543.

ON AN IDENTITY IN SPHERICAL TRIGONOMETRY.

[From the *Messenger of Mathematics*, vol. I. (1872), p. 145.]

In a spherical triangle, writing for shortness α, β, γ for the cosines and α', β', γ' for the sines, of the sides: also

$$\Delta^2 = 1 - \alpha^2 - \beta^2 - \gamma^2 + 2\alpha\beta\gamma;$$

we have

$$\cos A = \frac{\alpha - \beta\gamma}{\beta'\gamma'}, \quad \sin A = \frac{\Delta}{\beta'\gamma'},$$

with the like expressions in regard to the other two angles B, C respectively.

Hence

$$\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \&\text{c}.$$

$$= \frac{(\alpha - \beta\gamma)(\beta - \gamma\alpha)(\gamma - \alpha\beta) - \Delta^2(\alpha + \beta + \gamma - \beta\gamma - \gamma\alpha - \alpha\beta)}{(1-\alpha^2)(1-\beta^2)(1-\gamma^2)}.$$

The numerator is identically

$$= (1-\alpha)(1-\beta)(1-\gamma)[\Delta^2 - (1+\alpha)(1+\beta)(1+\gamma)],$$

viz. comparing the two expressions, we have

$$(1-\alpha)(1-\beta)(1-\gamma)\Delta^2 - (1-\alpha^2)(1-\beta^2)(1-\gamma^2)$$
$$= (\alpha-\beta\gamma)(\beta-\gamma\alpha)(\gamma-\alpha\beta) + \Delta^2(-\alpha-\beta-\gamma+\beta\gamma+\gamma\alpha+\alpha\beta);$$

or, what is the same thing,

$$(1-\alpha\beta\gamma)\Delta^2 = (1-\alpha^2)(1-\beta^2)(1-\gamma^2) + (\alpha-\beta\gamma)(\beta-\gamma\alpha)(\gamma-\alpha\beta),$$

which is the identity in question and can be immediately verified. We have thus

$$\cos(A+B+C) = \frac{\Delta^2 - (1+\alpha)(1+\beta)(1+\gamma)}{(1+\alpha)(1+\beta)(1+\gamma)},$$

and thence

$$1 + \cos(A+B+C) = \frac{\Delta^2}{(1+\alpha)(1+\beta)(1+\gamma)},$$

$$1 - \cos(A+B+C) = \frac{2(1+\alpha)(1+\beta)(1+\gamma) - \Delta^2}{(1+\alpha)(1+\beta)(1+\gamma)}$$

giving at once the values of $\cos^2\frac{1}{2}(A+B+C)$, $\sin^2\frac{1}{2}(A+B+C)$, $\sin(A+B+C)$, and $\tan^2\frac{1}{2}(A+B+C)$: these are known expressions in regard to the spherical excess.