498.

ON THE INVERSION OF A QUADRIC SURFACE.

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THE inversion intended to be considered is that by reciprocal radius vectors, viz. if x, y, z are rectangular coordinates, and $r^2 = x^2 + y^2 + z^2$, then x, y, z are to be changed into $\frac{x}{r^2}$, $\frac{y}{r^2}$, $\frac{z}{r^2}$. But it is convenient to introduce for homogeneity a fourth coordinate w, $=1$; and the change then is x, y, z into $\frac{xw^2}{r^2}$, $\frac{yw^2}{r^2}$, $\frac{zw^2}{r^2}$.

Starting from the quadric surface

$$(a, b, c, d, f, g, h, l, m, n \between x, y, z, w)^2 = 0,$$

or, what is the same thing,

$$\begin{aligned} &(a, b, c, f, g, h \between x, y, z)^2 \\ &+ 2w\,(lx + my + nz) \\ &+ dw^2 \qquad\qquad = 0, \end{aligned}$$

the equation of the inverse surface is

$$\begin{aligned} &w^2\,(a, b, c, f, g, h \between x, y, z)^2 \\ &+ 2w\,(lx + my + nz)\,r^2 \\ &+ dr^4 \qquad\qquad = 0, \end{aligned}$$

where $r^2 = x^2 + y^2 + z^2$. The inverse surface is thus a quartic having the nodal conic $w = 0$, $x^2 + y^2 + z^2 = 0$ (circle at infinity); and having the node $x = 0$, $y = 0$, $z = 0$ (the centre of inversion); or say it is a nodal bicircular quartic surface, or nodal anallagmatic.

For x, y, z write $x-\frac{1}{2}\frac{l}{d}w$, $y-\frac{1}{2}\frac{m}{d}w$, $z-\frac{1}{2}\frac{n}{d}w$, and put for shortness

$$lx+my+nz=u,\quad l^2+m^2+n^2=\alpha,$$

$$al+hm+gn=\mathrm{a},\quad (a, b, c, f, g, h \between l, m, n)^2=A,$$

$$hl+bm+fn=\mathrm{b},$$

$$gl+fm+cn=\mathrm{c},$$

then

$$r^2 \quad \text{becomes}\quad r^2-\frac{uw}{d}+\tfrac{1}{4}\frac{\alpha}{d^2}w^2,$$

$$lx+my+nz \quad ,, \quad u-\tfrac{1}{2}\frac{\alpha}{d}w,$$

$$(a, \ldots \between x, y, z)^2 \quad ,, \quad (a, \ldots \between x, y, z)^2-(\mathrm{a}x+\mathrm{b}y+\mathrm{c}z)\frac{w}{d}+\tfrac{1}{4}A\frac{w^2}{d^2}.$$

Hence the equation is

$$d\left\{r^4-2r^2\frac{uw}{d}+w^2\left(\tfrac{1}{2}\frac{\alpha}{d^2}r^2+\frac{u^2}{d^2}\right)-\tfrac{1}{2}\frac{\alpha uw^3}{d^3}+\tfrac{1}{16}\frac{\alpha^2}{d^4}w^4\right\}$$

$$+2\left(wr^2-\frac{uw^2}{d}+\tfrac{1}{4}\frac{\alpha}{d^2}w^3\right)\left(u-\tfrac{1}{2}\frac{\alpha}{d}w\right)$$

$$+w^2\left\{(a, \ldots \between x, y, z)^2-(\mathrm{a}x+\mathrm{b}y+\mathrm{c}z)\frac{w}{d}+\tfrac{1}{4}A\frac{w^2}{d^2}\right\}=0;$$

viz. arranging and reducing, this is

$$dr^4$$

$$+w^2\left\{-\tfrac{1}{2}\frac{\alpha}{d}r^2-\frac{u^2}{d}+(a, \ldots \between x, y, z)^2\right\}$$

$$+w^3\left\{\quad\frac{\alpha u}{d^2}\quad-\frac{1}{d}(\mathrm{a}x+\mathrm{b}y+\mathrm{c}z)\right\}$$

$$+w^4\left\{-\tfrac{3}{16}\frac{\alpha^2}{d^3}+\tfrac{1}{4}A\frac{1}{d^2}\right\}\qquad=0;$$

and we may without loss of generality assume

$$-\frac{mn}{d}+f=0,\ \text{that is}\quad df-mn=0,$$

$$-\frac{nl}{d}+g=0,\quad ,, \quad dg-nl=0,$$

$$-\frac{lm}{d}+h=0,\quad ,, \quad dh-lm=0.$$

The equation then is

$$\begin{aligned} &r^4 \\ &+ w^2 \left\{-\tfrac{1}{2}\frac{\alpha}{d^2}(x^2+y^2+z^2) + \left(\frac{a}{d} - \frac{l^2}{d^2}\right)x^2 + \left(\frac{b}{d} - \frac{m^2}{d^2}\right)y^2 + \left(\frac{c}{d} - \frac{n^2}{d^2}\right)z^2\right\} \\ &+ w^3 \left\{\frac{\alpha u}{d^3} - \frac{1}{d^2}(\mathrm{a}x + \mathrm{b}y + \mathrm{c}z)\right\} \\ &+ w^4 \left\{-\tfrac{3}{16}\frac{\alpha^2}{d^4} + \tfrac{1}{4}A\frac{1}{d^3}\right\} = 0. \end{aligned}$$

Write

$$\begin{aligned} ad - l^2 &= a'd, \\ bd - m^2 &= b'd, \\ cd - n^2 &= c'd. \end{aligned}$$

We have

$$\mathrm{a} = al + hm + gn = gn = al + \frac{lm^2}{d} + \frac{ln^2}{d} = \frac{l}{d}(ad - l^2 + \alpha),$$

that is

$$\mathrm{a} = la' + \frac{l\alpha}{d},$$

and similarly

$$\mathrm{b} = mb' + \frac{m\alpha}{d},$$

$$\mathrm{c} = nc' + \frac{n\alpha}{d}.$$

Hence also

$$A = l^2a' + m^2b' + n^2c' + \frac{\alpha^2}{d},$$

and the equation is

$$\begin{aligned} &r^4 \\ &+ w^2 \left\{\left(-\tfrac{1}{2}\frac{\alpha}{d^2} + \frac{a'}{d}\right)x^2 + \left(-\tfrac{1}{2}\frac{\alpha}{d^2} + \frac{b'}{d}\right)y^2 + \left(-\tfrac{1}{2}\frac{\alpha}{d^2} + \frac{c'}{d}\right)z^2\right\} \\ &+ w^3 \left\{ -\frac{la'}{d^2}x - \frac{mb'}{d^2}y - \frac{nc'}{d^2}z\right\} \\ &+ w^4 \left\{\frac{1}{4d^3}(l^2a' + m^2b' + n^2c') + \tfrac{1}{16}\frac{\alpha^2}{d^4}\right\} = 0. \end{aligned}$$

This is Kummer's form, say

$$r^4 = 4w^2\{\alpha_1x^2 + \beta_1y^2 + \gamma_1z^2 + \delta_1w^2 + 2w(a_1x + b_1y + c_1z)\},$$

where

$$-4\alpha_1 = -\tfrac{1}{2}\frac{\alpha}{d^2} + \frac{a'}{d},$$

$$-4\beta_1 = -\tfrac{1}{2}\frac{\alpha}{d^2} + \frac{b'}{d},$$

$$-4\gamma_1 = -\tfrac{1}{2}\frac{\alpha}{d^2} + \frac{c'}{d},$$

$$-4\delta_1 = \frac{1}{4d^3}(l^2a' + m^2b' + n^2c') + \tfrac{1}{16}\frac{\alpha^2}{d^4},$$

$$-8a_1 = -\frac{la'}{d^2},$$

$$-8b_1 = -\frac{mb'}{d^2},$$

$$-8c_1 = -\frac{nc'}{d^2}.$$

Hence Kummer's equation

$$\delta_1 + \lambda^2 = \frac{a_1{}^2}{\lambda + \alpha_1} + \frac{b_1{}^2}{\lambda + \beta_1} + \frac{c_1{}^2}{\lambda + \gamma_1},$$

or say

$$64\delta_1 + 64\lambda^2 = \frac{256a_1{}^2}{4\lambda + 4\alpha_1} + \frac{256b_1{}^2}{4\lambda + 4\beta_1} + \frac{256c_1{}^2}{4\lambda + 4\gamma_1},$$

becomes

$$64\lambda^2 - \frac{4}{d^3}(l^2a' + m^2b' + n^2c') - \frac{\alpha^2}{d^4} = \frac{4l^2a'^2}{d^4\left(\tfrac{1}{2}\frac{\alpha}{d^2} - \frac{a'}{d} + 4\lambda\right)} + \frac{4m^2b'^2}{d^4\left(\tfrac{1}{2}\frac{\alpha}{d^2} - \frac{b'}{d} + 4\lambda\right)} + \frac{4n^2c'^2}{d^4\left(\tfrac{1}{2}\frac{\alpha}{d^2} - \frac{c'}{d} + 4\lambda\right)},$$

which is satisfied by $4\lambda = -\tfrac{1}{2}\frac{\alpha}{d^2}$. Writing therefore

$$4\lambda + \tfrac{1}{2}\frac{\alpha}{d^2} = -\frac{\theta}{d},$$

that is

$$8\lambda = -\frac{2\theta}{d} - \frac{\alpha}{d^2},$$

$$64\lambda^2 = \frac{4\theta^2}{d^2} + \frac{4\theta\alpha}{d^3} + \frac{\alpha^2}{d^4};$$

the equation is

$$\frac{4\theta^2}{d^3} + \frac{4\theta\alpha}{d^3} - \frac{4}{d^3}(l^2a' + m^2b' + n^2c') = \frac{4l^2a'^2}{d^4\left(-\frac{\theta}{d} - \frac{a'}{d}\right)} + \frac{4m^2b'^2}{d^4\left(-\frac{\theta}{d} - \frac{b'}{d}\right)} + \frac{4n^2c'^2}{d^4\left(-\frac{\theta}{d} - \frac{c'}{d}\right)},$$

viz. this is

$$l^2a' + m^2b' + n^2c' - \theta\alpha - \theta^2 d = \frac{l^2a'^2}{\theta + a'} + \frac{m^2b'^2}{\theta + b'} + \frac{n^2c'^2}{\theta + c'},$$

which is of course satisfied by $\theta = 0$. Moreover the derived equation

$$-\alpha - 2\theta d = -\frac{l^2a'^2}{(\theta+a')^2} - \frac{m^2b'^2}{(\theta+b')^2} - \frac{n^2c'^2}{(\theta+c')^2}$$

is also satisfied by $\theta = 0$, so that this is a double root. The equation in fact is

$$\{\theta^2 d + \theta\alpha - (l^2a' + m^2b' + n^2c')\}(\theta+a')(\theta+b')(\theta+c')$$
$$+\{l^2a'^2(\theta+b')(\theta+c') + m^2b'^2(\theta+c')(\theta+a') + n^2c'^2(\theta+a')(\theta+b')\} = 0,$$

or, expanding and dividing by θ^2, this is

$$\begin{aligned} &d(\theta+a')(\theta+b')(\theta+c') \\ &+ \alpha\{\theta^2 + \theta(a'+b'+c') + b'c' + c'a' + a'b'\} \\ &- \quad (l^2a' + m^2b' + n^2c')(\theta + a' + b' + c') \\ &+ \quad l^2a'^2 + m^2b'^2 + n^2c'^2 = 0, \end{aligned}$$

which gives the remaining three roots.

If $a' = b' = c'$ the equation is

$$(\theta + a' + \alpha)(\theta + a')^2 = 0.$$

I recall that we have

$$a,\ b,\ c,\ d,\ f = \frac{mn}{d},\ g = \frac{nl}{d},\ h = \frac{lm}{d}\quad l,\ m,\ n,$$

$$a' = a - \frac{l^2}{d},\quad b' = b - \frac{m^2}{d},\quad c' = c - \frac{n^2}{d},\quad \alpha = l^2 + m^2 + n^2,$$

so that the quadric surface is

$$d(a'x^2 + b'y^2 + c'z^2) + (lx + my + nz + dw)^2 = 0,$$

and that, α_1, β_1, γ_1, δ_1, a_1, b_1, c_1 denoting as before, the equation of the inverse surface (referred to a different origin) is

$$r^4 = 4w^2\{\alpha_1x^2 + \beta_1y^2 + \gamma_1z^2 + \delta_1w^2 + 2w(a_1x + b_1y + c_1z)\}.$$