531.

A "SMITH'S PRIZE" PAPER(1); SOLUTIONS.

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1. *If* $\sqrt{\{(x-a)^2+(y-b)^2\}}$, $\sqrt{\{(x-a')^2+(y-b')^2\}}$, *c, are respectively real and positive, show that in the equation* $\pm\sqrt{\{(x-a)^2+(y-b)^2\}} \pm \sqrt{\{(x-a')^2+(y-b')^2\}} = c$, *considered as representing a curve, the signs cannot either of them be assumed at pleasure to be + or to be −: and distinguish the cases of the ellipse and the hyperbola.*

Writing the equation in the form $\pm\sqrt{(S)} \pm \sqrt{(S')} = c$, we find

$$\pm 2c\sqrt{(S)} = c^2 + S - S',$$

$$\pm 2c\sqrt{(S')} = c^2 - S + S',$$

and thence

$$4c^2 S = (c^2 + S - S')^2,$$

$$4c^2 S' = (c^2 - S + S')^2,$$

either of which equations is the rational equation of the curve (the equation being of the second order, inasmuch as $S - S'$ is a linear function of the coordinates). But writing the rational equation under these two forms respectively and passing back to the last preceding forms, it is clear that for any given point of the curve, $c^2 + S - S'$ and $c^2 - S + S'$, *quà* rational functions of the coordinates, have each of them a completely determinate value; the ambiguous sign therefore cannot be assumed at pleasure, but in the equation $\pm 2c\sqrt{(S)} = c^2 + S - S'$ it must be taken to be + or to be − according as the value of $c^2 + S - S'$ is positive or is negative; and the like for the other equation. It is with the signs so determined that the two irrational equations hold good, and inasmuch as from them we deduce the original equation $\pm\sqrt{(S)} \pm \sqrt{(S')} = c$, the signs in this equation are not arbitrary, but each of them has, at a given point of the curve, a determinate value, fixed as above.

[1] Set by me for the Master of Trinity, Feb. 3, 1869.

The condition for an ellipse is that the given length c shall be greater than the distance $\sqrt{\{(a-a')^2+(b-b')^2\}}$ between the two foci; for a hyperbola that it shall be less. In the former case, for any real point of the curve, it is obvious that the relation *must be* $\sqrt{(S)}+\sqrt{(S')}=c$; in the latter case, for any real point of the curve it *must be* either $\sqrt{(S)}-\sqrt{(S')}=c$ or $-\sqrt{(S)}+\sqrt{(S')}=c$, viz. one of these equations holds for one branch, the other for the other branch of the hyperbola. To see this *à posteriori*, observe that in the ellipse, starting from the equation

$$\pm\sqrt{\{(x-ae)^2+y^2\}}\pm\sqrt{\{(x+ae)^2+y^2\}}=2a,$$

we find

$$\pm\sqrt{\{(x-ae)^2+y^2\}}=a-ex,$$

$$\pm\sqrt{\{(x+ae)^2+y^2\}}=a+ex,$$

but here, e being less than 1, and for every real point of the curve x being less in absolute magnitude than a, we have $a-ex$, $a+ex$ each positive for any real point whatever of the curve; the two signs are therefore each of them $+$, or we have $\sqrt{\{(x-ae)^2+y^2\}}+\sqrt{\{(x+ae)^2+y^2\}}=2a$; and a like verification applies to the hyperbola.

2. *In a system of curves defined by an equation containing a variable parameter, investigate at any point the normal distance between two consecutive curves; and determine the form of the equation for a system of parallel curves.*

Consider the system of curves $f(x, y, c)=0$; then if the point x, y belongs to the curve $f(x, y, c)=0$, and the point $x+\delta x$, $y+\delta y$ to the curve $f(x, y, c+\delta c)=0$, we have

$$\frac{df}{dx}\delta x+\frac{df}{dy}\delta y+\frac{df}{dc}\delta c=0,$$

and if the point $x+\delta x$, $y+\delta y$ be on the normal at (x, y) to the curve $f(x, y, c)=0$, we have

$$\delta x\div\frac{df}{dx}=\delta y\div\frac{df}{dy},$$

or writing

$$\delta x=k\frac{df}{dx},\quad \delta y=k\frac{df}{dy},$$

we have

$$k\left\{\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2\right\}+\frac{df}{dc}\delta c=0,$$

wherefore

$$k=-\frac{df}{dc}\delta c\div\left\{\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2\right\},$$

and the normal distance at (x, y) of the curves c and $c+\delta c$ is $=\sqrt{(\delta x^2+\delta y^2)}$, viz., it is $=\pm k\sqrt{\left\{\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2\right\}}$, or finally it is

$$=\pm\frac{df}{dc}\delta c\div\sqrt{\left\{\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2\right\}},$$

where, if the distance in question be regarded as positive (that is, if we attend only to its absolute value), the sign is to be taken so that $\pm\frac{df}{dc}\delta c$ shall be positive.

If the system be given in the form $V - c = 0$ (V a function of (x, y)), we have the normal distance

$$= \pm \delta c \div \sqrt{\left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2\right\}}.$$

For parallel curves, the normal distance is everywhere the same, that is, $\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2$ must have a constant value for all values of (x, y) satisfying the relation $V = c$, viz. we must have identically $\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 = \phi(V)$, ϕ arbitrary, a partial differential equation to be satisfied by V in order that $V = c$ may be the equation of a system of parallel curves. Assuming the equation to be satisfied for any particular form of ϕ, we may, it is clear, find U a function of V, such that $\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2 = 1$, and inasmuch as the equation $f(V) = Const.$ is the same thing as $V = Const.$, it follows that the equation of the system of parallel curves may be taken to be $V = c$, where

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 = 1.$$

3. *Two cannons (each free to recoil) differ only in weight and in the weight of the ball; and it is assumed that at any instant during the explosion the explosive force depends only on the space occupied by the vapour of the gunpowder: compare the emerging velocities of the balls; and also the emerging velocities of balls fired from the same cannon when it is free to recoil, and when it is absolutely fixed.*

Consider a single cannon.

Take M for its mass, m for that of the ball, S, s for the spaces described, backwards by the cannon and forwards by the ball, at the time t during the explosion.

Then by hypothesis the explosive force, forwards on the ball and backwards on the cannon, is a function of $s + S$, $= \phi(s + S)$ suppose, or we have

$$m\frac{d^2s}{dt^2} = \phi(s + S),$$

$$M\frac{d^2S}{dt^2} = \phi(s + S),$$

and thence

$$\frac{d^2(s + S)}{dt^2} = \left(\frac{1}{m} + \frac{1}{M}\right)\phi(s + S).$$

Multiplying each side by $2\frac{d(s + S)}{dt}$, integrating from $s + S = 0$, to $s + S = a$, where a is the length of the tube, and observing that the initial velocities are each $= 0$, we find

$$\left\{\frac{d(s + S)}{dt}\right\}^2 = 2\left(\frac{1}{m} + \frac{1}{M}\right)\int_0^a \phi(s + S)(ds + dS),$$

where $\frac{ds}{dt}$, $\frac{dS}{dt}$ are the velocities of the ball and cannon respectively at the instant of emergence; $\int_0^a \phi(s+S)(ds+dS)$, is a constant (that is a quantity independent of the weights of the ball and cannon), and putting it $=\frac{1}{2}C$, the equation may be written

$$(v+V)^2 = \left(\frac{1}{m}+\frac{1}{M}\right)C,$$

where v, V are the velocities at the instant of emergence.

But from the equation

$$m\frac{d^2s}{dt^2} = M\frac{d^2S}{dt^2},$$

(the initial values of the velocities being each $=0$), we have

$$mv = MV;$$

and hence from the foregoing equation we obtain

$$v = \frac{M}{M+m}(V+v), = \frac{M}{M+m}\,\frac{\sqrt{(M+m)}}{\sqrt{(Mm)}}\sqrt{(C)} = \frac{\sqrt{(M)}}{\sqrt{(m)}\sqrt{(M+m)}}\sqrt{(C)} = \frac{\sqrt{(C)}}{\sqrt{(m)}\sqrt{\left(1+\frac{m}{M}\right)}},$$

or say

$$v\sqrt{(m)} = \frac{\sqrt{(C)}}{\sqrt{\left(1+\frac{m}{M}\right)}}.$$

And similarly for the other cannon, if m_1, M_1 be the mass of the ball and cannon, v_1 the velocity of the ball, we have

$$v_1\sqrt{(m_1)} = \frac{\sqrt{(C)}}{\sqrt{\left(1+\frac{m_1}{M_1}\right)}},$$

whence

$$v\sqrt{(m)} : v_1\sqrt{(m_1)} = \sqrt{\left(1+\frac{m_1}{M_1}\right)} : \sqrt{\left(1+\frac{m}{M}\right)}.$$

If $m_1 = m$, $M_1 = \infty$, then v, v_1 may be taken to be the velocities of the same ball fired from the cannon, mass M, when it is free to recoil and when it is absolutely fixed; and we have

$$v : v_1 = 1 : \sqrt{\left(1+\frac{m}{M}\right)},$$

viz. the velocity in the former case is equal to that in the latter case divided by

$$\sqrt{\left(1+\frac{m}{M}\right)}.$$

4. *If* $X : Y : Z = \eta\zeta' - \eta'\zeta : \zeta\xi' - \zeta'\xi : \xi\eta' - \xi'\eta$, *where* $\xi^2 + \eta^2 + \zeta^2 = 0$, $\xi'^2 + \eta'^2 + \zeta'^2 = 0$, *show that* $\xi\xi'$, $\eta\eta'$, $\zeta\zeta'$, $\eta\zeta' + \eta'\zeta$, $\zeta\xi' + \zeta'\xi$, $\xi\eta' + \xi'\eta$ *are proportional to quadric functions of* X, Y, Z.

We have

$$Y^2 + Z^2 = (\zeta\xi' - \zeta'\xi)^2 + (\xi\eta' - \xi'\eta)^2$$

$$= \xi'^2(\eta^2 + \zeta^2) + \xi^2(\eta'^2 + \zeta'^2) - 2\xi\xi'(\eta\eta' + \zeta\zeta'),$$

which, in virtue of the equations

$$\xi^2 + \eta^2 + \zeta^2 = 0, \quad \xi'^2 + \eta'^2 + \zeta'^2 = 0,$$

is

$$= -2\xi^2\xi'^2 - 2\xi\xi'(\eta\eta' + \zeta\zeta')$$

$$= -2\xi\xi'(\xi\xi' + \eta\eta' + \zeta\zeta').$$

And again

$$YZ = (\zeta\xi' - \zeta'\xi)(\xi\eta' - \xi'\eta)$$

$$= \xi\xi'(\eta\zeta' + \eta'\zeta) - \xi'^2\eta\zeta - \xi^2\eta'\zeta',$$

which, in virtue of the same equations, is

$$= \xi\xi'(\eta\zeta' + \eta'\zeta) + \eta\zeta(\eta'^2 + \zeta'^2) + \eta'\zeta'(\eta^2 + \zeta^2)$$

$$= (\eta\zeta' + \eta'\zeta)(\xi\xi' + \eta\eta' + \zeta\zeta'),$$

and, forming the analogous equations by symmetry, the factor $(\xi\xi' + \eta\eta' + \zeta\zeta')$ divides out, and we have

$$Y^2 + Z^2 : Z^2 + X^2 : X^2 + Y^2 : -2YZ : -2ZX : -2XY$$
$$= \xi\xi' : \eta\eta' : \zeta\zeta' : \eta\zeta' + \eta'\zeta : \zeta\xi' + \zeta'\xi : \xi\eta' + \xi'\eta,$$

which is the required theorem.

5. *Two tangents of a conic are harmonically related to a second conic: find the locus of the intersection of the two tangents.*

In plane geometry the angle in a semicircle is, in spherical geometry it is not, a right angle: show how these conclusions follow from the solution of the above problem.

Let the equation of the first conic be $x^2 + y^2 + z^2 = 0$; its equation in line coordinates is therefore $u^2 + v^2 + w^2 = 0$; or what is the same thing, the line $ux + vy + wz = 0$ will be a tangent of the conic if only $u^2 + v^2 + w^2 = 0$. Hence the equations of the two tangents being

$$\xi x + \eta y + \zeta z = 0,$$

$$\xi' x + \eta' y + \zeta' z = 0,$$

we have $\xi^2 + \eta^2 + \zeta^2 = 0$, $\xi'^2 + \eta'^2 + \zeta'^2 = 0$; and using X, Y, Z for the coordinates of the point of intersection of these tangents, we have

$$X : Y : Z = \eta\zeta' - \eta'\zeta : \zeta\xi' - \zeta'\xi : \xi\eta' - \xi'\eta,$$

and thence, by the last question,

$$\xi\xi' : \eta\eta' : \zeta\zeta' : \eta\zeta' + \eta'\zeta : \zeta\xi' + \zeta'\xi : \xi\eta' + \xi'\eta$$
$$= Y^2 + Z^2 : Z^2 + X^2 : X^2 + Y^2 : -2YZ : -2ZX : -2XY.$$

Suppose that the equation of the second conic in line coordinates is

$$(a, b, c, f, g, h)(u, v, w)^2 = 0;$$

the condition in order that the two lines $\xi x + \eta y + \zeta z = 0$, $\xi' x + \eta' y + \zeta' z = 0$, may be harmonically related to this conic is

$$(a, b, c, f, g, h)(\xi, \eta, \zeta)(\xi', \eta', \zeta') = 0;$$

or, what is the same thing, it is

$$a\xi\xi' + b\eta\eta' + c\zeta\zeta' + f(\eta\zeta' + \eta'\zeta) + g(\zeta\xi' + \zeta'\xi) + h(\xi\eta' + \xi'\eta) = 0,$$

and by what precedes we have

$$a(Y^2 + Z^2) + b(Z^2 + X^2) + c(X^2 + Y^2) - 2fYZ - 2gZX - 2hXY = 0,$$

or, what is the same thing,

$$(a + b + c)(X^2 + Y^2 + Z^2) - (a, b, c, f, g, h)(X, Y, Z)^2 = 0,$$

as the locus of the point of intersection of the two conics; the required locus is therefore a conic.

It is to be observed that the locus is that of a point which is such that the pairs of tangents from it to the two given conics respectively form a harmonic pencil; viz. if the equations of the given conics (in line coordinates) are

$$u^2 + v^2 + w^2 = 0,$$
$$(a, b, c, f, g, h)(u, v, w)^2 = 0;$$

then the equation of the required locus, or say the equation of the harmonic conic, is (in point coordinates)

$$(a + b + c)(x^2 + y^2 + z^2) - (a, b, c, f, g, h)(x, y, z)^2 = 0.$$

In particular, if the second conic be a point-pair or, say, if its equation be

$$(\alpha u + \beta v + \gamma w)(\alpha' u + \beta' v + \gamma' w) = 0,$$

then the equation of the harmonic conic is

$$(\alpha\alpha' + \beta\beta' + \gamma\gamma')(x^2 + y^2 + z^2) - (\alpha x + \beta y + \gamma z)(\alpha' x + \beta' y + \gamma' z) = 0,$$

which is satisfied by writing $(x, y, z) = (\alpha, \beta, \gamma)$, or $= (\alpha', \beta', \gamma')$; viz. the harmonic conic passes through the two points of the point-pair. And so, if the first conic is also a point-pair, the harmonic conic passes through the four points of the two point-pairs.

The equation of the first conic in point coordinates is $x^2+y^2+z^2=0$; hence, supposing as above, that the second conic is a point-pair, the intersections of the first conic with the harmonic conic are given by

$$x^2+y^2+z^2=0,\quad (\alpha x+\beta y+\gamma z)(\alpha' x+\beta' y+\gamma' z)=0\,;$$

whence the harmonic conic has double contact with the first conic, only if each of the lines $\alpha x+\beta y+\gamma z$, $\alpha' x+\beta' y+\gamma' z=0$, touches the first conic; or, what is the same thing, if each of the points (α, β, γ), $(\alpha', \beta', \gamma')$ lies on the first conic.

In plane geometry, we have (in the plane) a point-pair, the two circular points at infinity; any conic through these points is a circle: the two lines harmonic in regard hereto are at right angles. Hence, by what precedes, taking one of the given conics to be the circular points at infinity, and the other conic to be any two points P, Q; the locus of the intersection of lines through P, Q, cutting each other at right angles, is a circle; this is evidently the circle standing on PQ as diameter, or the angle in a semicircle is a right angle.

In spherical geometry we have (on the sphere) an imaginary conic $x^2+y^2+z^2=0$, called the *absolute;* any conic having double contact herewith is a circle (small circle of the sphere); two lines (arcs of great circles), harmonic in regard hereto, are at right angles. Hence, taking one of the conics to be the conic $x^2+y^2+z^2=0$ and the other to be the two points P, Q; the locus of the intersections of the lines (arcs of great circles) through P, Q, which cut at right angles, is a spherical conic; but it is *not a circle* unless the points P, Q, are each of them on the conic $x^2+y^2+z^2=0$, viz. it is not a circle for any two real points whatever: that is, in spherical geometry, the angle in a semicircle is not a right angle.

6. *A mass M attached to the end A of a chain AC is placed (with the chain) on a horizontal plane, in such wise that a portion AB of the chain forms a straight line, the remaining portion BC being heaped up at B: the mass M is then set in motion in the direction B to A with a given velocity, and so moves in a straight line, dragging the chain: determine the motion; and explain the peculiarity of the dynamical problem.*

The mass attached to the end of the chain is taken to be M, and the mass of a unit of length of the chain to be $=m$; suppose also that at the commencement of the motion the distance CA is $=a$, and that at the end of the time t this distance is $=a+x$, so that the length of chain then in motion is $=a+x$, ($x<l-a$, if l be the whole length of the chain). Suppose also that the velocity at the time t is $=v$; then we have a mass $=M+m(a+x)$ moving with a velocity v; and, in the element of time dt, this sets in motion with the velocity v a length of chain $dx=vdt$, or mass of chain $=mvdt$; if then the impulse backwards on the mass $M+m(a+x)$, and forwards on the element $mvdt$ be $=R$, we have

$$\{M+m(a+x)\}\,dv=-R,$$

$$mvdt\,.\,v=R,$$

that is,

$$\{M+m(a+x)\}\,dv+mv^2dt=0\,;$$

or, observing that $vdt=dx$, this may be written

$$(M+ma)\,dv+md\,.\,xv=0,$$

that is,

$$\{M+m\,(a+x)\}\,v=\text{constant},$$
$$=(M+ma)\,V,$$

if V be the velocity at the commencement of the motion. The equation gives, in terms of the space x, the velocity v at any time t before the whole chain is set in motion: writing it in the form

$$\{M+m\,(a+x)\}\,\frac{dx}{dt}=(M+ma)\,V,$$

we have

$$(M+ma)\,x+\tfrac{1}{2}mx^2=(M+ma)\,Vt,$$

and putting herein $x=l-a$, the equation gives the value of t at the instant when the whole chain is set in motion: after this epoch, the mass and chain will (it is clear) move on with a uniform velocity

$$=\frac{(M+ma)\,V}{M+ml}.$$

The foregoing equation

$$\{M+m\,(a+x)\}\,v=(M+ma)\,V$$

might have been obtained at once by the consideration that the momentum is constant throughout the motion; but the method employed puts more clearly in evidence the peculiarity of the dynamical problem; viz. it is, so to speak, a problem of continuous impulse: in each element of time dt an infinitesimal element of mass has its velocity abruptly altered (in the present problem from 0 to v), but, for the very reason that it is an infinitesimal element of mass which undergoes this abrupt change of velocity, the effect is a continuous, not an abrupt, change of velocity of the whole finite mass which is then in motion.

7. *Show how an ellipse may be constructed as the envelope of a variable circle having its centre upon either of the axes; and examine the geometrical peculiarities which occur according as the major or the minor axis is made use of.*

Considering the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

the equation

$$(x-\alpha)^2+y^2=\gamma^2,$$

where α and γ are in the first instance arbitrary parameters, represents a circle having its centre on the major axis: in order that this may touch the ellipse, a relation must be established between α and γ. When this is done we obtain a variable circle (the equation of which contains a single arbitrary parameter), and this circle has the ellipse for its envelope. (On account of the symmetry in regard to the axis of x, the circle will, it is clear, touch the ellipse in *two* points, situate symmetrically in regard to this axis.) Now to make the circle touch the ellipse, eliminating y, we have

$$(x-\alpha)^2+b^2\left(1-\frac{x^2}{a^2}\right)-\gamma^2=0,$$

that is,

$$x^2\left(1-\frac{b^2}{a^2}\right)-2\alpha x+\alpha^2-\gamma^2+b^2=0,$$

which equation, considered as an equation in x, must have equal roots; that is, we must have

$$\left(1-\frac{b^2}{a^2}\right)(\alpha^2-\gamma^2+b^2)-\alpha^2=0;$$

or, what is the same thing,

$$-b^2\alpha^2+(a^2-b^2)(b^2-\gamma^2)=0;$$

consequently

$$\gamma^2=b^2-\frac{b^2}{a^2-b^2}\alpha^2;$$

or the required equation of the circle is

$$(x-\alpha)^2+y^2=b^2-\frac{b^2}{a^2-b^2}\alpha^2;$$

or, as this may also be written

$$(x-\alpha)^2+y^2=(1-e^2)\left(a^2-\frac{\alpha^2}{e^2}\right).$$

It is to be observed that at the points of contact of the ellipse and circle, we have

$$x=\frac{a^2\alpha}{a^2-b^2}=\frac{\alpha}{e^2}.$$

By simply interchanging a and b, it appears that when the centre is on the minor axis, the equation of the variable circle is

$$x^2+(y-\beta)^2=a^2+\frac{a^2}{a^2-b^2}\beta^2,$$

and that at the points of contact of the ellipse and circle we have

$$y=-\frac{b^2\beta}{a^2-b^2}.$$

In the case where the centre is on the major axis, then attending to positive values of α, the circle remains real so long as α is $\not> ae$, but if α is $=ae$, that is, if the centre be at the focus, the radius of the circle is $=0$. But from the formula $x=\frac{\alpha}{e^2}$, we have $x=a$ for $\alpha=ae^2$, and when α is greater than ae^2, $x>a$, and the points of contact are imaginary. That is, as α passes from 0 to ae^2, the circle is real, and has real contact with the ellipse; for $\alpha = ae^2$ the circle becomes the circle of maximum curvature, touching the ellipse at the extremity of the major axis; as α increases from ae^2 to ae, the circle is still real but its contact with the ellipse is imaginary; for $\alpha = ae$, the circle reduces itself to the focus considered as an evanescent circle; and for greater values of α, the circle is imaginary.

When the centre is on the minor axis it appears in like manner that the circle is always real; but, attending to negative values of β, the circle has real contact with the ellipse only so long as $\beta \not> b - \frac{a^2}{b}$; for this value of β, the circle touches the ellipse at the extremity of the minor axis, being in fact the circle of minimum curvature; and for greater negative values of β, the contact is imaginary.

8. *Show that the number of ways in which n things can be arranged so that no one of them occupies its original position is of the form $(n-1)A_n$; and that we have $A_1 = 0$, $A_2 = 1$, $A_n = (n-2)A_{n-1} + (n-3)A_{n-2}$: show also that*

$$A_n = (n-1)A_{n-1} - \frac{1}{n-1}\{A_{n-1} + (-)^{n-1}1\}.$$

Supposing the n things arranged as required, we may by a single transposition of two things bring a given thing, say n, into its original place (the last place): and conversely every arrangement of the required form can be obtained from an arrangement in which n occupies the last place, by a transposition of n with another of the things. Now in the arrangement which thus gives rise to an arrangement of the required form, either all the things $1, 2, 3, \ldots, (n-1)$ are out of their original places; and we can then transpose n with any one of the things $1, 2, 3, \ldots, n-1$: or else only one thing is in its original place; say 1 is in its original place, and we can then transpose 1 and n; or 2 is in its original place, and we can transpose 2 and n; ... or $n-1$ is in its original place, and we can transpose $n-1$ and n. And these are the only ways in which an arrangement of the required form is obtained. Hence if for n things we denote by U_n the number of arrangements in which no one thing occupies its original place, we have, by what precedes,

$$U_n = (n-1)(U_{n-1} + U_{n-2}),$$

and it thus appears that U_n is of the form $(n-1)A_n$; and writing accordingly $U_n = (n-1)A_n$, and therefore

$$U_{n-1} = (n-2)A_{n-1}, \quad U_{n-2} = (n-3)A_{n-2},$$

we have

$$A_n = (n-2)A_{n-1} + (n-3)A_{n-2};$$

which is the required equation for A_n; we have, it is clear, $A_1 = 0$, $A_2 = 1$, and the successive values $A_3 = 1$, $A_4 = 3$, &c. can then be calculated. But the equation of differences of the second order may be integrated into one of the first order, viz writing the equation in the form

$$\{(n-1)A_n - n(n-2)A_{n-1}\} + \{(n-2)A_{n-1} - (n-1)(n-3)A_{n-2}\} = 0,$$

the integral is

$$(n-1)A_n - n(n-2)A_{n-1} = (-)^{n-1}C,$$

and writing $n=2$, we have $C=-1$, wherefore

$$(n-1)A_n = n(n-2)A_{n-1} - (-)^{n-1}1,$$

or, what is the same thing,

$$A_n = (n-1)A_{n-1} - \frac{1}{n-1}\{A_{n-1} + (-)^{n-1}1\},$$

which is the other equation for A_n.

The foregoing very elegant proof of the equation

$$U_n = (n-1)(U_{n-1} + U_{n-2}) = (n-1)A_n$$

was unknown to me; but was given in the Examination; my own proof of the equation $U_n = (n-1)A_n$ was derived from the well-known formula

$$U_n = 1\,.\,2\ldots n\left\{1 - \frac{1}{1} + \frac{1}{1\,.\,2}\ldots + (-)^n\frac{1}{1\,.\,2\,.\,3\ldots n}\right\}.$$

The theorem itself, and the two equations of differences for A_n are due to Euler, see his memoir, "Sur une espèce particulière de Carrés Magiques," *Comm. Arith. Coll.*, t. I., p. 359.

9. *If in any covariant of a binary form $(a, b, c, \ldots, k)(x, y)^n$ the coefficients $a, b, \ldots, k$ are replaced by $ax+by$, $bx+cy$, $\ldots$, $kx+ly$ respectively, show that the result is a covariant of the next superior form $(a, b, c, \ldots, l)(x, y)^{n+1}$: and determine the covariant obtained by thus operating on the discriminant of the cubic form $(a, b, c, d)(x, y)^3$.*

A covariant of the binary form $U, = (a, b, \ldots, k)(x, y)^n$, is either given by a single symbolical expression of the form $\overline{12}^{\alpha}\,\overline{13}^{\beta}\ldots\overline{23}^{\gamma}\ldots U_1U_2U_3\ldots$, or it is the sum of a number of such expressions, each multiplied by a constant (numerical factor): in the latter case each of the expressions in question is a covariant of U. Any such expression is at once seen to be a function of U and of its differential coefficients (i.e. it may contain the differential coefficients of each or any of the orders $0, 1, 2, \ldots, n$), homogeneous as regards the differential coefficients of the same order. But writing $U' = (a, b, c, \ldots, l)(x, y)^{n+1}$, the differential coefficients of any order of U are by the change of $a, b, \ldots, k$ into $ax+by$, $bx+cy$, $\ldots$, $kx+ly$, converted into the differential coefficients of the same order of U', each multiplied into the same merely numerical coefficient; the result (disregarding numerical factors) is thus the same derivative $\overline{12}^{\alpha}\,\overline{13}^{\beta}\ldots\overline{23}^{\gamma}\ldots U_1'U_2'U_3'\ldots$ of U'; viz. it is a covariant of U'; and making the like change in any sum of such expressions (each into a numerical factor), the result is a like sum of expressions referring to U'; that is, making the change in any covariant of U, the result is a covariant of U'.

Or the same thing may be otherwise proved thus; if to fix the ideas we write $U = (a, b, c, d)(x, y)^3$; any covariant Θ of U is reduced to zero by each of the operations $a\delta_b + 2b\delta_c + 3c\delta_d - y\delta_x$; $3b\delta_a + 2c\delta_b + d\delta_c - x\delta_y$; and, conversely, any function Θ which is thus reduced to zero is a covariant of U. Attending to the first operator, we have for any covariant Θ of U,

$$(a\delta_b + 2b\delta_c + 3c\delta_d - y\delta_x)\,\Theta = 0;$$

write $ax + by = a'$, $bx + cy = b'$, $cx + dy = c'$, $dx + ey = d'$, and let Θ' be the function obtained from Θ by changing a, b, c, d into a', b', c', d'. We ought to have

$$(a\delta_b + 2b\delta_c + 3c\delta_d + 4d\delta_e - y\delta_x)\,\Theta' = 0,$$

or considering Θ' as a function of a', b', c', d', x, y this is

$$\{a(y\delta_{a'} + x\delta_{b'}) + 2b(y\delta_{b'} + x\delta_{c'}) + 3c(y\delta_{c'} + x\delta_{d'}) + 4dy\delta_{d'} - y(a\delta_{a'} + b\delta_{b'} + c\delta_{c'} + d\delta_{d'}) - y\delta_x\}\,\Theta' = 0,$$

or, what is the same thing,

$$\{(ax + by)\,\delta_{b'} + 2(bx + cy)\,\delta_{c'} + 3(cx + dy)\,\delta_{d'} - y\delta_x\}\,\Theta' = 0,$$

that is,

$$\{a'\delta_{b'} + 2b'\delta_{c'} + 3c'\delta_{d'} - y\delta_x\}\,\Theta' = 0,$$

an equation which, Θ' being the same function of a', b', c', d', x, y that Θ is of a, b, c, d, x, y, is satisfied identically; and thus Θ' is reduced to zero by the operation $a\delta_b + 2b\delta_c + 3c\delta_d + 4d\delta_e - y\delta_x$; and similarly it is reduced to zero by the operation $4b\delta_a + 3c\delta_b + 2d\delta_c + e\delta_d - x\delta_y$; and it is thus a covariant of $(a, b, c, d, e \between x, y)^4$.

The discriminant of $(a, b, c, d \between x, y)^3$ is

$$= a^2d^2 + 4ac^3 + 4b^3d - 6abcd - 3b^2c^2;$$

making the substitution in question, it is

$$= (ax + by)^2 (dx + ey)^2 + \&c.,$$

viz. it is

$$= (a^2d^2 + 4ac^3 + 4b^3d - 6abcd - 3b^2c^2)\,x^4 + \&c.$$

But there is no such irreducible covariant of the quartic function $(a, b, c, d, e \between x, y)^4$; and observing that we have identically

$$(ae - 4bd + 3c^2)(ac - b^2) - (ace - ad^2 - b^2e - c^3 + 2bcd)\,a = a^2d^2 + 4ac^3 + 4b^3d - 6abcd - 3b^2c^2,$$

the covariant in question must be

$$= (ae - 4bd + 3c^2)\,[(ac - b^2)\,x^4 + \&c.] - (ace - ad^2 - b^2e - c^3 + 2bcd)\,(a, b, c, d, e \between x, y)^4,$$

where $(ac - b^2)\,x^4 + \&c.$ is the Hessian of the quartic function $(a, b, c, d, e \between x, y)^4$.

10. *From the equation of the curves of curvature of an ellipsoid, or otherwise, determine the curves of curvature of a paraboloid: show also that for the paraboloid $xy = cz$ (the parallel sections $z = Const.$ being rectangular hyperbolas) the curves of curvature are the intersections of the paraboloid by the system of surfaces*

$$h = \sqrt{(x^2 + z^2)} \pm \sqrt{(y^2 + z^2)}.$$

The curves of curvature of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

are given as the intersection of the surface with the cónfocal surface

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1;$$

transforming to the vertex, or writing $z-c$ in place of z, these become

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2-2cz}{c^2}=0,$$

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2-2cz-\lambda}{c^2+\lambda}=0,$$

or multiplying by c and writing $\frac{a^2}{c}=l$, $\frac{b^2}{c}=m$, $\frac{\lambda}{c}=\theta$, the equations are

$$\frac{x^2}{l}+\frac{y^2}{m}+\frac{z^2}{c}-2z=0,$$

$$\frac{x^2}{l+\theta}+\frac{y^2}{m+\theta}+\frac{z^2}{c+\theta}-\frac{2z+\theta}{1+\frac{\theta}{c}}=0,$$

or, putting herein $c=\infty$, the equations are

$$\frac{x^2}{l}+\frac{y^2}{m}-2z=0,$$

$$\frac{x^2}{l+\theta}+\frac{y^2}{m+\theta}-2z-\theta=0,$$

viz. these equations, where θ is a variable parameter, determine the curves of curvature of the paraboloid $\frac{x^2}{l}+\frac{y^2}{m}-2z=0$. The second equation may be replaced by

$$\frac{x^2}{l(l+\theta)}+\frac{y^2}{m(m+\theta)}+1=0.$$

Write in the equations $l=-m=c$; they become

$$x^2-y^2-2cz=0,$$

$$\frac{x^2}{c+\theta}+\frac{y^2}{c-\theta}+c=0;$$

or, substituting herein $\frac{x+y}{\sqrt{2}}$ for x, and $\frac{x-y}{\sqrt{2}}$ for y, the equations are

$$xy-cz=0,$$

$$\frac{(x+y)^2}{c+\theta}+\frac{(x-y)^2}{c-\theta}+2c=0$$

the second of which is

$$c(x^2+y^2) - 2\theta xy + c(c^2-\theta^2) = 0,$$

which is the equation for determining the curves of curvature of the surface $xy - cz = 0$.

Writing in this equation $\theta = \sqrt{(h^2+c^2)}$, it becomes

$$c(x^2+y^2) - 2xy\sqrt{(h^2+c^2)} - ch^2 = 0,$$

or, as this may be written,

$$(c^2+x^2)(c^2+y^2) - x^2y^2 - 2cxy\sqrt{(h^2+c^2)} - c^2h^2 - c^4 = 0,$$

that is,

$$\pm\sqrt{(c^2+x^2)}\sqrt{(c^2+y^2)} = xy + c\sqrt{(h^2+c^2)},$$

and thence

$$\begin{aligned} c^2h^2 &= (c^2+x^2)(c^2+y^2) + x^2y^2 - c^4 \mp 2xy\sqrt{(c^2+x^2)}\sqrt{(c^2+y^2)} \\ &= x^2(c^2+y^2) + y^2(c^2+x^2) \mp 2xy\sqrt{(c^2+x^2)}\sqrt{(c^2+y^2)}, \end{aligned}$$

that is,

$$ch = x\sqrt{(c^2+y^2)} \pm y\sqrt{(c^2+x^2)};$$

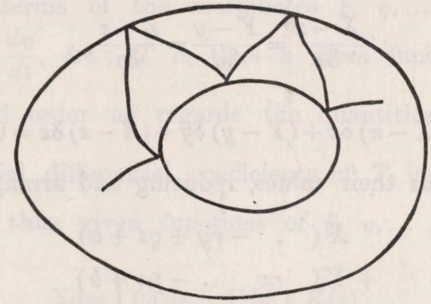
or combining with the equation $xy - cz = 0$ of the paraboloid, this is

$$h = \sqrt{(x^2+z^2)} \mp \sqrt{(y^2+z^2)},$$

or the curves of curvature are given as the intersections of the paraboloid by the series of surfaces represented by this equation.

11. *Explain for a surface such as the ellipsoid the form of the curve of given constant slope; and in an ellipsoid having one of its principal planes horizontal determine the limits within which the curve is situate.*

At any point of a surface, the direction of the line of greatest slope is evidently at right angles to the level curve, or, what is the same thing, to the trace of the tangent plane on the horizontal plane; and the inclination of the element of the line of greatest slope is equal to that of the tangent plane to the horizontal plane. The line of given constant slope must therefore lie entirely on that portion of the surface for which the inclination of the tangent plane has a value not less than the given constant slope of the curve; viz. on a closed surface such as the ellipsoid, it will lie upon a certain zone of the surface, included between two boundaries, which boundaries are the curves such that at any point thereof the inclination of the tangent plane is *equal* to the given constant slope; and by what precedes, the direction of the line of given constant slope, at any point where it meets the boundary, is coincident with the direction of greatest slope; viz. it will in general meet the boundary at a finite angle; this implies that the point is a cusp on the curve of given constant slope, and the curve in question will be a curve as shown in the figure, passing continually from one boundary to the other, and when it reaches either boundary, turning back cusp-wise to the other boundary; the curve may in particular cases, after making a circuit, or any number of circuits of the zone, re-enter upon itself, forming a closed curve; but in general this is not the case, and the curve will go on as above between the two boundaries, *ad infinitum*.



In the case of the ellipsoid, the plane of xy being as usual horizontal, and the equation being

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

then if λ be the given constant inclination, the boundaries are the series of points at which the inclination of the normal to the axes of z is $=\lambda$; that is, we have

$$\frac{z^2}{c^4}=\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)\cos^2\lambda,$$

or, what is the same thing,

$$\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)\cos^2\lambda-\frac{z^2}{c^4}\sin^2\lambda=0,$$

the boundaries are thus two detached ovals, meeting the principal sections $x=0$, $y=0$, in the points given by

$$\frac{z}{y}=\pm\frac{c^2}{b^2}\cot\lambda\,;\ \frac{z}{x}=\pm\frac{c^2}{a^2}\cot\lambda,$$

and the general form is thus at once perceived.

12. *To every point of space there corresponds a plane, viz. considering the several points as belonging to a solid body which is infinitesimally displaced in any manner, the plane which corresponds to a point is the plane drawn through the point at right angles to the direction of its motion; determine the plane which corresponds to a given point: and connect the result with any general geometrical theory.*

The displacements, in the directions of the axes, of a point whose coordinates are (x, y, z) are given by the ordinary formulæ

$$\delta x=a+qz-ry,$$
$$\delta y=b+rx-pz,$$
$$\delta z=c+py-qx,$$

where a, b, c, p, q, r are constants which determine the particular displacement; hence if X, Y, Z are current coordinates, the plane corresponding to the point (x, y, z) is the plane through this point at right angles to the line

$$\frac{X-x}{\delta x}=\frac{Y-y}{\delta y}=\frac{Z-z}{\delta z},$$

viz. it is the plane

$$(X-x)\,\delta x+(Y-y)\,\delta y+(Z-z)\,\delta z=0,$$

or, substituting for δx, δy, δz their values, reducing and arranging, this is

$$\begin{aligned} &X(\ \ .\ \ -ry+qz+a)\\ +&Y(\ rx\ \ .\ \ -pz+b)\\ +&Z(-qx+py\ \ .\ \ +c)\\ +&\ \ (-ax-by-cz\ \ .\ \ )=0; \end{aligned}$$

that is, the equation of the plane corresponding to the point (x, y, z) is an equation linear in regard to the coordinates (x, y, z) of this point; and such that arranging as above in the form of a square, the coefficients which are symmetrically situate in regard to the dexter diagonal are equal and of opposite signs; or say that the coefficients form a skew symmetrical matrix.

More generally, corresponding to the point (x, y, z) we may have the plane

$$AX+BY+CZ+D=0,$$

where A, B, C, D are any linear functions whatever $(=\alpha x+\beta y+\gamma z+\delta$, &c.) of the coordinates (x, y, z); this is the general relation which is the analytical basis of the theory of duality; it in fact appears that if the point be a variable point situate in a line or a plane, then the corresponding plane is a variable plane passing through a line or a point, &c., &c. In the general case, to a given point there corresponds a plane not in general passing through the point; the case above considered is distinguished by the circumstance that for any point whatever the corresponding plane does pass through the point.

13. *Write down the Lagrangian equations of motion, explaining the notation, and mode of applying them; and by way of illustration deduce the equations of motion of three particles, connected so as to form an equilateral triangle (of variable magnitude), moving in a plane under the action of any forces.*

The Lagrangian equations of motion are

$$\frac{d}{dt}\cdot\frac{dT}{d\xi'}-\frac{dT}{d\xi}=\frac{dU}{d\xi},$$

$$\frac{d}{dt}\cdot\frac{dT}{d\eta'}-\frac{dT}{d\eta}=\frac{dU}{d\eta},$$

&c.,

$\xi, \eta, \ldots$ are here any independent coordinates (in the most general sense of the word) which serve to determine the position of the system at the time t; T is the *vis-viva* function, or half-sum of the mass of each particle of the system into the square of its velocity, expressed in terms of the coordinates $\xi, \eta, \ldots$ and of their differential coefficients $\xi', = \frac{d\xi}{dt}$; $\eta', = \frac{d\eta}{dt}$, &c.; T is thus a *given* function of $\xi, \eta, \ldots \xi', \eta', \ldots$, homogeneous of the second order as regards the quantities $\xi', \eta', \ldots$; $\frac{dT}{d\xi}, \frac{dT}{d\eta}$, &c., $\frac{dT}{d\xi'}, \frac{dT}{d\eta'}$, &c. are the partial differential coefficients of T in regard to $\xi, \eta, \ldots \xi', \eta', \ldots$ respectively, and they are thus given functions of $\xi, \eta, \ldots \xi', \eta', \ldots$; U is the force-function or sum

$$\Sigma dm \int (Xdx + Ydy + Zdz)$$

(it being assumed that $Xdx + Ydy + Zdz$ is a complete differential, that is, that there exists a force-function U) expressed in terms of the coordinates $\xi, \eta, \ldots$ and being thus a *given* function of these coordinates; $\frac{dU}{d\xi}, \frac{dU}{d\eta}$, &c. are the partial differential coefficients of U in regard to $\xi, \eta, \ldots$ respectively; and are thus given functions of these coordinates. The terms $\frac{d}{dt}\frac{dT}{d\xi'}$, contain, it is clear, (and that linearly) the second differential coefficients $\xi'', = \frac{d^2\xi}{dt^2}$, &c. ..., the equations thus establish between the coordinates $\xi, \eta, \ldots$, and their first and second differential coefficients in regard to the time a system of relations, the number of which is equal to that of the coordinates, and which therefore would by integration lead to the expression of the coordinates $\xi, \eta, \ldots$ in terms of the time.

It has been tacitly assumed that T, U were functions of $\xi, \eta, \ldots \xi', \eta', \ldots$, and of $\xi, \eta, \ldots$, not containing the time t; this is the ordinary case, but there are cases in which T and U or either of them may also contain the time t.

In the proposed case of the three particles, if m_1, m_2, m_3 be their masses, r the side of the equilateral triangle, x_1, y_1 the coordinates of m_1, θ, $\theta + 60°$ (or write for convenience $\theta + \alpha$) the inclinations of m_1m_2, m_1m_3 to the axis of x, then the coordinates of m_2, m_3 will be

$$x + r\cos\theta, \quad x + r\cos(\theta + \alpha),$$
$$y + r\sin\theta, \quad y + r\sin(\theta + \alpha).$$

We have

$$\begin{aligned} T = {} & \tfrac{1}{2} m_1 (x'^2 + y'^2) \\ & + \tfrac{1}{2} m_2 [(x' + r'\cos\theta - r\sin\theta \,.\, \theta')^2 + (y' + r'\sin\theta + r\cos\theta \,.\, \theta')^2] \\ & + \tfrac{1}{2} m_3 [\{x' + r'\cos(\theta+\alpha) - r\sin(\theta+\alpha) \,.\, \theta'\}^2 + \{y' + r'\sin(\theta+\alpha) + r\cos(\theta+\alpha) \,.\, \theta'\}^2] \\ = {} & \tfrac{1}{2}(m_1 + m_2 + m_3)(x'^2 + y'^2) \\ & + m_2 [x'(r'\cos\theta - r\sin\theta \,.\, \theta') + y'(r'\sin\theta + r\cos\theta \,.\, \theta')] \\ & + m_3 [x'\{r'\cos(\theta+\alpha) - r\sin(\theta+\alpha) \,.\, \theta'\} + y'\{r'\sin(\theta+\alpha) + r\cos(\theta+\alpha) \,.\, \theta'\}] \\ & + \tfrac{1}{2}(m_2 + m_3)(r'^2 + r^2\theta'^2), \end{aligned}$$

a given function of $x, y, r, \theta, x', y', r', \theta'$; also

$$U = m_1 \int (Xdx + Ydy) + \&c.$$

will be a given function of x, y, r, θ; and the equations of motion will be

$$\frac{d}{dt}\frac{dT}{dx'} - \frac{dT}{dx} = \frac{dU}{dx},$$

$$\frac{d}{dt}\frac{dT}{dy'} - \frac{dT}{dy} = \frac{dU}{dy},$$

$$\frac{d}{dt}\frac{dT}{dr'} - \frac{dT}{dr} = \frac{dU}{dr},$$

$$\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} = \frac{dU}{d\theta}.$$

14. *From the integrals* $x = a\cos t + a'\sin t$, $y = b\cos t + b'\sin t$, *of the dynamical equations* $\frac{d^2x}{dt^2} = -x$, $\frac{d^2y}{dt^2} = -y$, *deduce a simultaneous solution of the two partial differential equations*

$$\frac{dS}{dt} + \tfrac{1}{2}\left\{\left(\frac{dS}{dx}\right)^2 + \left(\frac{dS}{dy}\right)^2\right\} = -\tfrac{1}{2}(x^2 + y^2),$$

$$\frac{dS}{dt} + \tfrac{1}{2}\left\{\left(\frac{dS}{da}\right)^2 + \left(\frac{dS}{db}\right)^2\right\} = -\tfrac{1}{2}(a^2 + b^2).$$

Writing $U = -\tfrac{1}{2}(x^2 + y^2)$, we have

$$x = a\cos t + a'\sin t,$$

$$y = b\cos t + b'\sin t,$$

as the integrals of the equations of motion

$$\frac{d^2x}{dt^2} = \frac{dU}{dx}, \quad \frac{d^2y}{dt^2} = \frac{dU}{dy};$$

moreover

$$x' = -a\sin t + a'\cos t,$$

$$y' = -b\sin t + b'\cos t,$$

where a, b, a', b' are the initial values of x, y, x', y' respectively $\left(x' = \frac{dx}{dt}, y' = \frac{dy}{dt}, \text{as usual}\right)$. Hence the Principal Function

$$S = \int_0^t (T + U)\,dt$$

$$= \tfrac{1}{2}\int_0^t (x'^2 + y'^2 - x^2 - y^2)\,dt,$$

expressed as a function of x, y, a, b, t, will satisfy simultaneously the proposed partial differential equations.

We have

$$x'^2 + y'^2 - x^2 - y^2 = (a'^2 + b'^2 - a^2 - b^2)\cos 2t - 2(aa' + bb')\sin 2t.$$

Hence

$$4S = (a'^2 + b'^2 - a^2 - b^2)\sin 2t + 2(aa' + bb')(\cos 2t - 1)$$
$$= (a'^2 + b'^2)\sin 2t - (a^2 + b^2)\sin 2t - 4(aa' + bb')\sin^2 t.$$

But

$$a' \sin t = x - a\cos t,$$
$$b' \sin t = y - b\cos t.$$

Hence

$$(aa' + bb')\sin t = ax + by - (a^2 + b^2)\cos t,$$
$$(a'^2 + b'^2)\sin^2 t = x^2 + y^2 - 2(ax + by)\cos t + (a^2 + b^2)\cos^2 t,$$

and substituting these values

$$\begin{aligned} 4S = \frac{2\sin t\cos t}{\sin^2 t}\{x^2 + y^2 - 2(ax+by)\cos t + (a^2+b^2)\cos^2 t\} \\ - (a^2+b^2)\,2\sin t\cos t \\ - 4\sin t\,\{ax + by - (a^2+b^2)\cos t\} \\ = 2(x^2+y^2)\frac{\cos t}{\sin t} - \frac{4}{\sin t}(ax+by) + 2(a^2+b^2)\frac{\cos t}{\sin t};\end{aligned}$$

or, what is the same thing,

$$S = \tfrac{1}{2}(x^2 + y^2 + a^2 + b^2)\cot t - 2(ax + by)\operatorname{cosec} t,$$

which value of S satisfies the two partial differential equations.

More generally, the two equations are satisfied by

$$S = c + \text{foregoing value},$$

(c an arbitrary constant) which new value, considered as a solution of the first equation, contains the three arbitrary constants c, a, b, and is thus a *complete* solution; and similarly considered as a solution of the second equation, it contains the three arbitrary constants c, x, y, and is thus a *complete* solution.

I venture to add a few remarks in illustration of what is required in the papers sent up in an Examination.

In the latter part of question (2) (form of the equation for a system of parallel curves) it is worse than useless to say $\frac{df}{dc} \div \left\{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2\right\}^{\frac{1}{2}}$ [p. 441] must be constant: a good and sufficient answer would be that it must be constant *in virtue of the given equation* $f(x, y, c) = 0$. So in question (13) (the Lagrangian equations of motion), it is *quite essential* to explain [p. 455] that T, U are *given* functions of $\xi, \eta, \ldots, \xi', \eta', \ldots$ and of $\xi, \eta, \ldots$ respectively; but for this the equations might be partial differential equations for the determination of T, U, or nobody knows what: it is natural and proper to explain further that T is homogeneous of the second order in regard to the derived functions $\xi', \eta', \ldots$. In question (14) the answer [p. 456] that $S = \frac{1}{2}\int_0^t (x'^2 + y'^2 - x^2 - y^2)\,dt$ expressed as a function of x, y, a, b, t will satisfy simultaneously the proposed equations—would be, not of course a complete answer, but a good and creditable one; without the words "expressed as a function of x, y, a, b, t" it would be altogether worthless. A clear and precise indication of a process of solution is very much better than a detailed solution incorrectly worked out.