NOTES AND REFERENCES.

518. RIBAUCOUR, *C. R.*, t. LXXV. (1872), pp. 533—536, referring to my Note remarks that the condition can be (by means of the imaginary coordinates of M. Ossian Bonnet) expressed in a simple form communicated by him to the Philomathic Society, May, 1870. I reproduce this investigation, although it is not easy to present it in a quite intelligible form. We take $p=f(x, y, z)$ to represent a family of surfaces belonging to a triply orthotomic system, and consider two neighbouring surfaces (A) and (A') corresponding to the values z and $z+dz$; A and A' the two points where they meet the trajectories of the surfaces; AT, $A'T'$ the tangents to the curves of curvature of the same system at A, A', respectively. Then according to the remark of M. Lévy, it is to be expressed that these lines meet, and this is done by expressing that *along the trajectory AA', the variation of the angle of AT with the osculating plane at A is equal to the angle of the osculating planes at A, A' respectively.*

Let B' be the spherical image of A', the plane OBB' is parallel to the osculating plane at A of the trajectory, and the angle of the two osculating planes measures the geodesic curvature of BB': denote this by $d\gamma$.

Let β be the angle of BB' with BX, θ the angle of AT with BX, $\beta-\theta$ is the angle of AT with the osculating plane at A of the trajectory: $d\beta-d\theta=d\gamma$. Introducing the symmetric imaginary coordinates x and y, we write

$$a=\frac{dp}{\lambda^2\,dx},\quad b=\frac{dp}{\lambda^2\,dy},\quad c=\frac{1}{\lambda^2}\,\frac{d^2p}{dx\,dy},\quad ds^2=4\lambda^2\frac{da}{dx}\,\frac{db}{dy}\,dx\,dy.$$

But dx and dy being the increments of x, y corresponding to dz in the passage from A to A', then by a theorem of M. Liouville

$$d\gamma=d\beta-i\left(\frac{d\lambda}{\lambda\,dx}\,dx-\frac{d\lambda}{\lambda\,dy}\,dy\right);$$

the condition thus is

$$d\theta=i\left(\frac{d\lambda}{\lambda\,dx}\,dx-\frac{d\lambda}{\lambda\,dy}\,dy\right),$$

and the formula

$$e^{-2i\theta} = \pm \sqrt{\frac{da}{dx}} \div \sqrt{\frac{db}{dy}},$$

enables this to be written in the definitive form

$$dx \frac{d}{dx} l \left(\lambda^4 \frac{db}{dy} \div \frac{da}{dx}\right) + dy \frac{d}{dy} l \left(\frac{db}{dy} \div \lambda^4 \frac{da}{dx}\right) + dz \left\{\frac{d}{dz}\left(l \frac{db}{dy}\right) - \frac{d}{dz}\left(l \frac{da}{dx}\right)\right\} = 0.$$

We have

$$dx\left(\tfrac{1}{2}p + c\right) + dy \frac{db}{dy} \qquad + dz \frac{db}{dz} = 0,$$

$$dx \frac{da}{dx} \qquad + dy\left(\tfrac{1}{2}p + c\right) + dz \frac{da}{dz} = 0,$$

and thence eliminating dx, dy, dz, we have

$$\begin{vmatrix} \frac{d}{dx} l \left(\lambda^4 \frac{db}{dy} \div \frac{da}{dx}\right), & \frac{d}{dy} l \left(\frac{db}{dy} \div \lambda^4 \frac{da}{dx}\right), & \frac{d}{dz} l \left(\frac{db}{dy} \div \frac{da}{dx}\right) \\ \tfrac{1}{2}p + c \quad , & \frac{db}{dy} \quad , & \frac{db}{dz} \\ \frac{da}{dx} \quad , & \tfrac{1}{2}p + c \quad , & \frac{da}{dz} \end{vmatrix} = 0,$$

which defines the triply orthotomic system.

END OF VOL. VIII.