525.

AN EXAMPLE OF THE HIGHER TRANSFORMATION OF A BINARY FORM.

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THE quartic

(1) $$(a, b, c, d, e)(x, y)^4$$

is by means of the two quadrics

(2) $$(\alpha, \beta, \gamma)(x, y)^2 \text{ and } (\alpha', \beta', \gamma')(x, y)^2$$

transformed into

(3) $$(a_1, b_1, c_1, d_1, e_1)(x_1, y_1)^4,$$

that is, eliminating x, y from

$$(a, b, c, d, e)(x, y)^4 = 0,$$

$$x_1(\alpha, \beta, \gamma)(x, y)^2 + y_1(\alpha', \beta', \gamma')(x, y)^2 = 0,$$

we obtain

$$(a_1, b_1, c_1, d_1, e_1)(x_1, y_1)^4 = 0.$$

It is required to express the invariants of (3) in terms of the simultaneous invariants of (1) and (2).

Write

$$P, Q, R = \alpha x_1 + \alpha' y_1, \quad \beta x_1 + \beta' y_1, \quad \gamma x_1 + \gamma' y_1;$$

the equations from which (x, y) have to be eliminated are

$$(a, b, c, d, e)(x, y)^4 = 0, \quad (P, Q, R)(x, y)^2 = 0,$$

and the result of the elimination therefore is

$$\left|\begin{matrix} & & a, & 4b, & 6c, & 4d, & e \\ & a, & 4b, & 6c, & 4d, & e & \\ & & & & P, & 2Q, & R \\ & & & P, & 2Q, & R & \\ & & P, & 2Q, & R & & \\ & P, & 2Q, & R & & & \end{matrix}\right| = 0,$$

viz. this determinant is the transformed quartic $(a_1, b_1, c_1, d_1, e_1)(x_1, y_1)^4$.

The developed expression of the determinant is

$$a^2R^4 - 8abQR^3$$
$$+\begin{pmatrix}-12\,ac\\+16\,b^2\end{pmatrix} PR^3 - 24\,acQ^2R^2 + \begin{pmatrix}24\,ad\\-48\,bc\end{pmatrix} PQR^2 - 32\,adQ^3R$$
$$+\begin{pmatrix}2\,ae\\-32\,bd\\+36\,c^2\end{pmatrix} P^2R^2 + \begin{pmatrix}-16\,ae\\+64\,bd\end{pmatrix} PQ^2R + 16\,aeQ^4$$
$$+\begin{pmatrix}24\,be\\-48\,cd\end{pmatrix} P^2QR - 32\,bePQ^3 + \begin{pmatrix}-12\,ce\\+16\,d^2\end{pmatrix} P^3R \quad + 24\,ceP^2Q^2$$
$$- 8\,deP^3Q + e^2P^4,$$

so that writing for P, Q, R their values, we have the transformed function $(a_1, b_1, c_1, d_1, e_1)(x_1, y_1)^4$, the coefficients being of the forms

$$a_1 = (a, b, c, d, e)^2 (\alpha, \beta, \gamma)^4$$
$$b_1 = (\quad ,, \quad)^2 (\alpha, \beta, \gamma)^3 (\alpha', \beta', \gamma')$$
$$\vdots$$
$$e_1 = (\quad ,, \quad)^2 \quad . \quad . \quad (\alpha', \beta', \gamma')^4.$$

Writing I, J for the invariants of the quartic (1), and

$$A = 4(\alpha\beta' - \alpha'\beta)(\beta\gamma' - \beta'\gamma) - (\gamma\alpha' - \gamma'\alpha)^2,$$
$$B = (e, c, a, b, c, d)(\alpha\beta' - \alpha'\beta, \gamma\alpha' - \gamma'\alpha, \beta\gamma' - \beta'\gamma)^2,$$

we have I, J, A, B simultaneous invariants of the forms (1) and (2). Putting moreover $\nabla = I^3 - 27J^2$, and writing I_1, J_1, ∇_1, for the like invariants of the form (3), I find

$$I_1 = 4(4\,IB^2 + 12\,JAB + \tfrac{1}{3}I^2A^2),$$
$$J_1 = 8\{8\,JB^3 + \tfrac{4}{3}I^2AB^2 + 2\,IJA^2B + (2J^2 - \tfrac{1}{27}I^3)A^3\},$$

and thence

$$\nabla_1 = 256(4\,B^3 - IA^2B - JA^3)\,\nabla.$$

As a verification, suppose $(a, b, c, d, e)(x, y)^4 = x^4 + y^4$ (whence $I = 1, J = 0$). And take $x_1(x+y)^2 - y_1(x-y)^2 = 0$ for the transforming equation, that is, $(\alpha, \beta, \gamma) = (1, 1, 1)$ and $(\alpha', \beta', \gamma') = (-1, 1, -1)$. We have $P = R = x_1 - y_1$ and $Q = x_1 + y_1$, and thence

$$\begin{aligned} \text{Det.} &= (P^2 + R^2)^2 - 16\, PQ^2R + 16\, Q^4 \\ &= (2P^2 - 4Q^2)^2 = (-2x_1^2 - 12\, x_1y_1 - 2y_1^2)^2, \end{aligned}$$

that is,

$$(a_1, b_1, c_1, d_1, e_1)(x_1, y_1)^4 = 4\,(x_1^2 + 6x_1y_1 + y_1^2)^2,$$

whence

$$I_1 = \frac{4096}{3}, \quad = \frac{2^{12}}{3}; \quad J_1 = -\frac{262144}{27}, \quad = -\frac{2^{18}}{27};$$

also

$$A = -16, \quad B = 8,$$

and the equations for I_1, J_1 become

$$\frac{4096}{3} = 4\,(4\,.\,64 + \tfrac{1}{3}\,256),$$

$$-\frac{262144}{27} = 8\,(\tfrac{4}{3}\,.\, -16\,.\,64 + \tfrac{1}{27}\,4096),$$

which are true. More generally, assuming

$$(a, b, c, d, e)(x, y)^4 = x^4 + 6\Theta x^2y^2 + y^4,$$

(whence $I = 1 + 3\Theta^2$, $J = \Theta - \Theta^3$), and the same transforming equation, we have

$$(a_1, b_1, c_1, d_1, e_1)(x_1, y_1)^4 = 4\,\{(1 + 3\Theta)\, x_1^2 + (3 - 3\Theta)\, 2x_1y_1 + (1 + 3\Theta)\, y_1^2\}^2,$$

whence

$$I_1 = \frac{2^{12}}{3}(1 - 3\Theta)^2, \quad J_1 = -\frac{2^{18}}{27}(1 - 3\Theta)^3;$$

also

$$A = -16, \quad B = 8\,(1 - \Theta).$$

Substituting these different values in the equations for I_1, J_1, we obtain

$$16\,(1 - 3\Theta)^2 = 12\,(1 + 3\Theta^2)(1 - \Theta)^2 - 72\,(\Theta - \Theta^3)(1 - \Theta) + 4\,(1 + 3\Theta^2)^2,$$

and

$$\begin{aligned} -8\,(1 - 3\Theta)^3 = {} & 27\,(\Theta - \Theta^3)(1 - \Theta)^3 - 9\,(1 + 3\Theta^2)^2(1 - \Theta)^2 \\ & + 27\,(1 + 3\Theta^2)(\Theta - \Theta^3)(1 - \Theta) - 54\,(\Theta - \Theta^3)^2 + (1 + 3\Theta^2)^3, \end{aligned}$$

which are in fact satisfied identically.

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