494.

EXAMPLE OF A SPECIAL DISCRIMINANT.

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If we have a function $(a, \ldots \between x, y, z)^n$, where the coefficients $(a, \ldots)$ are such that the curve $(a, \ldots \between x, y, z)^n = 0$ has a node, and *à fortiori* if this curve has any number of nodes or cusps, the discriminant of the function (that is, the discriminant of the general function $(* \between x, y, z)^n$, substituting in such discriminant for the coefficients their values for the particular function in question) vanishes *identically*. But the particular function has nevertheless a *special discriminant*, viz. this is a function of the coefficients which, equated to zero, gives the condition that the curve may have (besides the nodes or cusps which it originally possesses) one more node; and the determination of this special discriminant (which, observe, is not deducible from the expression of the discriminant of the general function $(* \between x, y, z)^n$) is an interesting problem. I have, elsewhere, shown that if the curve in question $(a, \ldots \between x, y, z)^n = 0$ has δ nodes and κ cusps, then the degree of the special discriminant in regard to the coefficients a, &c., of the function is $= 3(n-1)^2 - 7\delta - 11\kappa$: and I propose to verify this in the case of a quartic curve with two cusps.

Consider the curve

$$6nx^2y^2 + 12rz^2xy + (4gx + 4iy + cz)\, z^3 = 0,$$

where $x = 0$ is the tangent at a cusp; $y = 0$ the tangent at a cusp; and $z = 0$ the line joining the two cusps.

For the special discriminant we have

$$3nxy^2 + 3ryz^2 + gz^3 = 0,$$

$$3nx^2y + 3rxz^2 + iz^3 = 0,$$

$$z\,\{6rxy + (3gx + 3iy + 4cz)\, z\} = 0\,;$$

the last of which may be replaced by the equation of the curve.

Assume $x = \lambda z$, $y = \mu z$, the first two equations give

$$3(n\lambda\mu + r)\mu + g = 0,$$

$$3(n\lambda\mu + r)\lambda + i = 0,$$

whence also

$$6n\lambda^2\mu^2 + 6r\lambda\mu + g\lambda + i\mu = 0,$$

and the equation of the curve gives

$$6n\lambda^2\mu^2 + 12r\lambda\mu + 4g\lambda + 4i\mu + c = 0,$$

whence eliminating $g\lambda + i\mu$ we find

$$18n\lambda^2\mu^2 + 12r\lambda\mu - c = 0.$$

Moreover the first two equations give

$$9(n\lambda\mu - r)^2\lambda\mu - ig = 0,$$

or putting $\lambda\mu = \theta$ we have

$$18n\theta^2 + 12r\theta - c = 0,$$

$$9(n\theta + r)^2\theta - ig = 0,$$

from which θ is to be eliminated.

The equations are

$$18n\theta^2 + 12r\theta - c = 0,$$

$$9n^2\theta^3 + 18nr\theta^2 + 9r^2\theta - ig = 0,$$

and thence

$$\begin{aligned} 18n^2\theta^3 + 36nr\theta^2 + 18r^2\theta - 2ig &= 0,\\ 18n^2\theta^3 + 12nr\theta^2 - cn\theta \quad &= 0,\\ 24nr\theta^2 + (18r^2 + cn)\theta - 2ig \quad &= 0,\\ 18nr\theta^2 \quad + \quad 12r^2\theta - cr \quad &= 0,\\ (6r^2 + 3cn)\theta - 6ig + 4cr \quad &= 0, \end{aligned}$$

$$\theta = \frac{6ig - 4cr}{6r^2 + 3cn} = \tfrac{2}{3}\,\frac{3ig - 2cr}{2r^2 + cn};$$

or substituting in $18n\theta^2 + 12r\theta - c = 0$, this is

$$8n(3ig - 2cr)^2 + 8r(3ig - 2cr)(2r^2 + cn) - c(2r^2 + cn)^2 = 0.$$

Hence, developing, the special discriminant is

$$\begin{aligned} \square = &- \ \ 1\,c^3n^2\\ &+ 12\,c^2nr^2\\ &- 72\,cginr\\ &- 36\,cr^4\\ &+ 72\,g^2i^2n\\ &+ 48\,gir^3, \end{aligned}$$

which is as it should be of the degree $5, = 3 \,.\, 3^2 - 11 \,.\, 2$.