529.

A "SMITH'S PRIZE" PAPER[1]; SOLUTIONS.

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1. *Find the form of a function of a given number of letters, which has two and only two values.*

It is required to find the general form of a function $\phi(a, b, c, \ldots, k)$, rational and integral, which for all permutations whatever of the letters has two and only two values.

Suppose that any particular permutation of the letters changes $\phi(a, b, c, \ldots, k)$ into $\phi_1(a, b, c, \ldots, k)$; then any permutation of the letters will either leave the functions ϕ, ϕ_1 each of them unaltered, or it will change ϕ into ϕ_1, and ϕ_1 into ϕ. Hence $\phi + \phi_1$ is a symmetrical function of all the letters, say

$$\phi + \phi_1 = 2L;$$

$\phi - \phi_1$ is a function which by any permutation of the letters is either unaltered, or simply changes its sign; and it is to be shown that, writing for shortness V to denote the product $(a-b)(a-c)\ldots(b-c)\ldots$ of the differences of the letters, and denoting by $2M$ a symmetrical function of the letters, we have

$$\phi - \phi_1 = 2VM.$$

These equations give

$$\phi = L + VM,$$

which is the general form required; viz. the function ϕ has then only the two values $L + VM$, $L - VM$.

To prove the subsidiary theorem, observe that there is at least one interchange of two letters which changes $\phi - \phi_1$ (for otherwise $\phi - \phi_1$ would be a symmetrical function);

[1] Set by me, for the Master of Trinity, Thursday, January 30, 1868.

let this be (a, b). Then $\phi-\phi_1$, changing its sign for the interchange in question, must vanish for $a=b$, that is, $\phi-\phi_1$ must contain the factor $(a-b)$; let it contain it in the power $(a-b)^s$; then the quotient $\phi-\phi_1 \div (a-b)^s$, not vanishing for $a=b$, cannot change its sign by the interchange in question, and as by supposition $\phi-\phi_1$ does change its sign, it appears that the exponent s must be odd. But $\phi-\phi_1$, containing the factor $(a-b)^s$, must contain every other like factor $(a-c)^s$ or $(c-d)^s$; in fact, writing $\phi-\phi_1=K(a-b)^s$, then if the interchange (a, c) alters K into K_1, we have $K(a-b)^s = \pm K_1(a-c)^s$, and K consequently contains the factor $(a-c)^s$; it does not contain any higher power $(a-c)^{s+s'}$, for if it did, by reversing the process it would appear that $\phi-\phi_1$ contained (contrary to supposition) the factor $(a-b)^{s+s'}$. Similarly $\phi-\phi_1$ contains the factor $(c-d)^s$, but no higher power $(c-d)^{s+s'}$. Hence $\phi-\phi_1$ contains the product of all the factors $(a-b)^s$, that is, it contains V^s, and writing $\phi-\phi_1=2V^sM$, the quotient M does not contain any such factor as $(a-b)$; it therefore does not change its sign for any interchange whatever (a, b); and in consequence it remains unaltered for any such interchange, that is, M is a symmetrical function. Observing that any even power of V is a symmetrical function, we may without loss of generality include V^{s-1} in the symmetrical function M, and write therefore

$$\phi-\phi_1=2VM,$$

which is the subsidiary theorem in question.

2. *Express* $\dfrac{x^7-1}{x-1}$ *as the product of two cubic factors; and show generally that, for any prime exponent* p, *the function* $\dfrac{x^p-1}{x-1}$ *may be broken up into two factors each of the order* $\frac{1}{2}(p-1)$, *by means of a quadratic equation.*

Let r be any root of the given equation, then

$$r^6+r^5+r^4+r^3+r^2+r+1=0;$$

the roots of this equation are $r^1, r^2, r^3, r^4, r^5, r^6$.

Hence

$$x^6+x^5+x^4+x^3+x^2+x+1=(x-r^1)(x-r^2)(x-r^4).(x-r^3)(x-r^5)(x-r^6),$$

and denoting the two cubic factors by y_1, y_2, or writing

$$y_1=(x-r^1)(x-r^2)(x-r^4),$$
$$y_2=(x-r^3)(x-r^5)(x-r^6),$$

we have

$$y_1=x^3-x^2(r^1+r^2+r^4)+x(r^3+r^5+r^6)-1,$$
$$y_2=x^3-x^2(r^3+r^5+r^6)+x(r^1+r^2+r^4)-1,$$

and thence

$$y_1+y_2=2x^3+x^2-x-2.$$

Hence y_1, y_2 are the roots of the quadratic equation

$$y^2-y(2x^3+x^2-x-2)+x^6+x^5+x^4+x^3+x^2+x+1=0.$$

Solving the equation, observing that the quantity under the radical sign must of necessity be a square, and that its value is

$$(2x^3+x^2-x-2)^2-4(x^6+x^5+x^4+x^3+x^2+x+1),$$

which is in fact

$$=-7(x^2+x)^2,$$

we find that the roots y_1, y_2, that is, the required cubic factors, are

$$\tfrac{1}{2}\{2x^3+x^2-x-2\pm\sqrt{(-7)}\,(x^2+x)\}.$$

Generally for any prime exponent p, denoting any root by r, and writing

$$y_1=(x-r^{a_1})(x-r^{a_2})\ldots,$$

$$y_2=(x-r^{b_1})(x-r^{b_2})\ldots,$$

where a_1, $a_2\ldots$ are the quadratic residues, and b_1, $b_2\ldots$ the quadratic non-residues of p, we find

$$y^2-yP+Q=0,$$

where P is a function of x of the order $\frac{1}{2}(p-1)$, and Q is $=x^{p-1}+x^{p-2}\ldots+x+1$. Hence

$$y=\tfrac{1}{2}\{P\pm\sqrt{(P^2-4Q)}\},$$

where P^2-4Q is a perfect square, $=\epsilon pZ^2$, Z a rational function of a degree less than $\frac{1}{2}(p-1)$, and $\epsilon=+$ or $-$ according as p is $\equiv 1$ or $\equiv 3$ (mod. 4). Hence the required factors are

$$\tfrac{1}{2}\{P\pm\sqrt{(\epsilon p)}\,Z\}.$$

3. *In a Map of the World, wherein the meridians are projected into right lines meeting in a point, the inclination of any two of the lines being equal to that of the two meridians, and the parallels into circles about the point as centre, the radius of the circle being a given function of the colatitude: compare on the sphere and in the map* (1) *the corresponding elements of area,* (2) *the azimuths of corresponding linear elements; and explain what conveniences may be obtained by proper determinations of the above-mentioned function of the colatitude.*

Let the longitude and colatitude be

on the sphere l, c,

in the map l', c',

then the projection is such, that $l'=l$, $c'=f(c)$, a given function of c.

The lengths of corresponding linear elements in the direction of a meridian, and perpendicular to it, are

$$dc\,,\ \sin c\,dl,$$

$$dc',\quad c'dl:$$

whence, elements of area are

$$dc \,.\, \sin c\, dl,$$

$$dc' \,.\qquad c'dl;$$

tangents of azimuth are

$$dc \div \sin c\, dl,$$

$$dc' \div \qquad c'dl;$$

and substituting the values $l'=l$, $c'=f(c)$, we have $dl'=dl$, $dc'=f'(c)\,dc$; and thence

$$\text{elements of area are as } \sin c : f(c)f'(c),$$

$$\text{tangents of azimuth as } \frac{1}{\sin c} : \frac{f'(c)}{f(c)}.$$

By proper determinations of the function $f(c)$, we can make

(1) The ratio of the elements of area to be constant; this will be the case if

$$f(c)f'(c) = k\sin c, \text{ that is, } f^2(c) = \text{const.} - 2k\cos c$$
$$= 2k(1-\cos c),$$

since $f(c)$, $=c'$, must vanish for $c=0$; that is,

$$f(c) = 2\sqrt{(k)}\sin\tfrac{1}{2}c:$$

(2) corresponding azimuths to be equal; this will be the case if

$$\frac{f'(c)}{f(c)} = \frac{1}{\sin c}, \text{ that is, } \log f(c) = \text{const.} + \log\tan\tfrac{1}{2}c,$$

or say

$$f(c) = k\tan\tfrac{1}{2}c.$$

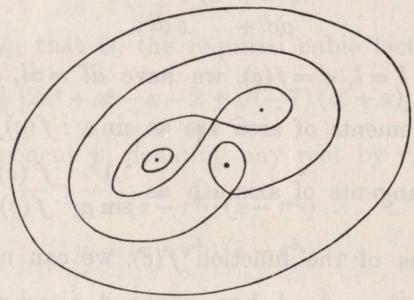
This is in fact the stereographic projection, in which (as is known) any indefinitely small figure is in the map represented without distortion.

4. *Explain the general configuration of the contour and slope lines in a tract of Lake and Mountain country.*

A contour line is the locus of points having a given altitude.

To fix the ideas, consider an island forming a two-headed mountain. The contour line at the sea level is a closed curve; at a sufficiently great altitude the contour line consists of two closed curves surrounding the two summits respectively; the transition from one form to the other takes place at the altitude of the pass between the two summits, the contour line then having a node at the top of the pass, and being in form a figure of eight. At the altitude of the lower summit one of the closed curves is reduced to a point, and for greater altitudes it disappears, the contour line being then a single closed curve surrounding the higher summit; and at the altitude of the higher summit this reduces itself to a point. (See fig. 1.) If there is on the breast of the mountain a lake, the contour line at the lake level is (as in the case of the pass) a curve with a node, being however here a closed curve with an interior loop as shown in the figure. The contour line for an altitude below the lake level includes as part of itself a closed curve lying within the contour of the lake,

Fig. 1.



and which for the altitude of the lowest point (or "imit") of the lake reduces itself to a point, and for smaller altitudes disappears.

The contour lines in the immediate neighbourhood of a summit are in general ellipses (geometrically, the indicatrix is an ellipse); they may however be circles (viz. if the summit be an umbilicus).

A slope line is a line of greatest inclination, and it is consequently an orthogonal trajectory to the series of contour lines. A slope line may be considered as always terminated in a summit or an imit; there is through each summit (or imit) an infinity of slope lines, and in general these all *touch* there; if, however, the contour lines in the immediate neighbourhood are circles, then the slope lines, instead of touching, pass from the point in all directions. Through the node at the top of a pass, or outlet of a lake, there are two intersecting slope lines, one ascending each way from the node, and being in general a "ridge-line"; the other descending each way, and being in general a "course-line," viz. in the case of a pass, it is in each direction the course of the principal stream of the valley; and in the case of a lake-outlet, it is in the direction away from the lake, the course of the out-flowing stream. The slope lines which thus pass through the several nodes mark out distinct regions, and so facilitate the tracing of the intermediate slope lines.

5. *Find the differential equations corresponding to the three integral equations respectively* (i) $(y+c)^2 = x(x-1)(x-2)$; (ii) $(y+c)^2 = x^2(x-1)$; *and* (iii) $(y+c)^2 = x^3$: *and discuss geometrically the singular solutions.*

Generally, for the equation $(y+c)^2 - X = 0$, the derived equation is $4X\left(\dfrac{dy}{dx}\right)^2 - X'^2 = 0$. If from the integral equation, differentiating in regard to c, we attempt to find the singular solution, we obtain $(y+c) = 0$; and thence $X = 0$ for the singular solution. It is however to be observed that, if X contain single and multiple factors,

$$X = (x+\alpha)(x+\beta)\dots(x+\gamma)^m\dots,$$

then it is only the single factors $x+\alpha=0$, $x+\beta=0$, ..., which are solutions of the differential equation when expressed in its proper form free from extraneous factors.

To explain how this is, write the differential equation in the form

$$X'^2\left(\frac{dx}{dy}\right)^2-4X=0;$$

for a single factor $x+\alpha$, X' does not contain the factor $x+\alpha$, and there is no division by $x+\alpha$. The equation $x+\alpha=0$ gives $\frac{dx}{dy}=0$, $X=0$, and the equation is thus satisfied; and by what precedes $x+\alpha=0$ is a singular solution. Contrariwise, for the multiple factor $(x+\gamma)^m$, X' will contain $(x+\gamma)^{m-1}$, and the equation $X'^2\left(\frac{dx}{dy}\right)^2-4X=0$, will divide by $(x+\gamma)^m$, and divested of this factor it will be of the form

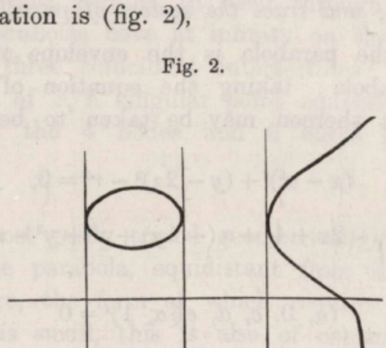
$$\frac{X'^2}{(x+\gamma)^m}\left(\frac{dx}{dy}\right)^2-\frac{4X}{(x+\gamma)^m}=0,$$

where $\frac{X'}{(x+\gamma)^m}$ contains the factor $(x+\gamma)^{m-2}$ (index is 0 or positive) but $\frac{X}{(x+\gamma)^m}$ does not contain $x+\gamma$; hence the equation $x+\gamma=0$, gives $\frac{dx}{dy}=0$, and therefore $\frac{X'}{(x+\gamma)^m}\left(\frac{dx}{dy}\right)^2=0$, but it does not give $\frac{X}{(x+\gamma)^m}=0$, and consequently fails to satisfy the differential equation. Reverting to the integral equation $(y+c)^2=(x+\alpha)(x+\beta)\ldots(x+\gamma)^m\ldots$, we see that $x+\alpha=0$ touches each of the series of curves, and is thus an envelope thereof: that $x+\gamma=0$ is not an envelope, but is the locus of a singular point on the series of curves.

Applying the foregoing considerations to the proposed question,

(i) The differential equation is (fig. 2),

Fig. 2.



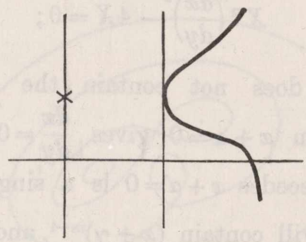
$$(3x^2-6x+2)^2\left(\frac{dx}{dy}\right)^2-4x(x-1)(x-2)=0,$$

having the singular solutions $x=0$, $x-1=0$, $x-2=0$.

(ii) The differential equation is (fig. 3),

$$(3x-2)^2\left(\frac{dx}{dy}\right)^2 - 4(x-1) = 0,$$

Fig. 3.



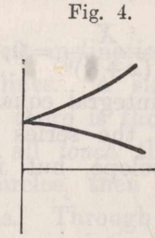
having the singular solution $x-1=0$. But $x=0$ is not a solution; it is the locus of the series of conjugate points of the curves $(y+c)^2 = x^2(x-1)$.

(iii) The differential equation is (fig. 4)

$$9x\left(\frac{dx}{dy}\right)^2 - 4 = 0,$$

which has no singular solution. But $x=0$ is the locus of the cusp of the curves $(y+c)^2 = x^3$.

Fig. 4.



6. *Show that the curve parallel to the parabola is of the order* 6 *and class* 4; *explain the reduction of class; and trace the system of parallel curves.*

The curve parallel to the parabola is the envelope of a circle of constant radius, having its centre on a parabola; taking the equation of the parabola to be $y^2 = 4x$, the coordinates of any point thereon may be taken to be α^2, 2α, and the equation of the variable circle is

$$(x-\alpha^2)^2 + (y-2\alpha)^2 - r^2 = 0,$$

that is,

$$\alpha^4 + \alpha^2(-2x+4) + \alpha(-4y) + x^2 + y^2 - r^2 = 0;$$

or multiplying by 6, this is

$$(a,\ 0,\ c,\ d,\ e\between \alpha,\ 1)^4 = 0,$$

where

$$a = 6,$$

$$c = -2x+4,$$

$$d = -6y,$$

$$e = 6(x^2+y^2-r^2).$$

The equation of the envelope is obtained by eliminating α between the equation in α and the derived equation; or, what is the same thing, by equating to zero the discriminant of the equation in α; we thus obtain

$$(ae+3c^2)^3 - 27(ace - ad^2 - c^3)^2 = 0,$$

which, substituting for a, c, d, e their values, is an equation in (x, y) of the order 6: the order of the curve is thus $= 6$.

The class is most easily obtained by geometrical considerations. Seeking the tangents which can be drawn from a given point, it at once appears that, describing about this point a circle radius r, then to each tangent through the point to the parallel curve, there corresponds a common tangent of the circle and the parabola, and reciprocally; whence the class of the curve is equal to the number of common tangents of the circle and the parabola, that is, it is $= 4$.

To the order 6 corresponds in general a class $= 30$, and the reduction from 30 to 4, $= 26$, is caused by the cusps and nodes of the curve.

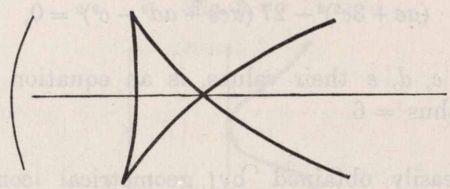
The cusps are given as the points of intersection of the curves $ae+3c^2=0$, $ace-ad^2-c^3=0$, which being respectively of the orders 2 and 3, give 6 cusps; it may be added that these cusps (2 real and 4 imaginary, or else all 6 imaginary) are, in regard to the parabola, the centres of curvature for those points at which the radius of curvature is $= r$.

There is on the axis a single point (always real) whose normal distance from the parabola is $= r$. Such point is a node of the parallel curve, viz., according to the value of r, either an acnode (conjugate point) or a crunode: we have thus 1 node. The parallel curve at infinity coincides with the two parabolas obtained by the displacement of the given parabola parallel to itself through the distances $+r$, $-r$ along the axis of y. Two such parabolas have at infinity on the axis of x a contact of the second order, equivalent to three coincident intersections; and the parallel curve has thus at infinity on the axis of x, a singular point equivalent to 3 nodes; the number of nodes is thus $= 4$; and the 4 nodes and 6 cusps give the required reduction $2.4 + 3.6, = 26$.

The parallel curve is most easily traced geometrically; there are two branches, one outside, the other inside the parabola, equidistant from it. The outside branch is a curve of continuous curvature, the form of which requires no explanation. As regards the inside branch, when r is small, this is also of continuous curvature, but there is on the axis of x inside the branch a real acnodal point (the node above referred to): when r becomes = radius of curvature at vertex (or twice the focal distance), the acnode coincides with the branch; and the point on the axis, although presenting no visible singularity, is really in the nature of a triple point composed of two cusps and a node; when r is greater than this limiting value, instead of the acnode we have a crunode;

and two real cusps present themselves; viz. the form of the inside branch is as shown in fig. 5.

Fig. 5.



7. *Show that a curve of the order n has at most* $\frac{1}{2}(n-1)(n-2)$ *double points; and that in any curve having this maximum number of double points, the coordinates* (x, y, z) *may be taken to be proportional to rational and integral functions of a variable parameter.*

A curve of the order n cannot have more than $\frac{1}{2}(n-1)(n-2)$ double points; for suppose it had one more, say

$$\tfrac{1}{2}(n-1)(n-2)+1, \; = \tfrac{1}{2}(n^2-3n)+2, \text{ double points};$$

then since a curve of the order $n-2$ can be drawn through

$$\tfrac{1}{2}(n-2)(n+1), \quad = \tfrac{1}{2}(n^2-n)-1 \text{ points},$$

suppose it drawn through the

$$\tfrac{1}{2}(n^2-3n)+2 \text{ double points};$$

and besides through $\quad n-3$ points on the curve,

together $\quad \frac{1}{2}(n^2-n)-1$ points,

it will cut the curve of the order n in the double points considered as

$$n^2-3n+4 \text{ points},$$

and besides in $\quad n-3$ points,

together $\quad n^2-2n+1$ points,

that is, we should have a curve of order $n-2$ meeting the curve of the order n in $n(n-2)+1$ points, which is of course impossible.

Consider now a curve of the order n with $\frac{1}{2}(n-1)(n-2)$ double points; draw a curve of the order $(n-2)$ through the double points

$$\tfrac{1}{2}(n^2-3n)+1 \text{ points},$$

and besides through $\quad n-3$ points on the curve,

together $\quad \frac{1}{2}(n^2-n)-2$ points;

this imposes on the curve $\frac{1}{2}(n^2-n)-2$ conditions; but as the curve might have been determined to satisfy $\frac{1}{2}(n^2-n)-1$ conditions, its equation will contain an indeterminate parameter λ: it meets the curve n in the double points considered as

	n^2-3n+2	points,
in the	$n-3$	points,
and besides in	1	point,
altogether	n^2-2n	points.

Hence since for any value whatever of λ, there is only one remaining intersection, the coordinates of this point must be rationally determinable in terms of the parameter λ; that is, the coordinates of *any* point on the given curve will be rational functions of λ; or using trilinear coordinates, the coordinates x, y, z of any point on the given curve will be proportional to rational and integral functions of λ.

8. *Show that three bodies attracting each other according to the law of gravitation may move in a line in such wise that the mutual distances are in a constant ratio the value of which depends on the masses.*

Taking the masses m_1, m_2, m_3, to be arranged in this order at distances x_1, x_2, x_3 from a fixed origin, the equations of motion are

$$\frac{d^2x_1}{dt^2}= \frac{m_2}{(x_2-x_1)^2}+\frac{m_3}{(x_3-x_1)^2},$$

$$\frac{d^2x_2}{dt^2}=-\frac{m_1}{(x_1-x_2)^2}+\frac{m_3}{(x_3-x_2)^2},$$

$$\frac{d^2x_3}{dt^2}=-\frac{m_1}{(x_1-x_3)^2}-\frac{m_2}{(x_2-x_3)^2},$$

whence

$$\frac{d^2(x_2-x_1)}{dt^2}=-\frac{m_1+m_2}{(x_2-x_1)^2}+\frac{m_3}{(x_3-x_2)^2}-\frac{m_3}{(x_3-x_1)^2},$$

$$\frac{d^2(x_3-x_1)}{dt^2}=-\frac{m_1+m_3}{(x_3-x_1)^2}-\frac{m_2}{(x_3-x_2)^2}-\frac{m_2}{(x_2-x_1)^2}.$$

Assume

$$x_3-x_1=\alpha(x_2-x_1),$$

and therefore

$$x_3-x_2=x_3-x_1-(x_2-x_1)=(\alpha-1)(x_2-x_1);$$

then the equations become

$$\frac{d^2(x_2-x_1)}{dt^2}=\left\{-m_1-m_2+\frac{m_3}{(\alpha-1)^2}-\frac{m_3}{\alpha^2}\right\}\frac{1}{(x_2-x_1)^2},$$

and

$$\frac{d^2(x_3-x_1)}{dt^2}=\left\{-\frac{m_1+m_3}{\alpha^2}-\frac{m_2}{(\alpha-1)^2}-m_2\right\}\frac{1}{(x_2-x_1)^2},$$

which equations will be one and the same equation if only

$$\alpha\left\{-m_1-m_2+\frac{m_3}{(\alpha-1)^2}-\frac{m_3}{\alpha^2}\right\}=-\frac{m_1+m_3}{\alpha^2}-\frac{m_2}{(\alpha-1)^2}-m_2,$$

an equation which (multiplying out) is of the fifth order, and gives at least one real value of α: and α being thus determined, we may from either equation obtain an equation of the form $\frac{d^2(x_2-x_1)}{dt^2}=\frac{C}{(x_2-x_1)^2}$, which determines x_2-x_1 in terms of t, and then $x_3-x_1, =\alpha(x_2-x_1)$, is also known; that is, the relative motions of the three bodies are determined.

9. *Write down the integral equations for the elliptic motion of a planet; and if r, s, v_1 denote the radius vector, tangent of the latitude, and reduced longitude respectively, show that*

$$\frac{\sqrt{(1+s^2)}}{r}=\frac{1}{a_1(1-e_1^2)}\{\sqrt{(1+s^2)}+e_1\cos(v_1-\varpi_1)\},$$

explaining the significations of the constants a_1, e_1, ϖ_1.

Write

a, the mean distance,

n, the mean motion $=\frac{\sqrt{(\mu)}}{a^{\frac{3}{2}}}$,

e, the eccentricity,

θ, longitude of node,

ϖ, longitude of pericentre in orbit,

ϕ, the inclination,

v, the longitude in orbit.

The position in the orbit is determined in terms of the time by means of an auxiliary quantity u, viz. writing

$$nt+c=u-e\sin u.$$

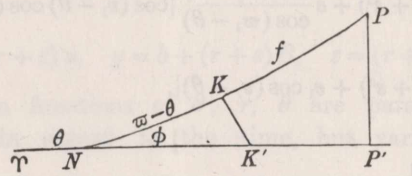
We then have the radius vector r, and the true anomaly f, given as functions of the time by the equations

$$\left.\begin{aligned}\cos f&=\frac{\cos u-e}{1-e\cos u},\\ \sin f&=\frac{\sqrt{(1-e^2)}\sin u}{1-e\cos u},\end{aligned}\right\}$$

$$r=\frac{a(1-e^2)}{1+e\cos f},\quad =a(1-e\cos u).$$

Take z, x, y the hypothenuse, base, and perpendicular of the right-angled triangle NPP', base angle $=\phi$ (see fig. 6), in which NP is the orbit, N the node, K the pericentre, P the planet; then

Fig. 6.



$z=\varpi-\theta+f$ gives z; and x, y are given in terms of z, ϕ, by the formulæ relating to the spherical triangle, so that we have

$$\begin{aligned}\text{longitude in orbit} &= z+\theta\,(=\varpi+f),\\ \text{reduced longitude} &= x+\theta,\\ \text{latitude} &= y.\end{aligned}$$

The expression for r, writing for f its value, $=v-\varpi$, gives

$$\frac{1}{r}=\frac{1}{a\,(1-e^2)}\{1+e\cos(v-\varpi)\},$$

and we have thence

$$\frac{\sqrt{(1+s^2)}}{r}=\frac{1}{a\,(1-e^2)}\{\sqrt{(1+s^2)}+\sqrt{(1+s^2)}\,e\cos(v-\varpi)\}.$$

But we have

$$v-\varpi=v-\theta-(\varpi-\theta),$$

and thence

$$\cos(v-\varpi)=\cos(v-\theta)\cos(\varpi-\theta)+\sin(v-\theta)\sin(\varpi-\theta).$$

Consequently

$$\begin{aligned}\sqrt{(1+s^2)}\cos(v-\varpi)=&\sqrt{(1+s^2)}\cos(\varpi-\theta)\cos(v-\theta)\\ &+\sqrt{(1+s^2)}\sin(\varpi-\theta)\sin(v-\theta);\end{aligned}$$

but by the right-angled triangle, we have

$$\begin{aligned}\sin(v-\theta)\sin\phi=\sin y &= \frac{s}{\sqrt{(1+s^2)}},\\ \sin(v_1-\theta) &= \cot\phi\tan y=s\cot\phi,\end{aligned}$$

and thence

$$\sqrt{(1+s^2)}\sin(v-\theta)=\frac{s}{\sin\phi}=\frac{1}{\cos\phi}s\cot\phi=\frac{\sin(v_1-\theta)}{\cos\phi},$$

$$\cos(v-\theta)=\cos(v_1-\theta)\cos y=\frac{\cos(v_1-\theta)}{\sqrt{(1+s^2)}},$$

that is,

$$\sqrt{(1+s^2)}\cos(v-\theta)=\cos(v_1-\theta).$$

The formula thus becomes

$$\frac{\sqrt{(1+s^2)}}{r} = \frac{1}{a(1-e^2)}\left[\sqrt{(1+s^2)} + e\left\{\cos(v_1-\theta)\cos(\varpi-\theta) + \sin(v_1-\theta)\frac{\sin(\varpi-\theta)}{\cos\phi}\right\}\right]$$

$$= \frac{1}{a\sqrt{(1-e^2)}}\left[\sqrt{(1+s^2)} + e\frac{\cos(\varpi-\theta)}{\cos(\varpi_1-\theta)}\{\cos(v_1-\theta)\cos(\varpi-\theta) + \sin(v_1-\theta)\sin(\varpi-\theta)\right]$$

$$= \frac{1}{a_1(1-e_1^2)}\{\sqrt{(1+s^2)} + e_1\cos(v_1-\theta)\},$$

where

$$\frac{1}{\cos\phi}\tan(\varpi-\theta) = \tan(\varpi_1-\theta),\ \text{gives}\ \varpi_1,$$

$$e_1 = e\frac{\cos(\varpi-\theta)}{\cos(\varpi_1-\theta)},\ \text{gives}\ e_1,$$

$$a_1(1-e_1^2) = a(1-e^2),\ \text{gives}\ a_1.$$

The first equation shows that, drawing the arc KK' at right angles to NP, then

$$\varpi_1 - \theta = NK',$$

and therefore

$$\varpi_1 = \theta + (\varpi_1 - \theta) = \Upsilon K'.$$

The other two equations then give e_1 and a_1: it may be added, that from the right-angled triangle NKK' we have $\cos(\varpi_1-\theta) = \cos(\varpi-\theta)\cos KK'$, and consequently that e_1 is $= e \div \cos KK'$.

10. *Find the differential equations for the motion of a material line acted upon by any forces and moving in a given ruled surface.*

If a, b are the coordinates of the point of intersection of any line of the ruled surface with the plane of xy; α, β, γ the cosines of the inclinations of the line to the three axes respectively; then writing

$$\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z}{\gamma}\,(=r),$$

and considering a, b, α, β, γ as given functions of a variable parameter θ, where α, β, γ satisfy the relation $\alpha^2+\beta^2+\gamma^2=1$, the equations in question, exclusive of the equation $(=r)$, determine the particular line on the surface; and taking account of the equation $(=r)$, they determine in terms of the parameters r, θ, the coordinates of a particular point in this line.

The motion of the material line is such that it comes successively to coincide with the several lines on the ruled surface; consider on the material line a fixed point, say its centre of gravity G, and imagine that in the course of the motion the material line comes to coincide with the line determined as above by the parameter θ, and the point G with the point determined as above by the parameters r, θ; consider on the material line any other point P whose distance from G is $=s$; the parameters of P will be $r+s$, θ; and the coordinates will consequently be given in terms of the variable parameters r, θ by the equations

$$\frac{x-a}{\alpha}=\frac{y-b}{\beta}=\frac{z}{\gamma}=r+s,$$

that is, we have

$$x=a+(r+s)\,\alpha,\quad y=b+(r+s)\,\beta,\quad z=(r+s)\,\gamma,$$

where a, b, α, β, γ are given functions of θ; r, θ are parameters varying with the time, and s is a constant in regard to the time, but varies with the point under consideration.

Hence, writing as usual T to denote the *Vis Viva* function

$$=\tfrac{1}{2}\Sigma dm\,(x'^2+y'^2+z'^2),$$

where dm is the element of the material line, and x', y', z' are the velocities of the element, it only remains to express T as a function of r, θ; and the equations of motion will be

$$\frac{d}{dt}\frac{\partial T}{\partial r'}-\frac{\partial T}{\partial r}=0,$$

$$\frac{d}{dt}\frac{\partial T}{\partial \theta'}-\frac{\partial T}{\partial \theta}=0.$$

Using a', b', α', β', γ' to denote the differential coefficients $\dfrac{\partial a}{\partial \theta}$, &c., but, as above, x', y', z', r', θ' to denote differential equations in regard to t, we have

$$x'=[a'+(r+s)\,\alpha'\,]\,\theta'+\alpha r',$$

$$y'=[b'+(r+s)\,\beta'\,]\,\theta'+\beta r',$$

$$z'=[\quad (r+s)\,\gamma'\,]\,\theta'+\gamma r',$$

and thence

$$x'^2+y'^2+z'^2=A\theta'^2+2Br'\theta'+r'^2,$$

if for shortness

$$A=a'^2+b'^2+2(r+s)\,(\alpha'a'+\beta'b')+(r+s)^2\,(\alpha'^2+\beta'^2+\gamma'^2),$$

$$B=a'\alpha+b'\beta+(r+s)\,(\alpha\,\alpha'+\beta\,\beta'+\gamma\gamma'),$$

$$=a'\alpha+b'\beta,$$

since

$$\alpha\alpha'+\beta\beta'+\gamma\gamma'=0.$$

Hence we have

$$T=\tfrac{1}{2}\theta'^2\Sigma A dm+r'\theta'\Sigma B dm+\tfrac{1}{2}r'^2\Sigma dm,$$

and, observing that in the sum $\Sigma A dm$, the terms involving $\Sigma s dm$ vanish in consequence of G being the centre of gravity, we have

$$\Sigma A dm=[a'^2+b'^2+2r\,(\alpha'a'+\beta'b')+r^2\,(\alpha'^2+\beta'^2+\gamma'^2)]\,\Sigma dm+(\alpha'^2+\beta'^2+\gamma'^2)\,\Sigma s^2 dm.$$

Or writing $\Sigma dm = M$, $\Sigma s^2 dm = Mk^2$, (M being therefore the mass of the line, and Mk^2 its moment about the centre of gravity), we have

$$\Sigma A dm = \tfrac{1}{2} M \left[a'^2 + b'^2 + 2r(\alpha' a' + \beta' b') + (r^2 + k^2)(\alpha'^2 + \beta'^2 + \gamma'^2)\right].$$

Moreover

$$\Sigma B dm = M(a'\alpha + b'\beta),$$

and hence we have

$$T = \tfrac{1}{2} M \left\{\theta'^2 \left[a'^2 + b'^2 + 2r(\alpha' a' + \beta' b') + (r^2 + k^2)(\alpha'^2 + \beta'^2 + \gamma'^2)\right] + 2\theta' r' (a'\alpha + b'\beta) + r'^2\right\},$$

a given function of r, θ, r', θ'; and the differential equations are therefore given as above.

11. *Explain the mutual connexion of the three theorems in conics:* (1) *the theorem* ad quatuor lineas; (2) *Pascal's theorem;* (3) *the theorem of the anharmonic relation of four points.*

The theorem *ad quatuor lineas* is that the locus of a point, such that the product of its distances from two given lines is always in a given ratio to the product of its distances from two other given lines, is a conic.

Consider the lines as forming a quadrilateral $ABCD$. Then A, B, C, D are points on the conic, and writing PAB to denote the perpendicular distance of a point P from the line AB, and so in other cases, the theorem is that the expression

$$\frac{PAB \, . \, PCD}{PAC \, . \, PBD}$$

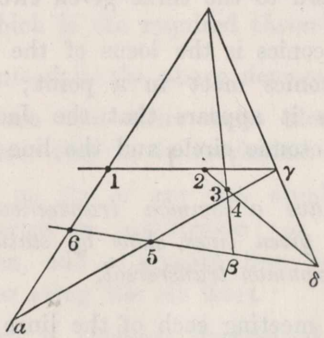
has a constant value for any point P whatever on the conic. Now the perpendicular distance PAB is $= 2\Delta PAB \div AB$, or, what is the same thing, it is $= PA \, . \, PB \sin PAB \div AB$; transforming the other perpendicular distances in the same manner, the distances PA, PB, PC, PD divide out and the foregoing expression becomes

$$= \frac{AC \, . \, BD \sin PAB \, . \sin PCD}{AB \, . \, CD \sin PAC \, . \sin PBD},$$

viz. omitting the constant factor, it appears that the expression $\dfrac{\sin PAB \, . \sin PCD}{\sin PAC \, . \sin PBD}$ is constant for all points P on the conic; or, what is the same thing, that the anharmonic ratio of the pencil $P(A, B, C, D)$ is constant for all points P on the conic. This is the anharmonic property of the points of a conic.

Pascal's theorem is that for any six points 1, 2, 3, 4, 5, 6 (fig. 7) on a conic, the intersections of the lines 12 and 45, the lines 23 and 56, the lines 34 and 61 lie in a line. Marking the points α, β, γ, δ as shown in the figure, it appears by the theorem that we have the two lines $(65\beta\delta)$, $(\alpha 54\gamma)$ meeting in 5, and such that the lines through the corresponding points 6 and α, β and 4, δ and γ meet in a point. Hence the ranges $(65\beta\delta)$ and $(\alpha 54\gamma)$ have the same anharmonic ratio; or, what is the same thing, the pencils $3(65\beta\delta)$ and $1(\alpha 54\gamma)$ have the same anharmonic ratio; that is, the pencils $3(6542)$ and $1(6542)$ have the same anharmonic ratio; or, considering 1 as a variable point, but 2, 3, 4, 5, 6 as fixed points on the conic, the pencil 1 (6542) of lines from the point 1 to the four points 6, 5, 4, 2 has the same anharmonic ratio for all

Fig. 7.



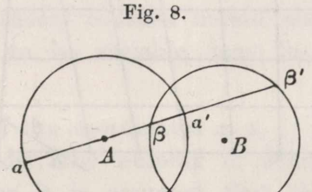
positions whatever of the variable point 1; which is the above-mentioned anharmonic property of the points of a conic.

12. *Show that any line through the centre of either of two orthotomic circles cuts the two circles harmonically; and connect this result with the theorem that the Jacobian of three circles is made up of the line infinity and the orthotomic circle.*

Drawing through the centre of the circle A, (fig. 8) the line $\alpha\beta\alpha'\beta'$, it appears by the figure that we have

$$\begin{aligned} A\beta \,.\, A\beta' &= \text{square of tangential distance of } A \text{ from the circle } B, \\ &= A\alpha'^2 \text{ since the circles cut at right angles,} \\ &= A\alpha \,.\, A\alpha'; \end{aligned}$$

Fig. 8.



that is, the points β, β' are inverse points in regard to the circle on the diameter $\alpha\alpha'$; and the points α, α'; β, β' are thus harmonically related to each other.

Hence the polar of α, in regard to the circle B, passes through the point α', which is the opposite of α in regard to the circle A; and is consequently a fixed point for all the circles B orthotomic to the circle A. That is, considering any three circles orthotomic to the circle A, the circle A is a locus of points α such that the polars of α in regard to the three circles meet in a point.

Moreover, all circles meet the line at infinity in two fixed points I, J, the circular points at infinity: hence for any point α on the line infinity, the polar of α in regard to any circle whatever meets the line infinity in a fixed point, the harmonic of α in regard to the two points I, J; whence the line infinity is also a locus of points α such that the polars of α in regard to the three given circles meet in a point.

The Jacobian of any three conics is the locus of the points α, such that the polars of α in regard to the three conics meet in a point; and it is in general a cubic curve. Hence by what precedes it appears that the Jacobian of three circles is the cubic curve made up of the orthotomic circle and the line infinity.

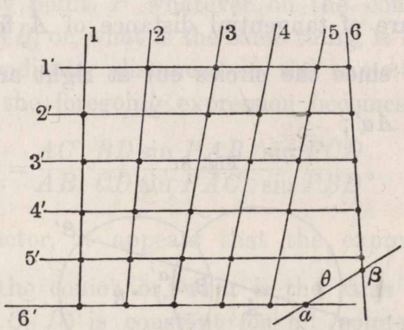
13. *If five given lines have a common transversal, then taking the remaining transversal of each four of the given lines, show by statical considerations or otherwise that the five transversals have a common transversal.*

Consider the line $6'$ (fig. 9) meeting each of the lines 1, 2, 3, 4, 5, and take

$1'$ the remaining transversal of (2, 3, 4, 5),
$2'$ „ „ (3, 4, 5, 1),
$3'$ „ „ (4, 5, 1, 2),
$4'$ „ „ (5, 1, 2, 3),
$5'$ „ „ (1, 2, 3, 4);

it is to be shown that the lines $1'$, $2'$, $3'$, $4'$, $5'$ have a common transversal 6.

Fig. 9.



Consider the line 6 as a line determined so as to meet the lines $2'$, $3'$, $4'$ and $5'$; and take any line θ meeting each of the lines 6, $6'$; since the six lines 1, 2, 3, 4, 5, θ have a common transversal $6'$, then considering the lines as fixed lines in a solid body, it is possible to find along the lines 1, 2, 3, 4, 5, θ forces which will be in equilibrium. Suppose for a moment that the line 6, determined as above, does not meet the line $1'$; then we have the lines $1'$, $2'$, $3'$, $4'$, $5'$, θ and 6 such that the line 6 meeting the lines $2'$, $3'$, $4'$, $5'$, and θ, does not meet the line 1. The six lines $1'$, $2'$, $3'$, $4'$, $5'$, and θ would be independent lines, such that there do not exist along them forces in equilibrium, and a force acting in any line whatever may be resolved into forces acting along the lines $1'$, $2'$, $3'$, $4'$, $5'$, and θ. Hence the original forces along the lines 1, 2, 3, 4, 5 can be each of them so resolved; and combining with the resolved forces the original force along θ, we have a system of forces along the lines $1'$, $2'$, $3'$, $4'$, $5'$, θ in equilibrium with each other; which is impossible if the lines in question are related as above; that is, the line 6 meeting the lines $2'$, $3'$, $4'$, $5'$, θ must also meet the line 1; which is the required theorem.

The statical principles assumed in the above demonstration are as follows:

(1) Six lines may be such that there exist along them forces in equilibrium; or say, for shortness, the six lines may be in involution.

(2) For any seven lines, no six or any less number of which are in involution, there exist along the lines forces in equilibrium; or, what is the same thing, given any six lines not in involution, and a seventh line; then a force along the seventh line may be resolved into forces along the six lines.

(3) Six lines which have a common transversal are in involution (remark in passing that it is not conversely true that if the six lines are in involution they have a common transversal), but if five of the six lines have a common transversal not meeting the sixth line, then the six lines are *not* in involution.

14. *In the theory of the variation of the arbitrary constants of a mechanical problem, state and explain the results obtained by Lagrange and Poisson respectively; and point out the peculiar advantage of Poisson's theory, in regard to the consequences which follow from the coefficients of his formulæ being independent of the time.*

In the theory of the variation of the arbitrary constants of a mechanical problem, it is assumed that the forces consist of principal and disturbing forces, each of them depending on a force function, say there is a principal force function and a disturbing function; (the assumption of a force function however in regard to the disturbing forces, though usual and convenient, is not essential); and that, when the disturbing forces are neglected, or say in the undisturbed problem, the equations of motion can be completely integrated; the theory consists herein that the same integral equations, taking the arbitrary constants to be variable, may be made to satisfy the disturbed equations of motion.

Suppose that the number of the coordinates $x, y, \ldots$ is $=p$, then since the differential equations are of the second order, the number of arbitrary constants $(a, b, c, \ldots)$ will be $=2p$. In Lagrange's solution it is assumed that the coordinates $x, y, \ldots$ (and consequently also the derived functions $x', y', \ldots$) are each of them given in terms of t and the $2p$ constants; the disturbing function Ω is given in the same form; and the expressions for the variations $\frac{da}{dt}$, &c. are obtained by the solution of a system of linear equations

$$\frac{dR}{da} = (a,\ b)\frac{db}{dt} + (a,\ c)\frac{dc}{dt} + \&c.,$$

where the coefficients (a, b) are functions involving $\frac{dx}{da}$, $\frac{dx'}{da}$, &c.; these coefficients are thus *primâ facie* functions of the $2p$ constants *and of the time;* but it is a result of the theory that they are in fact functions of the constants only, the time disappearing of itself.

In Poisson's theory it is assumed that the integrals of the undisturbed problem are obtained in the form

$$a = \phi\,(x,\ y, \dots x',\ y', \dots,\ t),\ \&c.,$$

viz. that each constant is given in terms of the coordinates, their derived functions, and the time (so that, if all the integrals are known, this is equivalent to Lagrange's assumption). The expressions for the variations of the constants are given in the form

$$\frac{da}{dt} = [a,\ b]\frac{dR}{db} + [a,\ c]\frac{dR}{dc} + \&c.,$$

where each coefficient $[a,\ b]$ is given as a function of $\frac{da}{dx}$, $\frac{da}{dx'}$, &c., $\frac{db}{dx}$, $\frac{db}{dx'}$, &c.: these coefficients are thus in the first instance expressed as functions of the coordinates $x, y, \dots$, the derived functions $x', y', \dots$, and the time; but it is a result of the theory that the coefficients $[a,\ b]$ are really constant; viz. that, if $x, y, \dots, x', y', \dots$ were expressed in terms of the $2p$ constants and the time, then that the time would disappear of itself and the coefficients $[a,\ b]$, &c. would be found to be functions of the constants only.

The formation of the value of any coefficient $[a,\ b]$ requires only the knowledge of the expressions of a, b in terms of $x, y, \dots, x', y', \dots, t$, or say the knowledge of the two integrals a, b. We thence obtain the expression of $[a,\ b]$ as a function of $x, y, \dots, x', y', \dots, t$, say $[a,\ b] = f(x,\ y, \dots,\ x',\ y', \dots,\ t)$. But, as already mentioned, $[a,\ b]$ is in fact a constant; calling it c, we have $c = f(x,\ y, \dots,\ x',\ y', \dots, t)$; that is, we have an integral of the equations of motion of the undisturbed problem. It may happen that the value of $[a,\ b]$ is found to be $=0$; or to be a function of $x, y, \dots, x', y', \dots, t$, which in virtue of the given values of a, b in terms of these same quantities reduces itself to a function of $(a,\ b)$; in either of these cases we obtain no new integral; but if (as may be) neither of the foregoing cases happen, then the equation $c = f(x,\ y, \dots,\ x',\ y', \dots,\ t)$ *is actually a new integral of the equations of motion* (in the undisturbed problem) *obtained by mere differentiations from the two given integrals* a, b. There is nothing analogous to this in Lagrange's theory.

15. *Write a short dissertation on the transformation of coordinates (rectangular in space of three dimensions); and in particular explain under what restriction it is true that two sets of rectangular axes about the same origin may be made to coincide by means of a rotation of either set about a certain axis; and from the formulae of transformation obtain expressions for the position of this axis and the amount of the rotation.*

Two sets of rectangular axes about the same origin (each axis considered, not as a line extending in two opposite senses, but as drawn from the origin in one sense only) are or are not displacements the one of the other; viz., making the axis of x_1 to coincide with that of x, and the axis of y_1 to coincide with that of y, then either the axis of z_1 will coincide with that of z, or it will be in the opposite direction; in the former case the two sets are, in the latter case they are not, displacements one of the other. The restriction referred to in the question, is that the two sets shall be displacements one of the other

In the problem of transformation, the two sets of axes, if not displacements, can always be made so by simply reversing the direction of one of the axes (writing for example $-z$ for z); and there is thus no real loss of generality in considering the two sets as displacements the one of the other; and it is in general convenient to make this assumption.

The transformation between two sets of rectangular axes is at once given by the diagram

	x	y	z
x_1	α	β	γ
y_1	α'	β'	γ'
z_1	α''	β''	γ''

where α is the cosine of the inclination of the axes x, x_1; and so for the rest of the nine quantities. The relation between the two sets of axes is obtained at pleasure by reading the diagram horizontally $x_1 = \alpha x + \beta y + \gamma z$, &c.; or by reading it vertically $x = \alpha x_1 + \alpha' y_1 + \alpha'' z_1$, &c.

We must have identically $x^2 + y^2 + z^2 = x_1^2 + y_1^2 + z_1^2$; and we thus obtain the two equivalent sets of equations

$$\begin{array}{l|l}
\alpha^2 + \beta^2 + \gamma^2 = 1, & \alpha^2 + \alpha'^2 + \alpha''^2 = 1, \\
\alpha'^2 + \beta'^2 + \gamma'^2 = 1, & \beta^2 + \beta'^2 + \beta''^2 = 1, \\
\alpha''^2 + \beta''^2 + \gamma''^2 = 1, & \gamma^2 + \gamma'^2 + \gamma''^2 = 1, \\
\alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'' = 0, & \beta\gamma + \beta'\gamma' + \beta''\gamma'' = 0, \\
\alpha''\alpha + \beta''\beta + \gamma''\gamma = 0, & \gamma\alpha + \gamma'\alpha' + \gamma''\alpha'' = 0, \\
\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0, & \alpha\beta + \alpha'\beta' + \alpha''\beta'' = 0.
\end{array}$$

Either set leads to the relation

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}^2 = 1, \text{ consequently } \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix} = \pm 1,$$

the distinction between the two cases $+1$, -1 being that above explained; viz. if the axes are not displacements the one of the other, the sign is $-$; if they are, (and in all that follows this is assumed to be the case) then the sign is $+$. The equations give further $\alpha = \beta'\gamma'' - \beta''\gamma'$, &c. (nine equations).

It is easy to see geometrically that, as stated in the question, two sets of axes (being as above mentioned displacements the one of the other) can be made to coincide by means of a rotation of either set about a certain axis; inasmuch as the position of the axis itself is not altered by the rotation, it is clear that for any point of the axis, the coordinates x, y, z and x_1, y_1, z_1 must be respectively equal; we thus have

$$\begin{aligned}(\alpha-1)\,x + \quad \beta y + \quad \gamma z &= 0,\\ \alpha' x + (\beta'-1)\,y + \quad \gamma' z &= 0,\\ \alpha'' x + \quad \beta'' y + (\gamma''-1)\,z &= 0,\end{aligned}$$

equations which must be equivalent to two equations; we in fact have

$$\begin{vmatrix} \alpha-1, & \beta & , & \gamma \\ \alpha' & , & \beta'-1, & \gamma' \\ \alpha'' & , & \beta'' & , & \gamma''-1 \end{vmatrix} = 0,$$

as is easily verified by means of the foregoing relations between the coefficients. Any two of the three equations will then determine the ratios $x:y:z$; taking the second and third, we have

$$x : y : z = (\beta'-1)(\gamma''-1) - \beta''\gamma' : \gamma'\alpha'' - \alpha'(\gamma''-1) : \alpha'\beta'' - \alpha''(\beta'-1),$$

reducible to

$$x : y : z = 1 + \alpha - \beta' - \gamma'' : \beta + \alpha' : \gamma + \alpha'',$$

and treating in the same way the third and first, and the first and second equations the system of formulæ is

$$\begin{aligned} x : y : z &= 1 + \alpha - \beta' - \gamma'' : \quad \beta + \alpha' \quad : \quad \gamma + \alpha'' \\ &= \quad \alpha' + \beta \quad : 1 - \alpha + \beta' - \gamma'' : \quad \gamma' + \beta'' \\ &= \quad \alpha'' + \gamma \quad : \quad \beta'' + \gamma' \quad : 1 - \alpha - \beta' - \gamma'', \end{aligned}$$

equations equivalent to each other; they determine the position of the axis in question, or resultant axis. The foregoing equations may also be written

$$x^2 : y^2 : z^2 : x^2+y^2+z^2 = 1+\alpha-\beta'-\gamma'' : 1-\alpha+\beta'-\gamma'' : 1-\alpha-\beta'+\gamma'' : 3-\alpha-\beta'-\gamma'',$$

and hence, if A, B, C be the inclinations of the resultant axis to the axes of x and x_1, y and y_1, z and z_1 respectively, we have

$$\cos^2 A = \frac{1+\alpha-\beta'-\gamma''}{3-\alpha-\beta'-\gamma''},$$

and thence also

$$\sin^2 A = \frac{2(1-\alpha)}{3-\alpha-\beta'-\gamma''},$$

$$\cos 2A = \frac{-1+3\alpha-\beta'-\gamma''}{3-\alpha-\beta'-\gamma''},$$

$$\alpha - \cos 2A = \frac{(1-\alpha)(1+\alpha+\beta'+\gamma'')}{3-\alpha-\beta'-\gamma''}.$$

Let θ be the amount of the rotation about the resultant axis; we have a spherical triangle the two sides whereof are A, A, the included angle θ, and the opposite side $\cos^{-1}\alpha$; that is, we have

$$\cos\theta = \frac{\alpha - \cos^2 A}{\sin^2 A},$$

and thence

$$4\cos^2\tfrac{1}{2}\theta = 2(1+\cos\theta) = \frac{2(\alpha-\cos 2A)}{\sin^2 A}$$

$$= 1+\alpha+\beta'+\gamma'',$$

that is, the amount of the rotation is given by the formula

$$4\cos^2\tfrac{1}{2}\theta = 1+\alpha+\beta'+\gamma''.$$

The nine cosines α, β, γ, &c. may be expressed in terms of the inclinations A, B, C, and the rotation θ, or putting λ, μ, ν equal to $\tan\frac{1}{2}\theta\cos A$, $\tan\frac{1}{2}\theta\cos B$, $\tan\frac{1}{2}\theta\cos C$, in terms of the three quantities λ, μ, ν; but the resulting formulæ for the transformation of coordinates can be more readily obtained by other methods.