theta^2}\frac{\partial\theta}{\partial X} - \theta\varkappa(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q})\bar{q}, \tag{5.10}$$

where (4.3) was used.

Under the symmetry assumption the balance and compatibility equations (3.3)–(3.5) become

$$\begin{aligned}
&\varrho_0\frac{\partial v}{\partial t} - S_\varepsilon\frac{\partial\varepsilon}{\partial X} - S_\theta\frac{\partial\theta}{\partial X} - S_{\bar{\sigma}}\frac{\partial\bar{\sigma}}{\partial X} - S_{\bar{\eta}}\frac{\partial\bar{\eta}}{\partial X} - S_{\bar{\psi}}\frac{\partial\bar{\psi}}{\partial X} = 0,\\
&\frac{\partial\varepsilon}{\partial t} - \frac{\partial v}{\partial X} = 0,\\
&\varrho_0\theta N_\varepsilon\frac{\partial\varepsilon}{\partial t} + \varrho_0\theta N_\theta\frac{\partial\theta}{\partial t} + \frac{\partial\bar{q}}{\partial X} + \varrho_0(F_{\bar{\sigma}} + \theta N_{\bar{\sigma}})\bar{S} + \varrho_0(F_{\bar{\eta}} + \theta N_{\bar{\eta}})\bar{N}\\
&\qquad + \varrho_0(F_{\bar{\psi}} + \theta N_{\bar{\psi}})\bar{F} = 0.
\end{aligned} \tag{5.11}$$

The system of the Eqs. (3.6)–(3.8), (5.10) and (5.11) — where F, S, N are of the form (5.1) and $\bar{\mu}$ is determined by (4.5) — under the real acceleration waves assumption is a complete quasi-linear hyperbolic system of equations.

Now, if in the Eq. (5.10) we take $\mu \to \infty$ and suppose that $\varkappa/\mu$ has a positive and finite limit, i.e.

$$\lim_{\mu\to\infty}\frac{\varkappa}{\mu} = \frac{1}{k}, \tag{5.12}$$

then

$$c_{\mathrm{I}}^2 \to \frac{S_\varepsilon}{\varrho_0} = c_m^2, \tag{5.13}$$

$$c_{\mathrm{II}}^2 \to \infty, \tag{5.14}$$

and

$$\bar{q} = -k\frac{\partial\theta}{\partial X}. \tag{5.15}$$

Therefore, we find the expected classical results.

The quantity

$$c_a^2 = \frac{S_\varepsilon}{\varrho_0} - \frac{N_\varepsilon^2}{N_\theta} \tag{5.16}$$

is sometimes called the adiabatic sound speed. This sound speed is obtained from the system of the Eqs. (3.6)–(3.8), (5.10) and (5.11) by putting $\bar{q} = 0$ in the Eq. $(5.11)_3$ and disregarding the Eq. (5.10). But, if we put $\bar{q} = 0$ in the Eq. (5.10), we find $\partial\theta/\partial X = 0$. The remaining system does not, generally, lead to $\partial\theta/\partial X = 0$. Thus, such a procedure might be applied to very special cases only.

If the body $\mathscr{B}$ is subjected to a temperature field only, i.e. if we put $v = 0$ and $\varepsilon = 0$ in (3.6)–(3.8), (5.10) and (5.11), we find a finite velocity of the heat propagation waves

(5.17)
$$c_h^2 = \frac{\mu}{\varrho_0 N_\theta}$$

with the constrain $\partial\sigma/\partial X = 0$.

If the body $\mathscr{B}$ is subjected to a motion only, i.e. if $\theta(X, t) = \text{const}$, then one can look for a solution of the system (3.6)–(3.8) and $(5.11)_{1,2}$. The Eqs. (5.10) and $(5.11)_3$ remain as constrains. We find a velocity of propagation as that given by (5.13).

The quantities $S_\varepsilon, N_\theta, N_\varepsilon = -S_\theta/\varrho_0, k, \tau = 1/\theta\varkappa$ have the following physical meaning: S_ε is the instantaneous tangent modulus, N_ε is the instantaneous coupling modulus, N_θ is the instantaneous specific heat, k is the thermal conductivity and τ is called by some authors the relaxation time.

To end this section, we observe that

(5.18)
$$c_{\mathrm{I}}^2 < c_m^2 < c_a^2 < c_{\mathrm{II}}^2$$

for any finite μ, and

(5.19)
$$c_{\mathrm{I}}^2 = \frac{1}{2}\left[c_a^2 + c_h^2 - \sqrt{(c_a^2 + c_h^2)^2 - 4c_h^2 c_m^2}\right],$$
$$c_{\mathrm{II}}^2 = \frac{1}{2}\left[c_a^2 + c_h^2 + \sqrt{(c_a^2 + c_h^2)^2 - 4c_h^2 c_m^2}\right].$$

6. Shock waves

6.1. General remarks

When at a point $P = (X, t)$, a shock discontinuity direction is crossed, the jump conditions can be written as (see CHEN and GURTIN [30])

(6.1)
$$\varrho_0 c[v] + [\sigma] = 0,$$

(6.2)
$$[v] + c[\varepsilon] = 0,$$

(6.3)
$$\varrho_0 c[e] = \frac{1}{2} c[\sigma][\varepsilon] + c\sigma[\varepsilon] + [q],$$

(6.4)
$$\varrho_0 c[\eta] \geqslant \left[\frac{q}{\theta}\right],$$

where $[\sigma] = \sigma_a - \sigma_b$, etc., $c(\neq \infty)$ is the slope of the discontinuity direction and

(6.5) $\quad e = E(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}) = F(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}) - \theta N(\varepsilon, \theta; \bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q}) = \psi + \theta\eta.$

The jump conditions (6.1)–(6.4) are obtained when $v, \varepsilon, \ldots, \bar{q}$ are regulated functions and verify the system (3.6)–(3.7), (5.10) and (5.11) in the weak sense (see SULICIU [6, 7]).

Since q is an internal parameter [see formula (3.15)] and it is continuous with respect to t for any fixed X (see Sec. 2), then by Proposition 2.5 from SULICIU [7],

$$[q] = 0 \quad \text{for any direction} \quad c \neq 0. \tag{6.7}$$

For a fixed state $s_b = (\varepsilon_b, \theta_b, g_b) = (\partial\theta/\partial X)_b; \sigma_b, \eta_b, \psi_b, q_b)$, the formulae (6.4) and (6.7) show that ε_a, θ_a are related by

$$\varrho_0 E(\varepsilon, \theta; s_b^*) - E(s_b)) = \frac{1}{2} S(\varepsilon, \theta; s_b^*) - S(s_b))(\varepsilon - \varepsilon_b) + S(s_b)(\varepsilon - \varepsilon_b), \tag{6.8}$$

where $s_b^* = (\bar{\sigma}_b, \bar{\eta}_b, \bar{\psi}_b, \bar{q}_b)$ and the index a is omitted. By formal differentiation of (6.8) with respect to ε, we find

$$\left(\varrho_0 \theta \frac{\partial N}{\partial \theta} - \frac{1}{2} \frac{\partial S}{\partial \theta}[\varepsilon]\right) \frac{\partial \Theta}{\partial \varepsilon} = \frac{1}{2} \frac{\partial S}{\partial \varepsilon}[\varepsilon] - \frac{1}{2}[S] - \varrho_0 \theta \frac{\partial N}{\partial \varepsilon}. \tag{6.9}$$

When $(\varepsilon, \theta) \to (\varepsilon_b, \theta_b)$, $\dfrac{\partial \Theta}{\partial \varepsilon}$ exists and is given by

$$\left.\frac{\partial \Theta}{\partial \varepsilon}\right|_{(\varepsilon_b, \theta_b)} = -\left.\frac{N_\varepsilon}{N_\theta}\right|_{s_b} \tag{6.10}$$

(6.10) tells us that there exists a neighbourhood of $(\varepsilon_b, \theta_b)$ in the (ε, θ)-plane where the relation (6.8) between θ and ε can be given explicit, i.e.

$$\theta = \Theta(\varepsilon), \quad \theta_b = \Theta(\varepsilon_b); \tag{6.11}$$

moreover, the shocks of small amplitude propagate with a velocity close to the adiabatic sound speed (5.16), i.e.

$$\varrho_0 c^2 = \frac{[S]}{[\varepsilon]} \approx S_\varepsilon(s_b) - \frac{N_\varepsilon(s_b) S_\theta(s_b)}{N_\theta(s_b)} = \varrho_0 c_a^2. \tag{6.12}$$

6.2. The Riemann problem in the linearized theory

It is supposed that there exists a neighbourhood of the relaxed state $s_0 = (\varepsilon = 0, \theta = \theta_0, g = 0; \sigma = 0, \eta = 0, \psi = 0, q = 0)$, where

$$\begin{aligned}
&\bar{S} = 0, \quad \bar{F} = 0, \quad \bar{N} = 0,\\
&\bar{\mu} = \bar{\mu}_0 = \text{const}, \quad \varkappa = \varkappa_0 = \text{const},\\
&F = \frac{1}{\varrho_0}\left(a_{02} T^2 + a_{20}\varepsilon^2 + a_{11}\varepsilon T + \frac{\theta_0}{2\varrho_0 \bar{\mu}_0} \bar{q}^2\right),\\
&S = \varrho_0 \frac{\partial F}{\partial \varepsilon} = 2a_{20}\varepsilon + a_{11} T,\\
&N = -\frac{\partial F}{\partial \theta} = -\frac{1}{\varrho_0 \theta_0}(2a_{02} T + a_{11}\varepsilon),
\end{aligned} \tag{6.13}$$

with

$$|\varepsilon| \ll 1,\; T = \frac{\theta-\theta_0}{\theta_0}, \quad |T| \ll 1, \quad |q| \ll 1. \tag{6.14}$$

Then, the system (3.6)–(3.8), (5.10) and (5.11) becomes a linear system of partial differential equations (compare with ACHENBACK [22]).

The Riemann problem for this system requires to find the solution of the initial value problem

$$\begin{aligned} &\varepsilon(X,0) = 0, \quad v = (X,0) \qquad T(X,0) = 0 \quad \text{for} \quad X > 0, \\ &\varepsilon(X,0) = \varepsilon_0, \quad v(X,0) = v_0, \quad T(X,0) = T_0 \quad \text{for} \quad X < 0. \end{aligned} \tag{6.15}$$

From the mechanical point of view this is the problem of the impact of two elastic bars.

We discuss here the discontinuities at the point (0, 0) from the X, t-plane for $t > 0$, only. We have four acceleration waves, two shock waves and a steady shock discontinuity at $X = 0$ (see Fig. 1). These discontinuities are propagating in an order determined by

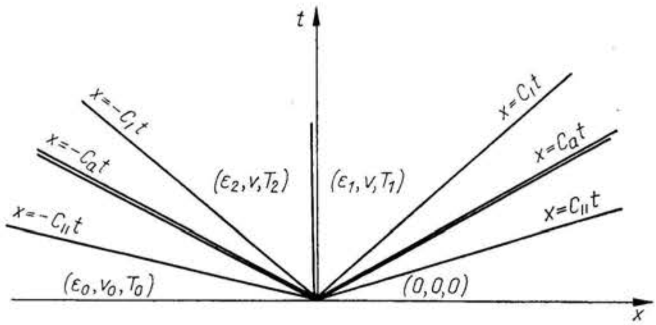

FIG. 6. Configuration of discontinuity directions: simple line—acceleration waves, double line—shock discontinuities.

(5.18). Shock waves are propagating with the adiabatic sound velocity given by (6.12). The steady shock discontinuity is characterized by continuity of stress and particle velocity. Condition (6.7) gives here $[\eta] = 0$, i.e.,

$$2a_{02}[T]+a_{11}[\varepsilon] = 0 \quad \text{for} \quad c \neq 0 \tag{6.16}$$

which is nothing else but condition (6.8). All the quantities at (0, 0) after the shocks passed can be written as

$$\begin{aligned} \varepsilon_1 &= \varepsilon_a(X = 0+, t = 0) = \sigma_0/2\varrho_0 c_a^2 - v_0/2c_a, \\ \varepsilon_2 &= \varepsilon_a(X = 0-, t = 0) = \varepsilon_0 - v_0/2c_a - \sigma_0/2\varrho_0 c_a^2, \\ T_1 &= T_a(X = 0+, t = 0) = \frac{a_{11}}{2a_{02}}\left(\frac{v_0}{2c_a} + \frac{\theta_0\eta_0}{2c_a^2}\right), \\ T_2 &= T_a(X = 0-, t = 0) = T_0 + \frac{a_{11}}{2a_{02}}\left(\frac{v_0}{2c_a} - \frac{\theta_0\eta_0}{2c_a^2}\right), \\ \eta_1 &= \eta_a(X = 0+, t = 0) = 0, \end{aligned} \tag{6.17}$$

(6.17) [cont.]

$$\eta_2 = \eta_a(X = 0-, t = 0) = \eta_0,$$

$$v = v_a(X = 0\pm, t = 0) = \frac{v_0}{2} - \frac{\sigma_0}{2\varrho_0 c_a},$$

$$q = q_a(X = 0\pm, t = 0) = 0,$$

$$\psi_1 = \frac{1}{\varrho_0}(a_{02} T_1^2 + a_{20}\varepsilon_1^2 + a_{11}\varepsilon_1 T_1),$$

$$\psi_2 = \frac{1}{\varrho_0}(a_{02} T_2^2 + a_{20}\varepsilon_2^2 + a_{11}\varepsilon_2 T_2),$$

$$\sigma_1 = \sigma_2 = \sigma = 2a_{20}\varepsilon_1 + a_{11} T_1,$$

$$\sigma_0 = 2a_{20}\varepsilon_0 + a_{11} T_0, \quad \eta_0 = \frac{-1}{\varrho_0\theta_0}(2a_{20} T + a_{11}\varepsilon).$$

Of course, one can compute the jump of derivatives across acceleration waves and study the way the quantities decay along shock waves, but the purpose of this example was only to point out the differences between this way of putting the problem and the way in which it was done by ACHENBACH [22] and POPOV [21].

In this framework we look for solutions which are regulated functions (i.e. which are functions defined at any point in the domain of interest). According to this point of view and to the fact that q is an internal state variable, we arrived at the conclusion (6.7) which implies (6.14). The above cited authors look for solutions which are distributions and *a* heat conduction constitutive equation is considered together with balance and compatibility equations, but not as a constitutive law for an internal state variable. They find a jump of heat flux proportional to the jump of temperature across the shock waves. The slopes found for shock waves are $c_{II}(s_0)$ and $c_I(s_0)$ while the shock wave of slope $c_n(s_0)$ does not appear at all.

However, the condition (6.7) is an often used hypothesis in wave propagation theory, and it was not weakened by experimental results (see e.g. [30, 31]).

7. Conclusions and remarks

We remark that the "hidden parameters" $(\bar{\sigma}, \bar{\eta}, \bar{\psi}, \bar{q})$ introduced here, play a similar role as the history parameter τ introduced by SULICIU [7], SULICIU, MALVERN and CRISTESCU [8], i.e. they can describe, for instance, how a state, reached instantaneously, moves to a relaxed one.

The hypotheses made for the existence of symmetric waves, push heat flux q between internal state variables. Thus q does not jump when a shock wave propagates through the material. The slope of the shock wave in the X, t plane, is the same as in the case when a Fourier heat conduction constitutive equation is considered.

Concerning the acceleration waves, it is proved, under certain assumptions (see Proposition 4.1) on the evolution equations (3.6) to (3.9), that the usual informations on the relaxed states [see (5.4) and (5.5)] and on the instantaneous response [see (5.6)] imply that the symmetric waves are real. These acceleration waves are coupled i.e. they carry jump discontinuities of both mechanical and thermal quantities. On the other hand, the hypothesis of real symmetric acceleration waves together with the informations (5.4) and (5.5) on the relaxed state imply (5.6).

When an elastic (linear) bar is impacted at one end, the first propagating wave is an acceleration one followed by a shock propagating with the adiabatic sound speed, and a second acceleration wave. This picture is different from the picture obtained by ACHENBACH [22].

We note that the symmetry conditions (3.11) and (3.12) can easily be written for the more general case treated by KOSIŃSKI and PERZYNA [2] and, in their notations, are

$$(\partial_\alpha \Psi + \theta \partial_\alpha N)\cdot \bar{A} = 0, \quad \partial_\alpha T \cdot \bar{A} = 0, \quad \partial_\theta Q = 0, \quad \partial_F Q = 0.$$

The discussion concerning the reality of acceleration waves is more difficult in this case. The conclusions, that the heat flux does not jump and that the shock waves of small amplitude propagate with a velocity close to the adiabatic sound speed still hold.

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