535.

NOTE ON THE PROBLEM OF ENVELOPES.

[From the *Messenger of Mathematics*, vol. I. (1872), pp. 3, 4.]

THERE is a mode of looking at the problem of Envelopes, which, so far as I am aware, has not been explicitly noticed. Let $U=(x, y, z)^m$ be a function of the coordinates (x, y, z), $\Theta=\Theta'=(x, y, z)^a(x', y', z')^{a'}$ a function of the two sets of coordinates (x, y, z) and (x', y', z'); it being understood that when we write Θ we regard (x, y, z) as the current coordinates, when Θ' we regard (x', y', z') as the current coordinates. Suppose that we have $U=0$; the curve $\Theta'=0$ is then a curve the equation whereof contains as parameters the coordinates (x, y, z) of a point P on the curve $U=0$; and we may seek for the envelope of the curve $\Theta'=0$ as P describes the curve $U=0$; the required envelope is of course obtained as an equation in (x', y', z') given by the elimination of x, y, z, λ from the equations (equivalent to four equations only)

$$U=0,\quad \Theta'=0,$$
$$d_x\Theta'+\lambda d_xU=0,$$
$$d_y\Theta'+\lambda d_yU=0,$$
$$d_z\Theta'+\lambda d_zU=0.$$

But, observe that the required envelope is the locus of the points of intersection of the curve $\Theta'=0$ belonging to a particular point (x, y, z) of the curve $U=0$, by the curve $\Theta'=0$ which belongs to a consecutive point of U. The curve $\Theta=0$, considering therein (x', y', z') as the coordinates of a given point of the plane, determines by its intersection with $U=0$ those points (x, y, z) on the curve $U=0$, to each of which belongs a curve $\Theta'=0$ passing through the point in question (x', y', z'). Hence, if the curve $\Theta=0$ touch the curve $U=0$, the point of contact, coordinates (x, y, z), is a point such that to it and to the consecutive point there belong curves, each of them passing through the given point (x', y', z'). Hence expressing that the curves

$\Theta = 0$, $U = 0$ touch each other, we have a relation in (x', y', z') which is the locus of the point of intersection of the curves $\Theta' = 0$ belonging to two consecutive points of the curve $U = 0$; that is, the equation of the required envelope is obtained as the condition that the curves $U = 0$, $\Theta = 0$ shall touch each other. But when the curves touch each other, they have at the point of contact their derived functions proportional, or we have simultaneously

$$\begin{aligned} U = 0, \quad \Theta &= 0, \\ d_x\Theta + \lambda d_x U &= 0, \\ d_y\Theta + \lambda d_y U &= 0, \\ d_z\Theta + \lambda d_z U &= 0, \end{aligned}$$

the same equations as before, since Θ and Θ' denote the same function.

It is to be added that, when $a = m$, the equations

$$\begin{aligned} d_x\Theta + \lambda d_x U &= 0, \\ d_y\Theta + \lambda d_y U &= 0, \\ d_z\Theta + \lambda d_z U &= 0, \end{aligned}$$

are homogeneous in (x, y, z), and we may by the elimination of (x, y, z) from these equations obtain an equation Disct. $(\Theta + \lambda U) = 0$, say for shortness $\Lambda = 0$, involving λ and also the coordinates (x', y', z'). Now it is a known theorem that the condition for the contact of the two curves $U = 0$, $\Theta = 0$ can be obtained by expressing that the equation $\Lambda = 0$ shall have a pair of equal roots, or, what is the same thing, by equating to zero the discriminant of the function Λ; this last-mentioned process leads therefore to the equation of the envelope of the curve $\Theta' = 0$, viz. (a being $= m$ as above) the equation of the envelope of the curve $\Theta' = 0$, is in fact

$$\text{Disct.}_{\lambda}\ \text{Disct.}_{(x, y, z)} (\Theta + \lambda U) = 0,$$

viz. we first take the discriminant of the function $\Theta + \lambda U$ in regard to the coordinates (x, y, z), and then taking the discriminant in regard to λ of this discriminant we equate it to zero. This is in many cases a more simple process than that of the direct elimination of x, y, z, λ from the five equations.