505.

ON THE SURFACES DIVISIBLE INTO SQUARES BY THEIR CURVES OF CURVATURE.

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PROFESSOR CAYLEY gave an account of an investigation recently communicated by him to the Academy of Sciences at Paris. The fundamental theorem is that, if the coordinates x, y, z of a point on a surface are expressed as functions of two parameters p, q (such expressions, of course replacing the equation of the surface); and if these parameters are such that $p = \text{const.}$, $q = \text{const.}$ are the equations of the two sets of curves of curvature respectively; then (writing for shortness

$$\frac{dx}{dp} = x_1,\quad \frac{dx}{dq} = x_2,\quad \frac{d^2x}{dp^2} = x_3,\quad \frac{d^2x}{dp\,dq} = x_4,\quad \frac{d^2x}{dq^2} = x_5,$$

and the like for y, z), the coordinates x, y, z, considered always as functions of p, q, satisfy the equations

$$x_1x_2 + y_1y_2 + z_1z_2 = 0,$$

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_4, & y_4, & z_4 \end{vmatrix} = 0.$$

The last equation is equivalent to

$$x_4 + Ax_1 + Bx_2 = 0,$$
$$y_4 + Ay_1 + By_2 = 0,$$
$$z_4 + Az_1 + Bz_2 = 0;$$

and if in the notation of Gauss we write

$$x_1^2 + y_1^2 + z_1^2 = E,$$
$$x_2^2 + y_2^2 + z_2^2 = G,$$

then adding the equations multiplied by x_1, y_1, z_1 respectively, and also adding the equations multiplied by x_2, y_2, z_2 respectively, we find

$$A = -\tfrac{1}{2}\frac{1}{E}\frac{dE}{dq}, \quad B = -\tfrac{1}{2}\frac{1}{G}\frac{dG}{dq}$$

and the equations thus become

$$2x_4 - \frac{1}{E}\frac{dE}{dq}x_1 - \frac{1}{G}\frac{dG}{dq}x_2 = 0,$$
$$\&c. \qquad \&c. \qquad \&c.,$$

which, in fact, agree with the equations (10 bis) in Lamé's "Leçons sur les coordonnées curvilignes," Paris (1859), p. 89. The surface will be divisible into squares if only $E : G$ is the quotient of a function of p by a function of q, or say if

$$E = \Theta P, \quad G = \Theta Q,$$

where Θ is any function of (p, q), but P and Q are functions of p and q respectively; we then have

$$\frac{1}{E}\frac{dE}{dq} = \frac{1}{\Theta}\frac{d\Theta}{dq}, \quad \frac{1}{G}\frac{dG}{dp} = \frac{1}{\Theta}\frac{d\Theta}{dp},$$

and the equations for x, y, z are

$$2x_4 - \frac{1}{\Theta}\frac{d\Theta}{dq}x_1 - \frac{1}{\Theta}\frac{d\Theta}{dp}x_2 = 0,$$
$$\&c. \qquad \&c. \qquad \&c.,$$

viz. x, y, z being functions of p, q such that $x_1x_2 + y_1y_2 + z_1z_2 = 0$, and which besides satisfy these equations, or say which each of them satisfy the equation

$$2u_4 - \frac{1}{\Theta}\frac{d\Theta}{dq}u_1 - \frac{1}{\Theta}\frac{d\Theta}{dp}u_2 = 0,$$

then the values of x, y, z in terms of (p, q) determine a surface which has the property in question.