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ON A DIFFERENTIAL FORMULA CONNECTED WITH THE THEORY OF CONFOCAL CONICS.

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THE following transformations present themselves in connexion with the theory of confocal conics.

The coordinates x, y of a point are considered as functions of the parameters h, k where

$$\frac{x^2}{a+h}+\frac{y^2}{b+h}=1,$$

$$\frac{x^2}{a+k}+\frac{y^2}{b+k}=1;$$

and then assuming $\xi=x+iy$, $\eta=x-iy$ $\left(i=\sqrt{(-1)}\right.$ as usual$\left.\right)$, and writing $c=a-b$, we find

$$h=\tfrac{1}{2}(-a-b+\xi\eta)+\tfrac{1}{2}\sqrt{\{(\xi^2-c)(\eta^2-c)\}},$$

$$k=\tfrac{1}{2}(-a-b+\xi\eta)-\tfrac{1}{2}\sqrt{\{(\xi^2-c)(\eta^2-c)\}},$$

whence if

$$H=(a+h)(b+h),\quad K=(a+k)(b+k),$$

we have

$$H=\tfrac{1}{4}\{\xi\sqrt{(\eta^2-c)}+\eta\sqrt{(\xi^2-c)}\}^2,$$

$$K=\tfrac{1}{4}\{\xi\sqrt{(\eta^2-c)}-\eta\sqrt{(\xi^2-c)}\}^2,$$

or, say

$$\sqrt{(H)}=\tfrac{1}{2}\{\xi\sqrt{(\eta^2-c)}+\eta\sqrt{(\xi^2-c)}\},$$

$$\sqrt{(K)}=\tfrac{1}{2}\{\xi\sqrt{(\eta^2-c)}-\eta\sqrt{(\xi^2-c)}\},$$

and thence

$$h + \tfrac{1}{2}(a+b) + \sqrt{(H)} = \tfrac{1}{2}\{\xi + \sqrt{(\xi^2 - c)}\}\{\eta + \sqrt{(\eta^2 - c)}\},$$

$$k + \tfrac{1}{2}(a+b) + \sqrt{(K)} = \tfrac{1}{2}\{\xi + \sqrt{(\xi^2 - c)}\}\{\eta - \sqrt{(\eta^2 - c)}\}$$

$$= \tfrac{1}{2}c\,\frac{\xi + \sqrt{(\xi^2 - c)}}{\eta + \sqrt{(\eta^2 - c)}}.$$

These also follow from the known differential formula

$$4(dx^2 + dy^2) = (h-k)\left(\frac{dh^2}{H} - \frac{dk^2}{K}\right),$$

that is,

$$\frac{4d\xi d\eta}{\sqrt{(\xi^2 - c)}\sqrt{(\eta^2 - c)}} = \frac{dh^2}{H} - \frac{dk^2}{K},$$

implying

$$\frac{2\alpha d\xi}{\sqrt{(\xi^2 - c)}} = \frac{dh}{\sqrt{(H)}} + \frac{dk}{\sqrt{(K)}},$$

$$\frac{2d\eta}{\alpha\sqrt{(\eta^2 - c)}} = \frac{dh}{\sqrt{(H)}} - \frac{dk}{\sqrt{(K)}},$$

where α is a constant. The foregoing integral formulæ give at once

$$\frac{dh}{\sqrt{(H)}} = \frac{d\xi}{\sqrt{(\xi^2 - c)}} + \frac{d\eta}{\sqrt{(\eta^2 - c)}},$$

$$\frac{dk}{\sqrt{(K)}} = \frac{d\xi}{\sqrt{(\xi^2 - c)}} - \frac{d\eta}{\sqrt{(\eta^2 - c)}},$$

and substituting these values we find $\alpha = 1$, and the differential formulæ are then satisfied.

We thence have

$$\text{const.} = \sqrt{\{(a+h)(b+h)\}} \pm \sqrt{\{(a+k)(b+k)\}},$$

as the integral of the differential equation

$$\frac{dh}{\sqrt{(H)}} \pm \frac{dk}{\sqrt{(K)}} = 0.$$