506.

ON THE MECHANICAL DESCRIPTION OF A CUBIC CURVE.

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If the coordinates x, y of a point on a curve are rational functions of $\sin\phi$, $\cos\phi$, $\sqrt{1-k^2\sin^2\phi}$, the curve has the deficiency 1, and conversely in any curve of deficiency 1 the coordinates x, y can be thus expressed in terms of the parameter ϕ. Hence writing $\sin\theta = k\sin\phi$, the coordinates will be rational functions of $\sin\phi$, $\cos\phi$, $\cos\theta$, or say of $\sin\phi$, $\cos\phi$, $\sin\theta$, $\cos\theta$; and for the mechanical representation of the relation $k\sin\phi = \sin\theta$, we require only a rod OA rotating about the fixed point O, and connected with it by a pin at A, a rod AB, the other extremity of which, B, moves in a fixed line Ox. The curve most readily obtained by such an arrangement is that described by a point C rigidly connected with the rod AB; this is however a quartic curve (with two dps., since its deficiency is $=1$). I first considered the cubic curve

$$xy - 1 = \sqrt{(1-x^2)(1-k^2x^2)},$$

or say

$$xy - 1 = -\sqrt{(1-x^2)(1-k^2x^2)};$$

writing herein $x = \sin\phi$, and as before $k\sin\phi = \sin\theta$, we have then $y\sin\phi = 1 - \cos\theta\cos\phi$; which values may be written

$$x = \sin\phi,$$

$$y = \frac{1-\cos(\theta+\phi)}{\sin\phi} - \sin\theta.$$

I found, however, that this was *not* the cubic curve most easily constructed; and I ultimately devised a mechanical arrangement consisting of

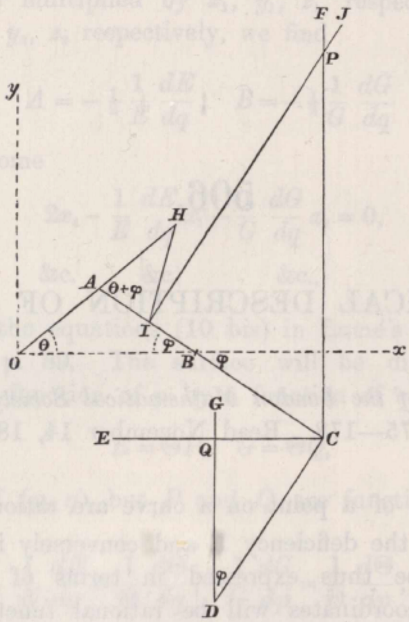
1. Rod OH, and connected with it by a pin at H, rod HI [1].

[1] There was a mechanical convenience in this, but observe that producing OH to meet IP in I', the single straight rod OHI' might have been made use of.

2 Square ACD, and connected with it by a pin at D, rod DG.

3. Square ECF; the two squares being connected by a pin at C.

4. Rod IJ.



The rod OH rotates about a pin at O; taking $HA = HI$, there is a pin at A connecting a fixed point of this rod with the extremity A of the square ACD: the fixed point B of this square moves along the line Ox. There is a pin at I connecting the extremities of the rods HI, IJ; and this slides along the leg AC of the square ACD, the rod IJ being always at right angles thereto: finally the legs of the square ECF are always parallel to Ox, Oy, and the rod DG at right angles to EC. I have omitted from the description the parallel-motion rods or other arrangements necessary for giving these fixed directions to the rod IJ, the square ECF, and the rod DG. It will be seen that the angles AOB, ABO are variable angles connected by an equation of the form above referred to; and that the lines IJ, CF determine by their intersection the point P; and the lines CE, DG determine by their intersection the point Q; the curve about to be considered is that determined by the relative motion of P in regard to Q; or say the curve the coordinates of a point of which are

$$x = QC, \quad y = CP.$$

I write

$$\angle AOB = \theta, \quad \angle ABO = \phi,$$

$$OA = a, \quad AB = b, \quad AC = c, \quad CD = d,$$

$$AH = HI = \tfrac{1}{2}h.$$

We then have $a \sin \theta = b \sin \phi$; and moreover the length AI being $= h \cos(\theta + \phi)$, and therefore $IC = c - h \cos(\theta + \phi)$, we have

$$y = CP = \frac{c - h \cos(\theta + \phi)}{\sin \phi},$$

$$x = QC = d \sin \phi;$$

whence also

$$xy = d \{c - h \cos(\theta + \phi)\};$$

or we have

$$xy = d \left\{c - h \sqrt{\left(1 - \frac{x^2}{d^2}\right)} \sqrt{\left(1 - \frac{b^2}{a^2} \frac{x^2}{d^2}\right)} + h \frac{b}{a} \frac{x^2}{d^2}\right\},$$

that is

$$x\left(y - \frac{bh}{ad} x\right) - cd = -dh \sqrt{\left(1 - \frac{x^2}{d^2}\right)} \sqrt{\left(1 - \frac{b^2}{a^2} \frac{x^2}{d^2}\right)}:$$

or rationalising and reducing, this is

$$x^2y^2 - \frac{2bh}{ad} x^3y - 2cdxy + \left\{2 \frac{b}{a} ch + h^2 \left(1 + \frac{b^2}{a^2}\right)\right\} x^2 + d^2 (c^2 - h^2) = 0,$$

a quartic curve with two dps.

In the particular case $a = b$, the relation between θ, ϕ is simply $\theta = \phi$; the curve should become unicursal.

Writing in the equation $\frac{b}{a} = 1$, the equation takes the form

$$\left\{x\left(y - \frac{2h}{d} x\right) - d(c - h)\right\} \{xy - d(c + h)\} = 0;$$

the second factor is extraneous, and the curve is the hyperbola

$$x\left(y - \frac{2h}{d} x\right) - d(c - h) = 0,$$

as at once appears from the foregoing irrational form of the equation.

In the particular case $h = c$, the equation contains the factor x, and omitting this it becomes

$$x\left(y^2 - \frac{2bc}{ad} x^2\right) - 2cdy + c\left(1 + \frac{b}{a}\right)^2 x = 0;$$

viz. we have here a cubic curve with three real asymptotes meeting in a point which is also the centre of the curve.

If simultaneously $a = b$ and $h = c$, then the equation is

$$x\left(y - \frac{2c}{d} x\right)(xy - 2cd) = 0,$$

the actual locus being in this case the line $y - \frac{2c}{d} x = 0$.

Writing $h=c$, and for greater convenience $h=c=d=1$; also to fix the ideas supposing $b<a$, and writing $\frac{b}{a}=k, =\sin\lambda$, then we have

$$\sin\theta = k\sin\phi,$$

$$x = \sin\phi,$$

$$y = \frac{1-\cos(\theta+\phi)}{\sin\phi};$$

that is

$$xy = 1-\sqrt{1-x^2}\sqrt{1-k^2x^2}+kx^2,$$

giving the rationalised equation

$$x(y^2-2kx^2)-2y+4x=0;$$

the angle ϕ may be anything whatever, but θ varies between the limits $\pm\lambda$, the simultaneous values of these angles and of the coordinates being

$\phi=0$	$\theta=0$	$x=0$	$y=0$
$\phi=90°$	$\theta=\lambda$	$x=1$	$y=1+\sin\lambda$
$\phi=180°$	$\theta=0$	$x=0$	$y=\pm\infty$
$\phi=270°$	$\theta=-\lambda$	$x=-1$	$y=-(1+\sin\lambda)$
$\phi=360°$	$\theta=0$	$x=0$	$y=0;$

and it thus appears that the mechanism gives the continuous branch which belongs to the asymptote $x=0$ of the cubic curve; the other two branches belong to $x=\sin\phi,\ y=\frac{1+\cos(\theta+\phi)}{\sin\phi}$, which would require a slight alteration in the arrangement of the mechanism.

I remark that if AH, HI had been unequal, then writing $\angle HIA=\chi$, this would be connected with $\theta+\phi$ by an equation of the form

$$\sin(\theta+\phi)=m\sin\chi,$$

and the coordinates x, y would be rational functions of the sines and cosines of θ, ϕ, χ; the deficiency is in this case >1.