NOTES AND REFERENCES.

384. THE conclusion arrived at Nos. 27—30 that the transformed curve of the order $D+1$ depends upon $4D-6$ parameters is at variance with Riemann's theorem according to which the number of parameters is $3p-3$, (p Riemann $=D$ Cayley), $=3D-3$, and this last is the correct value. My erroneous conclusion is referred to in the preface to Clebsch and Gordan's *Theorie der Abel'schen Functionen* (Leipzig, 1866), "Unter den von Riemann behandelten Theilen der Theorie haben wir die Frage nach der Anzahl der Moduln einer Klasse von Abel'schen Functionen ausschliessen zu müssen geglaubt. Diese Frage ist durch die scharfsinnigen Betrachtungen des Herrn Cayley Gegenstand der Controverse geworden: sie ist überhaupt wohl zunächst nur durch tiefe algebraische Untersuchungen endgültig zu entscheiden, für deren Schwierigkeiten die gegenwärtig bekannten Methoden nicht mehr auszureichen scheinen." In the case D (or p) $=3$, my value is 10, Riemann's is 9: that the latter is correct was shown by a direct proof in the paper Brill, "Note bezüglich der Zahl der Moduln einer Klasse von algebraischen Gleichungen," *Math. Ann.*, t. I. (1869), pp. 401—406: the explanation of my error is given in the paper, Cayley, "Note on the Theory of Invariants," *Math. Ann.*, t. III. (1871), pp. 268—271.

400. The question here considered, viz., the expression of a binary sextic f in the form v^2-u^3, v and u a cubic and a quadric respectively, forms the basis of the very interesting investigations contained in the Memoir, Clebsch "Zur Theorie der binären Formen sechster Ordnung und zur Dreitheilung der hyperelliptischen Functionen," *Gött. Abh.*, t. XIV. (1869), pp. 1—59. Considering f as a given sextic it is remarked that the number of solutions, or what is the same thing the number of the functions u or v, although at first sight $=45$, is really $=40$; supposing that there is a given solution u, v, or that the sextic function is in the first instance given in the form v^2-u^3, then if any other solution is u', v', we have $v^2-u^3=v'^2-u'^3$, where v', u' are functions to be determined: there are in all 39 solutions, a set of 27 and a set of 12 solutions: viz. writing the equation in the form $(v+v')(v-v')=(u-u')(u-\epsilon u')(u-\epsilon^2 u')$, ϵ an imaginary cube root of unity, then either the $v+v'$ and the $v-v'$ contain each of them as a factor one of the quadric functions $u-u'$, $u-\epsilon u'$, $u-\epsilon^2 u'$ (which gives the set of 27 solutions) or else the $v+v'$ and the $v-v'$ are each of them the product of three linear factors of the quadric functions respectively (which gives the set of 12

solutions). It may be added that the 27 solutions form 9 groups of 3 each and that these 9 groups depend upon Hesse's equation of the order 9 for the determination of the inflexions of a cubic curve; and that the 12 solutions are determined by an equation of the order 12 which is the known resolvent of this order arising from Hesse's equation and is solved by means of a quartic equation with a quadrinvariant $=0$. As appears by the title of the memoir, the question is connected with that of the trisection of the hyperelliptic functions.

401, 403. On the subject of Pascal's theorem, see Veronese, "Nuove teoremi sull' hexagrammum mysticum," *R. Accad. dei Lincei* (1876—77), pp. 7—61; Miss Christine Ladd (Mrs Franklin), "The Pascal Hexagram," *Amer. Math. Jour.*, t. II. (1879), pp. 1—12, and Veronese, "Interprétations géométriques de la théorie des substitutions de n lettres, particulièrement pour $n=3$, 4, 5, en relation avec les groupes de l'Hexagramme Mystique," *Ann. di Matem.*, t. XI. 1882—83, pp. 93—236. See also Richmond, "A Symmetrical System of Equations of the Lines on a Cubic Surface which has a Conical Point," *Quart. Math. Jour.*, t. XXII. (1889), pp. 170—179, where the author discusses a perfectly symmetrical system of the lines on the cubic surface and deduces from them equations of the lines relating to a Pascal's hexagon: there are of course through the conical point 6 lines lying on a quadric cone and these by their intersections with the plane give the six points of the hexagon: the interest of the paper consists as well in the connexion established between the two theories as in the perfectly symmetrical form given to the equations.

406, 407. A correction was made by Halphen to the fundamental theorem of Chasles that the number of the conics $(X, 4Z)$ is $=\alpha\mu+\beta\nu$, he finds that a diminution is in some cases required, and thus that the general form is, Number of conics $(X, 4Z)=\alpha\mu+\beta\nu-\Gamma$: see Halphen's two Notes, *Comptes Rendus*, 4 Sep. and 13 Nov., 1876, t. LXXXIII. pp. 537 and 886, and his papers "Sur la théorie des caractéristiques pour les coniques," *Proc. Lond. Math. Soc.*, t. IX. (1877—1878), pp. 149—170, and "Sur les nombres des coniques qui dans un plan satisfont à cinq conditions projectives et indépendantes entre elles," *Proc. Lond. Math. Soc.*, t. X. (1878—79), pp. 76—87: also Zeuthen's paper "Sur la revision de la théorie des caractéristiques de M. Study," *Math. Ann.*, t. XXXVII. (1890), pp. 461—464, where the point is brought out very clearly and tersely.

The correction rests upon a more complete development of the notion of the line-pair-point, viz. this degenerate form of conic seems at first sight to depend upon three parameters only, the two parameters which determine the position of the coincident lines, and a third parameter which determines the position therein of the coincident points: but there is really a fourth parameter. {Compare herewith the point-pair, or indefinitely thin conic, which working with point-coordinates presents itself in the first instance as a coincident line-pair depending on two parameters only, but which really depends also on the two parameters which determine the position therein of the vertices.} As to the fourth parameter of the line-pair-point the most simple definition is a metrical one; taking the semiaxes of the degenerate conic to be a and b $(a=0, b=0)$ then we have two positive integers p and q prime to each other such that the ratio

$a^p : b^q$ is finite; and this being so the fractional or it may be integer number $p : q$ is the fourth parameter in question. But it is preferable to adopt Halphen's purely descriptive definition, viz. we consider a conic 1° in reference to three given points y, z, t on a given line, and take x, x' for the intersections of the conic with the line: we take $a = (y, z, t, x) - (y, z, t, x')$ for the difference of the corresponding anharmonic ratios of the three points with the points x, x' respectively; and 2° we consider the conic in reference to three given lines Y, Z, T through a given point and take X, X' for the tangents from the given point to the conic; we take $b = (Y, Z, T, X) - (Y, Z, T, X')$ for the difference of the corresponding anharmonic ratios of the three lines with the lines X, X' respectively $\left(\text{observe that these values are } a = \dfrac{x - x'}{z - x \,.\, z - x'} \div \dfrac{y - t}{z - y \,.\, z - t}\right.$, and $b = \left.\dfrac{X - X'}{Z - X \,.\, Z - X'} \div \dfrac{Y - T}{Z - Y \,.\, Z - T}\right)$. Here when the conic is a line-pair-point, $x = x'$ and $X = X'$, where $a = 0$ and $b = 0$, but we have as before the integers p and q such that $a^p : b^q$ is finite, and we have thus the fourth parameter $p : q$.

Halphen's correction is now as follows, starting from the formula number of conics $(X, 4Z) = \alpha\mu + \beta\nu$, we may have among the $\alpha\mu + \beta\nu$ conics line-pair-points any one of which if we disregard altogether the fourth parameter is a conic satisfying the five conditions, but which unless the fourth parameter thereof has its proper value is an improper solution of the problem and as such it has to be rejected: if the number of such solutions is $= \Gamma$, then there is this number to be subtracted, and the formula becomes, Number of conics $(X, 4Z) = \alpha\mu + \beta\nu - \Gamma$.

It may be asked in what way the fourth parameter comes into the question at all: as an illustration suppose that a, b denoting the semiaxes of a conic, or else the above mentioned descriptively defined quantities, then p, q, k denoting given quantities (p and q positive integers prime to each other) the condition X may be that the conic shall be such that $a^p \div b^q = k$; this implies $a^p : b^q$ finite, and hence clearly if the system of conics $(X, 4Z)$ contains line-pair-points, no such line-pair-point can be a proper solution unless this relation $a^p \div b^q = k$ is satisfied.

412. Zeuthen's Memoir of 1876 presently referred to contains applications to the theory of Cubic Surfaces, the numerical results given in the table p. 539 agree for the most part with those of the Memoir 412, see p. 363, but for the surfaces III, VI, IX and XII discrepancies occur in the values of r' and h' relating to the spinode develope. As to this observe that Zeuthen's h', or say $\bar{h}'$ includes actual as well as apparent double planes, and we have $r' = c'^2 - c' - 2\bar{h}' - 3\beta'$, my h' relates to apparent double planes only, but as I assume that there are no actual double planes the formula is $r' = c'^2 - c' - 2h' - 3\beta'$, and as the values of c' and β' agree we have in fact in each of the four cases $r' + 2h'$ (Cayley) $= r' + 2h'$ (Zeuthen). The values found are

		III	VI	IX	XII		III	VI	IX	XII
Cayley	n'	72	24	12	6	r'	42	24	32	9
Zeuthen	n'	84	30	24	7	r'	18	12	8	7

and assuming the correctness of Zeuthen's values it would seem to follow that the four forms of surface have

12, 6, 12, 1

actual double planes respectively.

413. In the equation No. 36, $\Omega = AP + BQ + CR + .. = 0$, it is implicitly assumed that the number of terms P, Q, R,.. is finite, viz. the implied theorem is that any given k-fold relation whatever (k of course a finite number) there is always a *finite* number of functions P, Q, R,... such that every onefold relation included in the k-fold relation is of the form in question Ω, $= AP + BQ + CR + \ldots$, $= 0$: this seems self-evident enough, but I never succeeded in finding a proof: a proof of the theorem has however been obtained by Hilbert, see his papers "Zur Theorie der algebraischen Gebilden (Erste Note)," *Gött. Nachr.* No. 16, (1888), pp. 450—457.

411, 415, 416. The first and second of these papers precede in date Zeuthen's Memoir of 1871 referred to in 416, but I ought in that paper to have referred also to his later Memoir, "Revision et extension des formules numériques de la théorie des surfaces réciproques," *Math. Ann.* t. x. (1876), pp. 446—546. I compare the notations as follows, viz. for the unaccented letters we have

Cayley.	Zeuthen.
$n, a, \delta, \kappa, \rho, \sigma$	$n, a, \delta, \kappa, \rho, \sigma$
b, q, k, t, γ	$b, q, \bar{k}, t, \gamma$; s
$c, r, h, \beta, \theta, \omega$	$c, r, \bar{h}, \beta$; m
j, χ	j, χ
C, B	B, U, O
f, i	f, i, d, g, e
23 letters in all.	27 letters in all.

Here for Zeuthen's k, h, I have written $\bar{k}$, $\bar{h}$, viz. these numbers represent the Plückerian equivalents of the number of double points for the nodal and cuspidal curves respectively. Zeuthen considers also the general node, say $\mathfrak{C}(\mu, \nu, y + \eta, z + \zeta, u, v)$, see 416, this includes the cnicnode C and off-point ω, and accordingly he includes under it and takes no special notice of these singularities, but it does not properly include, and he takes special notice of, the binode B; it does not extend to the case where the tangent cone breaks up into cones each or any of them more than once repeated, and accordingly not to the case of a unode U where the tangent cone is a pair of coincident planes. He introduces this singularity, and also the singularity of the osculating point O which is understood rather more easily by means of the reciprocal singularity of the osculating plane O', this is a tangent plane meeting the surface in a curve having the point of contact for a triple point; and he disregards my unexplained singularity θ. The letters s, m do not denote singularities; s is the class of the envelope of the osculating planes of the nodal curve, m the class of the envelope of the osculating planes of the cuspidal curve. Finally d denotes the number of stationary points (cusps) of the nodal curve, exclusive of the points γ which lie on the cuspidal curve; and g and e denote, g the number of ordinary actual double points of the cuspidal curve, e the number of stationary points (cusps) of the same curve, exclusive of the points β which lie on the nodal curve.

Moreover with Zeuthen, the nodal curve has

$$3t+f+3O'+\Sigma' \text{ double points}$$

($\bar{k}=k+3t+f+3O'+\Sigma'$, if k denotes, as with me, the number of apparent double points of the curve), and it has

$$\gamma+d+\Sigma' \text{ stationary points.}$$

The cuspidal curve has

$$g+6\chi'+12B'+U'+4O'+\Sigma+\Sigma' \text{ double points}$$

($\bar{h}=h+g+6\chi'+12B'+U'+4O'+\Sigma+\Sigma'$, if h denotes, as with me, the number of apparent double points of the curve), and it has

$$\beta+e+2O' \text{ stationary points}$$

and the nodal and cuspidal curves intersect in

$$3\beta+2\gamma+i+12O'+\Sigma+\Sigma' \text{ points;}$$

where I have written Σ and Σ' to denote sums (different in the different equations) determined by Zeuthen, and depending on the singularities $\mathfrak{C}$ and $\mathfrak{C}'$ respectively.

For comparison of my formulæ with Zeuthen's it is thus proper in my formulæ to write $C=0$, $\omega=0$, $\theta=0$ (but in the first instance I retain θ) and in his formulæ to write $U=0$, $O=0$, $d=0$, $g=0$, $e=0$, $\Sigma=0$, $\Sigma'=0$. Doing this the last mentioned formulæ give as with me $3t+f$ double points and γ stationary points for the nodal curve, but they give for the cuspidal curve $6\chi'+12\beta'$ (instead of 0) double points and β stationary points; and the two curves intersect (as with me) in $3\beta+2\gamma+i$ points. There is a real discrepancy in the number $6\chi'+12\beta'$ of double points on the cuspidal curve.

I compare his $(6+26+1=)\,33$ relations:

(1) $a=a'$. $d=d'$.

$f=f'$. $g=g'$.

$i=i'$. $h=h'$.

(6) $n(n-1)=a+2b+3c$.

(7) $a(a-1)=n+2\delta'+3\kappa'$.

(8) $c-\kappa'=3(n-a)$.

(9) $b(b-1)=q+2\bar{k}+3\gamma+3d+\Sigma'$.

(10) $[3(b-q)=\gamma+d-s+\Sigma'$, determines $s]$.

(11) $c(c-1)=r+2\bar{h}+3\beta+6O'+3e$.

(12) $[3(c-r)=\beta+e-m+2O'+\Sigma'$, determines $m]$.

(13) $a(n-2)=\kappa-B+\rho+2\sigma+\Sigma$.

(14) $b(n-2)=\rho+2\beta+3\gamma+3t+9O'+\Sigma$.

(15) $c(n-2)=2\sigma+4\beta+\gamma+8\chi'+16B'+12O'+\Sigma$.

(16) $a(n-2)(n-3)=2(\delta-3U)+3(ac-3\sigma-\chi)+2(ab-2\rho-j)$.

(17) $b(n-2)(n-3)=4(\bar{k}-3t-f)+(ab-2\rho-j)+3(bc-3\beta-2\gamma-i)+39O'+\Sigma+\Sigma'$.

(18) $c(n-2)(n-3)=6(\bar{h}-6\chi'-12B'-U'-4O'-g)+(ac-3\sigma-\chi)+2(bc-3\beta-2\gamma-i)$.

$$-30\,O'+\Sigma+\Sigma',$$

with the like reciprocal equations (6) to (18);

(19) $\sigma+m-r-\beta-4j'-3\chi'-14U'+\Sigma'$

$$=\sigma'+m'-r'-\beta'-4j-3\chi-14U+\Sigma.$$

where $\bar{k}=$

$\bar{h}=$

and my $(3+22+1=)\ 26$ relations as follows:

(1) $a=a'$.

(2) $f=f'$.

(3) $i=i'$.

(4) $a = n(n-1)-2b-3c$.

(5) $\kappa'=3n(n-2)-6b-8c$.

(6) $\delta'=\frac{1}{2}n(n-2)(n^2-9)-\&c.$

(A) (13) $q=b^2-b-2k-2f-3\gamma-6t$.

(B) (14) $r=c^2-c-2h-3\beta$.

(C) (7) $a(n-2)=k-B+\rho+2\sigma+3\omega$.

(D) (8) $b(n-2)=\rho+2\beta+3\gamma+3t$.

(E) (9) $c(n-2)=2\sigma+4\beta+\gamma+\theta+\omega$.

(F) (10) $a(n-2)(n-3)=2(\delta-C-3\omega)+3(ac-3\sigma-\chi-3\omega)+2(ab-2\rho-j\quad)$.

(G) (11) $b(n-2)(n-3)=\quad 4k\quad+\quad(ab-2\rho-j\quad)+3(bc-2\beta-2\gamma-i)$.

(H) (12) $c(n-2)(n-3)=\quad 6h\quad+\quad(ac-3\sigma-\chi-3\omega)+2(bc-3\beta-2\gamma-i)$,

with the like reciprocal equations (4) to (14);

(I) (26) $2(n-1)(n-2)(n-3)-12(n-3)(b+c)+6q+6r+24t+42\beta+30\gamma-\frac{3}{2}\theta$.
$=2(n'-1)(n'-2)(n'-3)-12(n'-3)(b'+c')+6q'+6r'+24t'+42\beta'+30\gamma'-\frac{3}{2}\theta'$.

$k+3f+3t+3O'+\Sigma$.

$h+g+6\chi'+12B'+U'+4O'+\Sigma+\Sigma'$.

Substituting for $\bar{k}$, $\bar{h}$ their values we have instead of (A), (B), (C), (D) the equations

$$(A')\quad b^2 - b \qquad = q + 2k + 2f + 6t + 6O' + 3\gamma + 3\delta + \Sigma + \Sigma'.$$

$$(B')\quad c^2 - c \qquad = r + 2h + 2g + 12\chi' + 24B' + 2U' + 14O' + 3\beta + 3e + \Sigma + \Sigma'.$$

$$(G')\quad b(n-2)(n-3) = 4k + 27O' + (ab - 2\rho - j) + 3(bc - 3\beta - 2\gamma - i) + \Sigma + \Sigma'.$$

$$(H')\quad c(n-2)(n-3) = 6h - 30O' + (ac - 3\sigma - \chi) + 2(bc - 3\beta - 2\gamma - i) + \Sigma + \Sigma'.$$

Writing as before $C = 0$, $\omega = 0$; $U = 0$, $O = 0$, $d = 0$, $g = 0$, $e = 0$, and neglecting the terms in Σ, Σ', the two equations (E) become

$$\text{Zeuthen} \qquad c(n-2) = 2\sigma + 4\beta + \gamma + 8\chi' + 16B',$$

$$\text{Cayley} \qquad c(n-2) = 2\sigma + 4\beta + \gamma + \theta,$$

which can be made to agree by writing $\theta = 8\chi' + 16B'$. But we have

$$\text{Zeuthen}\ (B') \quad c^2 - c \quad = r + 2h + 3\beta + 12\chi' + 24B',$$

$$\text{Cayley}\ (B\ ) \quad c^2 - c \quad = r + 2h + 3\beta,$$

values which differ by the terms $12\chi' + 24B'$, or if θ has the value just written down, the term $\frac{3}{2}\theta$.

I refrain from a comparison of the two equations (I.), and of the expressions for the deficiency given by these two equations respectively—but I notice here the expression for the deficiency obtained by Zeuthen in the last section (XIV.) of his Memoir, viz. this is

$$\begin{aligned} 24(D+1) = \ & c' - 12a + 24n + \beta + 3r - 15c + 2\sigma + 6\chi + 12\chi' + 6g + 9e \\ & + 8B + 24B' + 18U + 6U' + 6O' \\ & + \Sigma(3\nu + 3z + 8\eta + 13\zeta) + \Sigma'(6\zeta). \end{aligned}$$

The problem is a very difficult one, and it cannot be held that as yet a complete solution has been obtained. Take in plane geometry the question of reciprocal curves: here, using throughout point-coordinates, we start with a curve represented by the general equation $(x, y, z)^n = 0$, such a curve has only isolated singularities, viz. the line-singularities of the inflexion and the double tangent, we know the expression in point-coordinates of any such singularity (inflexion or double tangent as the case may be), viz. we can at once write down the equation of a curve of the order n having a given stationary tangent and point of contact therewith, or a given double tangent and two points of contact therewith. Returning to the general curve $(x, y, z)^n = 0$, we know that the reciprocal curve has other isolated singularities, viz. the point-singularities which correspond to these, the double point (or node) and the stationary point (or cusp), and we know the expression of any such singularity (node or cusp as the case may be), viz. we can at once write down the equation of a curve of the order n having at a given point a node with given tangents, or a cusp with given tangent. And then starting afresh with a curve of the order n having a node or a cusp we obtain the effect thereof as regards the line-singularities of the inflexion and the double tangent. We are thus led to consider as ordinary singularities in the theory the above-mentioned four singularities of the inflexion, the double tangent, the node and the cusp: and we know further that any other singularity whatever of a plane curve is compounded in a definite manner of a certain number of some or all of these singularities.

But in the theory of surfaces, starting in like manner with the general equation $(x, y, z, w)^n = 0$, such a surface has torse-singularities, the node-couple torse, and the spinode-torse; each of these is in general an indecomposable torse of a certain kind (but there is the new cause of complication that it may break into two or more separate torses), but we do not know the analytical expression of these singularities, nor consequently the analytical expression of the curve-singularities which correspond to them, the nodal curve and the cuspidal curve. Thus if we attempt to start with a surface $(x, y, z, w)^n = 0$ having a nodal curve, we can indeed write down the equation in its most general form, viz. if the nodal curve has for its complete expression the k equations $P = 0$, $Q = 0$, $R = 0$, &c. (viz. if the curve is such that every surface whatever through the curve is of the form $\Omega, = AP + BQ + CR + \ldots, = 0$) then the most general equation of the surface having this curve for a nodal curve is $(A, B, C, \ldots \between P, Q, R, \ldots)^2 = 0$, but this form is far too complicated to be worked with; and if for simplicity we take the nodal curve to be a complete intersection $P = 0$, $Q = 0$, and consequently the equation of the surface to be $(A, B, C \between P, Q)^2 = 0$, then it is by no means clear that we do not in this way introduce limitations extraneous to the general theory. The same difficulty applies of course, and with yet greater force, to the cuspidal curve; and even if we could deal separately with the cases of a surface having a given nodal curve, and a given cuspidal curve, this would in no wise solve the problem for the more general case of a surface having a given nodal curve and a given cuspidal curve. It is to be added that the general surface of the order n has no plane- or point-singularities, and thus that such singularities (which correspond most nearly to the singularities considered in the theory of reciprocal curves) present themselves in the theory of reciprocal surfaces as extraordinary singularities.

END OF VOL. VI.