Archives of Mechanics • Archiwum Mechaniki Stosowanej • 27, 5-6, pp. 701-706, Warszawa 1975

On the existence of solutions in viscoplasticity(*)

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PROOFS of the existence of solutions to boundary-value problems in mechanics of continua are known to be difficult to establish. Such a proof may be sketched by starting from the Dirichlet principle which is based on the assumption that the existence of a lower bound is equivalent to the existence of a minimum. The extremum principles derived for the relatively slow viscoplastic flows lead to an inequality which is then considered as a problem of minimum; that problem yields, under certain conditions, the possibility of proving the existence theorem.

Dowody istnienia rozwiązań brzegowych w mechanice ośrodków ciągłych nie są, jak wiadomo, łatwe do przeprowadzenia. Dowód taki naszkicować można wychodząc z zasady Dirichleta, która opiera się na założeniu, że istnienie granicy dolnej jest równoważne istnieniu minimum. Zasady ekstremalne, odnoszące się do stosunkowo powolnych przepływów lepkoplastycznych, prowadzą do pewnej nierówności, którą rozważa się jako problem minimalizacji pewnego funkcjonału; z problemu tego można, przy pewnych założeniach, wywieść wniosek o istnieniu rozwiązań.

Доказательства существования решений краевых задач в механике сплошных сред, как известно, не легко провести. Такое доказательство можно наметить исходя из принципа Дирихле, который опирается на предположении, что существование нижнего предела эквивалентно существованию минимума. Экстремальные принципы, относящиеся к сравнительно медленным вязкопластическим течениям, приводят к некоторому неравенству, которое рассматривается как задачу минимизации некоторого функционала; из этой задачи можно сделать, при некоторых предположениях, вывод о существовании решений.

1. Introduction

THE INVESTIGATIONS and research aimed at deriving the constitutive equations which satisfy certain imposed conditions have proved to be successful in recent years as regards one domain of the viscoplasticity theory; it embraces the behaviour of such materials subject to dynamic loading which start to exhibit their viscous properties after the purely elastic straining process is terminated. Papers [1, 2] present a detailed physical and thermodynamic motivation of the constitutive equations. In [3] the fundamental inequalities are derived and discussed, taking as base classical plasticity theory. Paper [4] presents the uniqueness theorem for the solutions of the boundary-value problem of relatively slow viscoplastic flows; it develops not only the minimum principle for velocities but also a maximum principle for stresses. It is thus possible to establish lower and upper bounds for the velocity and stress fields. By using the uniqueness considerations in [4] it was shown that a boundary-value problem in viscoplastic domains has, at the most, one solution; however, there is no information available on the existence of such a solution. From the physical

(*) The paper has been presented at the *EUROMECH* 53 *COLLOQUIUM* on "*THERMOPLASTICITY*", Jabłonna, September 16–19, 1974.

point of view this seems to be obvious but the problem consists in answering the question whether the corresponding constitutive equations allow for the necessary proof of the existence of the solutions.

The proofs of the existence of solutions to boundary-value problems in continuum mechanics are not simple to find; mathematically well-grounded proofs exist only in classical linear elasticity. A more or less heuristic procedure was presented by KOITER [5] for elastic-plastic continua by means of the Dirichlet principle. By using that principle it will be shown here that also in the case of a boundary value problem of a viscoplastic medium the solution exists and is unique.

2. Fundamental equations and constitutive law

2.1. Fundamental equations

Let us confine the description of continua to small strains, displacements and their derivatives and assume, for the sake of simplicity, that the materials considered are isotropic and incompressible. In addition, the elastic components of the strain field will be disregarded.

The mixed stress tensor σ^i_k is decomposed into the deviatoric and spherical components, τ^i_k and σ^r_r:

$$\sigma^i_k = \tau^i_k + \frac{1}{3}\sigma^r_r \delta^i_k. \tag{2.1}$$

The second invariant of deviatoric stresses to be used in future considerations is equal to

$$I_2 = \frac{1}{2}\tau^i_k \tau^k_i. \tag{2.2}$$

The symmetric strain rate tensor $\dot{\varepsilon}^i_k$ is assumed to be derivable from the velocity field v^i, and namely

$$\dot{\varepsilon}^i_k = \frac{1}{2}(v^i|_k + g^{im} g_{ks} v^s|_m), \tag{2.3}$$

where the vertical lines denote differentiation with respect to the convective coordinate θ_k. Here also holds the decomposition into the deviatoric $\dot{\gamma}^i_k$ and spherical components $\dot{\varepsilon}^r_r$, the latter denoting — due to the small deformations assumed — the volume expansion;

$$\dot{\varepsilon}^i_k = \dot{\gamma}^i_k + \frac{1}{3}\dot{\varepsilon}^r_r \delta^i_k. \tag{2.4}$$

The second invariant of the strain rate deviator is

$$\dot{I}_2 = \frac{1}{2}\dot{\gamma}^i_k \dot{\gamma}^k_i. \tag{2.5}$$

The momentum theorem yields the following equilibrium conditions to be satisfied inside the continuum

$$\sigma^k_i|_k + F_i = 0, \tag{2.6}$$

inertia forces being disregarded and F_i denoting the body forces.

2.2. Constitutive law

For the constitutive law describing the behaviour of viscoplastic materials, let us select the relations derived in [4],

$$\text{(2.7)}\qquad \begin{aligned} &\dot{\gamma}^i_k = \gamma \frac{F}{\sqrt{J_2}}[1-\gamma^2\varphi_2 F^2]\tau^i_k && \text{for} \quad F>0, \\ &\dot{\gamma}^i_k = 0 && \text{for} \quad F \leqslant 0. \end{aligned}$$

Here

$$\text{(2.8)}\qquad F = \frac{\sqrt{J_2}}{K} - 1;$$

K is a constant which follows from the von Mises flow rule. γ denotes a viscosity parameter and φ_2 — a material parameter to be determined from experiment.

Solving the Eqs. (2.7) for the stresses we obtain

$$\text{(2.9)}\qquad \tau^i_k = \frac{K}{\sqrt{\dot{I}_2}}\dot{\gamma}^i_k + \frac{K}{\gamma}[1+\varphi_2 \dot{I}_2]\dot{\gamma}^i_k.$$

The considerations to follow will be based on these constitutive relations, attention being paid to certain additional assumptions, especially those regarding the stable behaviour of materials.

3. Existence of solutions

3.1. Boundary value problem

Let us consider a solid continuum subject to the action of body forces F^i and surface forces T^i applied to a portion a_T of the surface, or to prescribed surface displacements V_i at the portion a_v of the surface; the continuum is deformed in a certain manner in the viscoplastic domain. In general, one part of the continuum undergoes viscoplastic deformation, while another part may remain rigid.

In [4], the uniqueness theorem was used to prove that in the deformed continuum there exists, at the most, one solution for stresses which satisfy following conditions:

1. The stresses satisfy the equilibrium equations inside the body

$$\text{(3.1)}\qquad \sigma^k_i|_k + F_i = 0.$$

2. The stresses satisfy the boundary conditions on a_T

$$\text{(3.2)}\qquad \sigma^k_i n_k = T_i,$$

n_k denoting the unit vector normal to the surface.

3. The strain rates $\dot{\gamma}^i_k$ related to the stresses by means of the constitutive law (2.7) are derivable from the velocity field v^i

$$\text{(3.3)}\qquad \dot{\gamma}^i_k = \frac{1}{2}(v^i|_k + v_k|^i).$$

4. The velocities satisfy the boundary conditions on a_v

$$v^i = V^i. \tag{3.4}$$

5. The velocity components satisfy the incompressibility condition

$$v^r|_r = 0. \tag{3.5}$$

3.2. General remarks on the proof of existence and the governing inequality

The proof of uniqueness of the solution does not contain any statement concerning its existence in the case of the boundary value problem formulated in the preceding section. It was mentioned in the introduction that a mathematically correct proof of existence of the solution to a boundary value problem should be difficult. This was true in the case of elastic media, and the more so in plasticity. Thus, we shall select a more intuitive approach originally used by KOITER [5] and following from the Dirichlet principle. The principle is based on the assumption that the existence of a lower bound of a functional is equivalent to the existence of a minimum (cf. also [6]).

In order to construct the proof it is necessary to derive, first of all, the fundamental inequality. From the minimum principle for velocities and the maximum principle for stresses, as derived in [4], the inequality is obtained in the form

$$\int_v \left[2K\sqrt{\dot{I}_2^*}+\frac{K}{\gamma}\dot{I}_2^*+\frac{1}{2}\frac{K}{\gamma}\varphi_2\dot{I}_2^{*2}\right]dv-\int_{a_T} T^i v_i^*\,da-\int_v F^i v_i^*\,dv$$
$$+\frac{1}{4}\gamma K\int_v (|\mathring{F}|+\mathring{F})^2\,dv-\frac{1}{32}\gamma^3\varphi_2 K\int_v (|\mathring{F}|+\mathring{F})^4\,dv-\int_{a_v}\mathring{T}^i V_i\,da \geqslant 0. \tag{3.6}$$

It is true for all kinematically admissible strain rate fields $\dot{\gamma}_k^{i*}$ and all statically admissible stress fields $\mathring{\tau}_k^i$. Inequality (3.6) was derived under the assumption of the existence of a solution to the boundary value problem; thus, it would be of little value for the existence theorem if we were not able to prove that the inequality remains valid also without that basic assumption. This is actually true, and in order to prove it let us multiply the statically admissible stresses $\mathring{\tau}_i^k$ by the kinematically admissible velocity vector $\mathbf{v}^*$ and integrate the result over the entire volume of the continuum. After certain mathematical transformations we obtain

$$\int_v \mathring{\tau}_i^k\dot{\gamma}_k^{i*}\,dv-\int_a \mathring{T}^i v_i^*\,da-\int_v \mathring{F}^i v_i^*\,dv = 0. \tag{3.7}$$

Now we have

$$\int_a \mathring{T}^i v_i^*\,da = \int_{a_v}\mathring{T}^i V_i\,da+\int_{a_T} T^i v_i^*\,da, \tag{3.8}$$

since the kinematically admissible velocity field v_i^* on a_v is equal to V_i, and the statically admissible stress field equals T^i on a_T. In addition

$$\int_v \mathring{F}^i v_i^*\,dv = \int_v F^i v_i^*\,dv,$$

provided the body forces remain unchanged. Following these considerations we transform the inequality (3.6) to obtain

$$(3.9)\qquad \int\limits_{v}\left[2K\sqrt{\dot{I}_2^*}+\frac{K}{\gamma}\dot{I}_2^*+\frac{1}{2}\frac{K}{\gamma}\varphi_2\dot{I}_2^{*2}+\frac{1}{4}\gamma K(|\mathring{F}|+\mathring{F})^2-\frac{1}{32}\gamma^3\varphi_2K(|\mathring{F}|+\mathring{F})^4-\mathring{\tau}_i^k\dot{\gamma}_k^{i*}\right]dv\geqslant 0.$$

According to the procedure outlined in [4] it is now easily shown that (3.9) remains true even without the knowledge of the extremum principles.

3.3. The proof of existence

Let the problem of minimum

$$(3.10)\qquad M=\int\limits_{v}\left[2K\sqrt{\dot{I}_2^*}+\frac{K}{\gamma}\dot{I}_2^*+\frac{1}{2}\frac{K}{\gamma}\varphi_2\dot{I}_2^{*2}+\frac{1}{4}\gamma K(|\mathring{F}|+\mathring{F})^2-\frac{1}{32}\gamma^3\varphi_2K(|\mathring{F}|+\mathring{F})^4-\mathring{\tau}_i^s\dot{\gamma}_t^{i*}\right]dv=\text{minimum}$$

be given as a functional of kinematically admissible strain rates $\dot{\gamma}_k^{i*}$ at a prescribed, statically admissible value of stresses $\mathring{\tau}_k^i$. Strain rates $\dot{\gamma}_k^{i**}$ denote now the strain rates $\dot{\gamma}_k^{i*}$ at which the expression (3.10) attains its minimum, and τ_k^{i**} is the corresponding stress following from the constitutive law. Consequently, the strain rates $\dot{\gamma}_k^{i*}$ are considered as a family of closely related functions and lead to the real minimum; hence

$$(3.11)\qquad \dot{\gamma}_k^{i*}=\dot{\gamma}_k^{i**}+\alpha\dot{\beta}_k^i;$$

$\dot{\beta}_k^i$ denoting a compatible strain rate field with velocities vanishing on a_v, and α — a small coefficient independent of the metric of the system. On substituting the Eq. (3.11) in the minimum problem (3.10) we obtain

$$(3.12)\qquad M=\int\limits_{v}\left[2K\sqrt{\dot{I}_2^{**}+\alpha\dot{\gamma}_k^{i**}\dot{\beta}_k^i+\frac{1}{2}\alpha^2\dot{\beta}_k^i\dot{\beta}_i^k}+\frac{K}{\gamma}\left(\dot{I}_2^{**}+\alpha\dot{\gamma}_k^{i**}\dot{\beta}_i^k+\frac{1}{2}\alpha^2\dot{\beta}_k^i\dot{\beta}_i^k\right)+\frac{1}{2}\frac{K}{\gamma}\varphi_2\left(\dot{I}_2^{**}+\alpha\dot{\gamma}_k^{i**}\dot{\beta}_i^k+\frac{1}{2}\alpha^2\dot{\beta}_k^i\dot{\beta}_i^k\right)^2+\frac{1}{4}\gamma K(|\mathring{F}|+\mathring{F})^2-\frac{1}{32}\gamma^3\varphi_2K(|\mathring{F}|+\mathring{F})^4-\mathring{\tau}_i^k(\dot{\gamma}_k^{i**}+\alpha\dot{\beta}_k^i)\right]dv=\text{minimum}.$$

The necessary condition of minimum has now the form

$$(3.13)\qquad \left(\frac{\partial M}{\partial\alpha}\right)_{\alpha=0}=0.$$

Evaluation of the second derivative at $\alpha=0$ shows that this is really a minimum; the second derivative is always positive at that point.

Applying the condition (3.13) to the minimum problem (3.12) we obtain

$$(3.14)\qquad \left(\frac{\partial M}{\partial \dot{\alpha}}\right)_{\alpha=0} = \int_v \left[\left(\frac{K}{\sqrt{\dot{I}_2^{**}}}\dot{\gamma}_k^{i**} + \frac{K}{\gamma}\dot{\gamma}_k^{i**} + \frac{K}{\gamma}\varphi_2 \dot{I}_2^{**}\dot{\gamma}_k^{i**}\right)\dot{\beta}_i^k - \mathring{\tau}_i^k\dot{\beta}_k^i\right]dv = 0.$$

This condition of minimum combined with the constitutive equations (2.9) assumes the form

$$(3.15)\qquad \int_v (\tau_i^{k**} - \mathring{\tau}_i^k)\dot{\beta}_k^i\, dv.$$

According to the virtual work principle $\tau_k^{i**} - \mathring{\tau}_i^k$ is a distribution of stresses with surface forces vanishing on a_T. It follows that τ_i^{k**} is a statically admissible stress distribution. The distribution satisfies the equilibrium condition (3.1) inside the continuum, and condition (3.2) — on a_T; the corresponding strain rates satisfy the kinematic field equations (3.3) and the geometric boundary conditions (3.4), as also the incompressibility condition (3.5). Thus the stress field τ_i^{k**} represents the solution τ_i^k of the boundary value problem.

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