527.

ON A THEOREM IN COVARIANTS.

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THE proof given in Clebsch "Theorie der binären algebraischen Formen" (Leipzig, 1872) of the finite number of the covariants of a binary form depends upon a subsidiary proposition which is deserving of attention for its own sake.

I use my own hyperdeterminant notation, which is as follows: Considering a function $U=(a,..)(x, y)^n$, (viz. $U_1=(a,..)(x_1, y_1)^n$ &c.), and writing $\overline{12}=\partial_{x_1}\partial_{y_2}-\partial_{y_1}\partial_{x_2}$, &c., then the general form of a covariant of the degree m is

$$k\,(\overline{12}^{\alpha}\,\overline{13}^{\beta}\,\overline{23}^{\gamma}\ldots)\,U_1U_2\ldots U_m,$$

where k is a merely numerical factor, the indices α, β, γ,.. are positive integers, and after the differentiations each set of variables (x_1, y_1), .., (x_m, y_m) is replaced by (x, y). I say that the general form of a covariant is as above; viz. a covariant is equal to a single term of the above form, or a sum of such terms.

Attending to a single term: the sum of the indices of all the duads which contain a particular number 1, 2,.. as the case may be is called an index-sum; each index-sum is at most $=n$; so that, calling the index-sums $\sigma_1, \sigma_2, \ldots, \sigma_m$ respectively, we have $n-\sigma_1$, $n-\sigma_2$, ..., $n-\sigma_m$ each of them zero or positive: the term, before the several sets of variables are each replaced by (x, y), is of the orders $n-\sigma_1$, $n-\sigma_2$, ..., $n-\sigma_m$ in the several sets of variables respectively.

The term may be expressed somewhat differently: for writing $\nabla_1=x\partial_{x_1}+y\partial_{y_1}$, $\nabla_2=x\partial_{x_2}+y\partial_{y_2}$ &c.—then (except as to a numerical factor) it is for a function $(*)(x_1, y_1)^p$ the same thing whether we change (x_1, y_1) into (x, y), or operate on this function with $\nabla_1{}^p$, and so for the other sets: the term may therefore be written

$$\nabla_1{}^{n-\sigma_1}\ldots\nabla_m{}^{n-\sigma_m}\,k\,(\overline{12}^{\alpha}\,\overline{13}^{\beta}\,\overline{23}^{\gamma}\ldots)\,U_1U_2\ldots U_m,$$

being now in the first instance a function of the single set (x, y) of variables.

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We may omit the operand $U_1U_2\dots U_m$, and consider only the symbol

$$k(\overline{12}^{\alpha}\,\overline{13}^{\beta}\,\overline{23}^{\gamma}\dots) \text{ or } \nabla_1{}^{n-\sigma_1}\dots\nabla_m{}^{n-\sigma_m}k(\overline{12}^{\alpha}\,\overline{13}^{\beta}\,\overline{23}^{\gamma}\dots),$$

which, under either of the two forms, I represent for shortness by $[12\dots m]$: observe that this is considered as a symbol involving the m symbolic numbers 1, 2, 3 ... m, even although in particular cases one or more of these numbers may be wanting from the actual expression of the symbol: thus [123] may denote $\overline{12}^{\alpha}$, but the operand to be supplied thereto is always $U_1U_2U_3$.

A sum of symbols is not in general equal to a single symbol: but a single symbol can be expressed in a variety of ways as a sum of symbols: the most simple transformation-formulæ relate to three or four symbolic numbers; viz. for three such numbers, say 1, 2, 3, we have

$$\nabla_1 . 23 + \nabla_2 . 31 + \nabla_3 . 12 = 0,$$

showing that in a symbol, which written with the ∇'s involves $\nabla_1 . 23$, this may be replaced by its value $\nabla_2 . 13 - \nabla_3 . 12$; and so in other cases.

For the four numbers 1, 2, 3, 4 we have a group of the like formulae

$$\begin{array}{rrrrl} . & -\nabla_2 . 34 & +\nabla_3 . 24 & -\nabla_4 . 23 & =0, \\ \nabla_1 . 34 & . & -\nabla_3 . 14 & -\nabla_4 . 31 & =0, \\ -\nabla_1 . 24 & +\nabla_2 . 14 & . & -\nabla_4 . 12 & =0, \\ \nabla_1 . 23 & +\nabla_2 . 23 & +\nabla_3 . 31 & . & =0, \end{array}$$

leading to

$$23 . 14 + 31 . 24 + 12 . 34 = 0,$$

which is a form not involving the ∇'s and consequently is applicable to the transformation of invariant-symbols where the numbers

$$n-\sigma_1,\ n-\sigma_2,\ \dots,\ n-\sigma_m$$

are all $=0$.

I establish the following definitions:

A symbol $[12\dots m]$ is *proximate* when each index-sum is $<n$; otherwise it is *ultimate*; viz. this is the case when any one or more of the index-sums is or are $=n$. We may say that the symbol is ultimate as to 1 if $\sigma_1 = n$; and that it is ultimate as to 1, 2 if σ_1 and σ_2 are each $=n$: and so in other cases.

A proximate symbol which has any one index-sum thereof $<\frac{1}{2}n$ is said to be *inferior*: thus if $\sigma_1 < \frac{1}{2}n$ the symbol is inferior in regard to 1; and so if σ_1 and σ_2 are each $<\frac{1}{2}n$, it is inferior in regard to 1 and 2: and the like in other cases.

Observe that if a symbol is inferior then in the covariant the order exceeds the

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degree by a number which is greater than $\frac{1}{2}n-1$: in fact, suppose it inferior in regard to 1, then the order is

$$(n-\sigma_1)+(n-\sigma_2)+\ldots+(n-\sigma_m),$$

where each term after the first is at least $=1$, that is, the order is at least $=n-\sigma_1+m-1$; hence order − degree is at least $=n-\sigma_1-1$; viz. σ_1 being less than $\frac{1}{2}n$, this is greater than $\frac{1}{2}n-1$.

Conversely, if for any symbol order-degree is $\overline{\gtrless}\frac{1}{2}n-1$, then the symbol is not inferior.

A symbol $[12\ldots m]$ is *sharp* when any index is $\overline{\gtrless}\frac{1}{2}n$; otherwise it is *flat;* viz. this is so when each index is $<\frac{1}{2}n$. The symbol is sharp as to any particular duad or duads when the index or indices thereof is or are each of them $\overline{\gtrless}\frac{1}{2}n$.

The subsidiary theorem is now as follows: "A symbol is inferior or sharp: or it can be expressed as a sum of symbols each of which is inferior or sharp"—or what is the same thing, the only symbols which need to be considered are those which are either inferior or sharp.

Thus for the degree 1 the symbol is [1] (which is simply unity) $\sigma_1=0$, and the symbol is inferior.

For the degree 2 the symbol is [2], $=\overline{12}^k$; if $k<\frac{1}{2}n$ the symbol is inferior, if $k\overline{\gtrless}\frac{1}{2}n$ then it is sharp.

A proof is first required for the degree 3, here $[123]=\overline{12}^\alpha\,\overline{13}^\beta\,\overline{23}^\gamma$ ($\beta+\gamma$, $\gamma+\alpha$, $\alpha+\beta$ each $=$ or $<n$) which may very well be neither inferior nor sharp; for instance, if $n=5$, we have $\overline{12}^2\,\overline{13}^2\,\overline{23}^2$, where each index being $=2$, the symbol is not sharp; and each index-sum being $=4$ the symbol is not inferior. But writing the symbol in the form $\nabla_1\nabla_2\nabla_3\overline{12}^2\,\overline{13}^2\,\overline{23}^2$, then by means of the relation

$$\nabla_1 . 23+\nabla_2 . 31+\nabla_3 . 12=0,$$

(or, what is the same thing, $\nabla_1 . 23=\nabla_2 . 13-\nabla_3 . 12$), the symbol becomes

$$\nabla_2\nabla_3\overline{12}^2\,\overline{13}^2\,\overline{23}^2(\nabla_2 . 13-\nabla_3 . 12),$$

$$=\nabla_2^2\nabla_3\overline{12}^2\,\overline{13}^3\,23-\nabla_2\nabla_3^2\,\overline{13}^3\,\overline{12}^2\,23,$$

where each term, as containing an index 3, is sharp. To complete the reduction, observe that calling the expression $\mathfrak{A}-\mathfrak{B}$, then in the term $\mathfrak{A}$ interchanging the numbers 2 and 3 we obtain $\mathfrak{A}=-\mathfrak{B}$, and thence $\mathfrak{A}-\mathfrak{B}=2\mathfrak{A}$; so that the whole is $2\nabla_2^2\nabla_3\overline{12}^2\,\overline{13}^3\,23$, viz. it is a multiple of $\overline{12}^2\,\overline{13}^3\,23$.

I prove the general case, substantially in the manner used by Dr Clebsch, as follows. We assume that the theorem is proved up to a particular degree m: that is, we assume that every symbol belonging to a degree not exceeding m can be

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expressed as a sum of terms each of which is sharp or inferior: and we have to prove this for the next following degree $m+1$, or writing for convenience p in place of $m+1$, (say for the degree p); that is, for a symbol

$$[12 \ldots mp], = \overline{p1}^{\lambda_1}\, \overline{p2}^{\lambda_2} \ldots \overline{pm}^{\lambda_m}\, [12 \ldots m]$$

$$= P\,[12 \ldots m] \text{ suppose.}$$

I write as before $\sigma_1, \sigma_2, \ldots, \sigma_m$ for the index-sums of $[12 \ldots m]$: those of $[12 \ldots mp]$ are therefore $\sigma_1+\lambda_1$, $\sigma_2+\lambda_2, \ldots, \sigma_m+\lambda_m$, and (for the duads involving p) $\sigma_p = \lambda_1+\lambda_2 \ldots + \lambda_m$.

If $[12 .. m]$ is sharp, then $[12 .. mp]$ is sharp, and the theorem is true.

If $\sigma_p < \frac{1}{2}n$, then $[12 \ldots mp]$ is inferior in regard to p; and the theorem is true.

The only case requiring a proof is when $[12 \ldots m]$ is not sharp (being therefore inferior) and when σ_p is $\overline{\overline{>}}\ \frac{1}{2}n$. And in this case if any one of the indices $\lambda_1, \ldots \lambda_m$ is $\overline{>}\ \frac{1}{2}n$ (or say if P is sharp) then the theorem is true.

Consider the expression

$$\overline{p1}^{\lambda_1}\, \overline{p2}^{\lambda_2} \ldots \overline{pm}^{\lambda_m}\, [12 \ldots m],$$

where $\sigma_1, \sigma_2, \ldots, \sigma_m$ are as before the index-sums for $[12 .. m]$ and therefore the numbers

$$n-\sigma_1-\lambda_1, \ldots, n-\sigma_m-\lambda_m$$

are none of them negative.

Assume that when $[12 \ldots m]$ is inferior, and when $\lambda_1 \ldots \lambda_m$ have any values such that their sum is not greater than a given value $\sigma_p - 1$, the expression is a sum of terms each of which is inferior or sharp: we wish to show that when $\lambda_1+\lambda_2 \ldots + \lambda_m$ has the next succeeding value, $=\sigma_p$, the case is still the same.

For this purpose, introducing the ∇'s I write

$$Q = \nabla_1{}^{n-\sigma_1-\lambda_1} \ldots \nabla_m{}^{n-\sigma_m-\lambda_m}\, \nabla_p{}^{n-\sigma_p}\, \overline{p1}^{\lambda_1}\, \overline{p2}^{\lambda_2} \ldots \overline{pm}^{\lambda_m}\, [12 \ldots m];$$

then supposing for a moment that λ_1 is not $=n-\sigma_1$ and λ_2 not $=0$, the expression contains the factor $\nabla_1 . p2$, which is equal to and may be replaced by $-\nabla_2 . p1 + \nabla_p . 12$: we have thus

$$Q = Q' + \Omega,$$

where omitting the ∇'s

$$Q' = j\, \overline{p1}^{\lambda_1+1}\, \overline{p2}^{\lambda_2-1}\, \overline{p3}^{\lambda_3} \ldots \overline{pm}^{\lambda_m}\, [12 \ldots m],$$

$$\Omega = k\, \overline{p1}^{\lambda_1}\, \overline{p2}^{\lambda_2-1}\, \overline{p3}^{\lambda_3} \ldots \overline{pm}^{\lambda_m}\, \overline{12}\, [12 \ldots m].$$

Now for Ω the sum of the indices λ_1, λ_2-1, $\lambda_3 .. \lambda_m$ is $\sigma_p - 1$, so that by hypothesis Ω is inferior or sharp: that is, the difference $Q-Q'$ is inferior or sharp: so that to prove that Q is inferior or sharp, we have only to prove this of Q', where Q' is derived from Q by increasing by unity the index of $p1$, at the expense of that of $p2$

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which is diminished by unity. Such change is possible so long as the index λ_1 has not attained its maximum value, $n-\sigma_1$ or σ_p as the case may be, and there is any other index $\lambda_2, \ldots, \lambda_m$ which is not $=0$: that is, we may pass from Q to Q', from Q' to Q'' and so on; and it will be sufficient to show that the last term of the series is inferior or sharp. We thus pass from Q to R, where

$$R = \overline{p1}^{\,n-\sigma_1}\ \overline{p2}^{\,\lambda_2-\alpha_2} \ldots \overline{pm}^{\,\lambda_m-\alpha_m}\ [12\ldots m]$$

and $\alpha_2+\alpha_3 \ldots + \alpha_m = n-\sigma_1-\lambda_1$; or else to

$$R = \overline{p1}^{\,\sigma_p}\ [12\ldots m],$$

according as $n-\sigma_1$ is not greater or is greater than σ_p.

Now let $[12\ldots m]$ be inferior; suppose it to be so in regard to 1, that is, let σ_1 be less than $\frac{1}{2}n$ or $n-\sigma_1$ greater than $\frac{1}{2}n$. Then if σ_p be less than $\frac{1}{2}n$ it is less than $n-\sigma_1$, that is, we have for R the last-mentioned form which is inferior in regard to p, viz. R is inferior; if σ_p is equal to or greater than $\frac{1}{2}n$, then R, whichever its form may be, is sharp as to $p1$, viz. R is sharp. Hence in either case Q is a sum of terms which are inferior or sharp; that is, assuming the theorem for a form for which $\lambda_1+\lambda_2 \ldots + \lambda_m$ does not exceed a given value σ_p-1, the theorem is true for the next succeeding value σ_p; or being true for the case $\sigma_p-1=0$, it is true generally.

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