491.

ON THE QUARTIC SURFACES $(*\between U, V, W)^2 = 0$.

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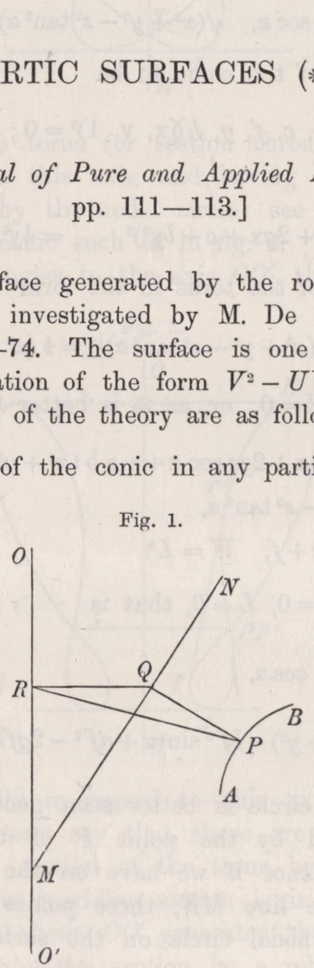
THE general Torus, or surface generated by the rotation of a conic about a fixed axis anywise situate, has been investigated by M. De La Gournerie, *Jour. de l'École Polyt.*, t. XXIII. (1863), pp. 1—74. The surface is one of the fourth order, having a nodal circle; and with its equation of the form $V^2 - UW = 0$, consequently of the form in question. The leading points of the theory are as follows:

Consider (fig. 1) the plane of the conic in any particular position thereof; let this

Fig. 1.



meet the axis of rotation OO' in the point M, and let the projection of OO' on the plane of the conic be MN. Take P any point of the conic; draw PQ in the plane of the conic, perpendicular to MN, and QR perpendicular to OO', and join PR: the point P of the conic describes a circle radius $RP, =\sqrt{(\overline{RQ}^2+\overline{QP}^2)}$. Hence if $\angle OMN=\alpha$, and if $MQ=\mathrm{x}$, $QP=\mathrm{y}$ are the coordinates in the plane of the conic of the point P; and if the coordinates x, y, z are measured, z upwards from M in the direction MO, and x, y in the plane at right angles to the axis OO': we have

$$z=\mathrm{x}\cos\alpha, \quad \sqrt{(x^2+y^2)}=\sqrt{(\mathrm{x}^2\sin^2\alpha+\mathrm{y}^2)};$$

or, what is the same thing,

$$\mathrm{x}=z\sec\alpha, \quad \mathrm{y}=\sqrt{(x^2+y^2-z^2\tan^2\alpha)}.$$

Hence the equation of the conic being $F(\mathrm{x}, \mathrm{y})=0$, that of the torus is

$$F\left(z\sec\alpha, \quad \sqrt{(x^2+y^2-z^2\tan^2\alpha)}\right)=0.$$

Thus taking the equation of the conic to be

$$(a, b, c, f, g, h\between\mathrm{x}, \mathrm{y}, 1)^2=0;$$

or, as this may be written,

$$(a\mathrm{x}^2+2g\mathrm{x}+c+b\mathrm{y}^2)^2 \qquad =4\mathrm{y}^2(h\mathrm{x}+f)^2,$$

we have at once the equation of the torus in the form

$$\{az^2\sec^2\alpha+2gz\sec\alpha+c+b(x^2+y^2-z^2\tan^2\alpha)\}^2=4(x^2+y^2-z^2\tan^2\alpha)(hz\sec\alpha+f)^2,$$

which is of the form $V^2-4UW=0$; or, as it is better to write it, $V^2-4UL^2=0$, where

$$\begin{aligned}V&=az^2\sec^2\alpha+2gz\sec\alpha+c+b(x^2+y^2-z^2\tan^2\alpha),\\ U&=x^2+y^2-z^2\tan^2\alpha,\\ L&=hz\sec\alpha+f, \quad W=L^2.\end{aligned}$$

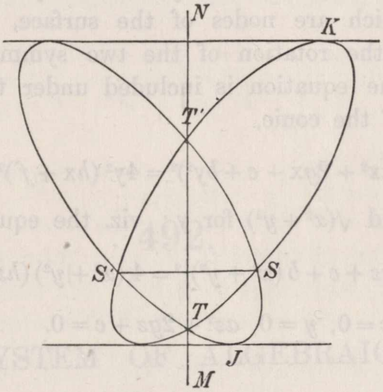
There is thus a nodal circle $V=0$, $L=0$, that is

$$z=-\frac{f}{h}\cos\alpha,$$

$$bh^2(x^2+y^2)-bf^2\sin^2\alpha+af^2-2gfh+ch^2=0.$$

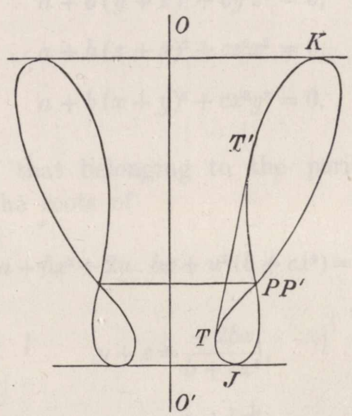
But the origin of this nodal circle is better seen geometrically. For observe that the radius of the circle described by the point P of the conic depends only on the *square* of the ordinate PQ: hence if we have on the conic two points S, S' situate symmetrically in regard to the line MN, these points S, S' will describe one and the same circle, which will be a nodal circle on the surface. And there is in fact one such pair of points S, S'; for (see fig. 2) considering in the plane of the conic the equal conic situate symmetrically thereto on the other side of the line MN, the two conics intersect in two points T, T' (real or imaginary) on the line MN, and in two other points S, S' (real or imaginary) situate symmetrically in regard to MN; we have thus the required pair of points which generate the nodal circle.

Fig. 2.



A meridian section of the torus (or section through the axis OO') is a quartic curve symmetrical in regard to this axis, and having two (real or imaginary) nodes the intersections of the plane by the nodal circle: see fig. 3, which shows the section for the surface generated by a conic such as in fig. 2. The quartic curve has 8 double tangents, 2 of them at right angles to the axis OO', the remaining 6 forming 3 pairs

Fig. 3.



of tangents situate symmetrically in regard to this axis; so that attending only to one tangent of each pair, we may say that there are 3 oblique bitangents: one of these is the line TT'; and the section of the torus by a plane through this line at right angles to the plane of the meridian section is in fact the two conics of fig. 2, either of which by its rotation about OO' generates the torus. But taking either of the other two oblique bitangents, the section by a plane through the bitangent at right angles to the meridian plane is in like manner a pair of conics situate symmetrically in regard to the bitangent, and such that either of them by its rotation about the axis OO' generates the torus. It thus appears that the same torus may be generated in three different ways by the rotation of a conic about the axis OO'.

In the particular case where the plane of the conic passes through the axis, the meridian section consists it is clear of two symmetrically situate conics, intersecting the axis in the points T, T', which are nodes of the surface, the surface having as before a nodal circle generated by the rotation of the two symmetrically situate intersections S, S' of the two conics. The equation is included under the foregoing form, but it is at once obtained from that of the conic,

$$(a\mathrm{x}^2 + 2g\mathrm{x} + c + b\mathrm{y}^2)^2 = 4\mathrm{y}^2 (h\mathrm{x} + f)^2,$$

by writing therein z for x and $\sqrt{(x^2 + y^2)}$ for y; viz. the equation of the torus here is

$$\{az^2 + 2gz + c + b(x^2 + y^2)\}^2 = 4(x^2 + y^2)(hz + f)^2,$$

and the two nodes thus are $x = 0$, $y = 0$, $az^2 + 2gz + c = 0$.