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ON POLYZOMAL CURVES, OTHERWISE THE CURVES $\sqrt{U}+\sqrt{V}+\&c.=0$.

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If U, V, &c., are rational and integral functions $(*\between x, y, z)^r$, all of the same degree r, in regard to the coordinates (x, y, z), then $\sqrt{U}+\sqrt{V}+\&c.$ is a polyzome, and the curve $\sqrt{U}+\sqrt{V}+\&c.=0$ a polyzomal curve. Each of the curves $\sqrt{U}=0$, $\sqrt{V}=0$, &c. (or say the curves $U=0$, $V=0$, &c.) is, on account of its relation of circumscription to the curve $\sqrt{U}+\sqrt{V}+\&c.=0$, considered as a girdle thereto ($\zeta\hat{\omega}\mu\alpha$), and we have thence the term "zome" and the derived expressions "polyzome," "zomal," &c. If the number of the zomes $\sqrt{U}$, $\sqrt{V}$, &c. be $=\nu$, then we have a ν-zome, and corresponding thereto a ν-zomal curve; the curves $U=0$, $V=0$, &c., are the zomal curves or zomals thereof. The cases $\nu=1$, $\nu=2$, are not, for their own sake, worthy of consideration; it is in general assumed that ν is $=3$ at least. It is sometimes convenient to write the general equation in the form $\sqrt{lU}+\&c.=0$, where l, &c. are constants. The Memoir contains researches in regard to the general ν-zomal curve; the branches thereof, the order of the curve, its singularities, class, &c.; also in regard to the ν-zomal curve $\sqrt{l(\Theta+L\Phi)}+\&c.=0$, where the zomal curves $\Theta+L\Phi=0$, all pass through the points of intersection of the same two curves $\Theta=0$, $\Phi=0$ of the orders r and $r-s$ respectively; included herein we have the theory of the depression of order as arising from the ideal factor or factors of a branch or branches. A general theorem is given of "the decomposition of a tetrazomal curve," viz. if the equation of the curve be $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}+\sqrt{pT}=0$; then if U, V, W, T are in involution, that is, connected by an identical equation $\mathrm{a}U+\mathrm{b}V+\mathrm{c}W+\mathrm{d}T=0$, and if l, m, n, p, satisfy the condition $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0$, the tetrazomal curve breaks up into two trizomal curves, each expressible by means of any three of the four functions U, V, W, T; for example, in the form $\sqrt{l'U}+\sqrt{m'V}+\sqrt{p'T}=0$. If, in this theorem, we take $p=0$, then the original curve is the trizomal $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$, T is any function $=-\frac{1}{\mathrm{d}}(\mathrm{a}U+\mathrm{b}V+\mathrm{c}W)$, where, considering l, m, n as given, a, b, c are quantities subject only to the condition $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$, and we have the theorem of "the variable zomal of a trizomal curve," viz. the equation of the trizomal $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$, may be expressed by means of any two of the three functions U, V, W, and of a function T determined as above, for example in the form $\sqrt{l'U}+\sqrt{m'V}+\sqrt{n'T}=0$; whence also it may be expressed in terms of three new functions T, determined as above. This theorem, which occupies a prominent position in the whole theory, was suggested to me by Mr Casey's theorem, presently referred to, for the construction of a bicircular quartic as the envelope of a variable circle.

In the ν-zomal curve $\sqrt{l(\Theta+L\Phi)}+\&\text{c.}=0$, if $\Theta=0$ be a conic, $\Phi=0$ a line, the zomals $\Theta+L\Phi=0$, &c. are conics passing through the same two points $\Theta=0$, $\Phi=0$, and there is no real loss of generality in taking these to be the circular points at infinity—that is, in taking the conics to be circles. Doing this, and using a special notation $\mathfrak{A}^\circ=0$ for the equation of a circle having its centre at a given point A, and similarly $\mathfrak{A}=0$ for the equation of an evanescent circle, or say of the point A, we have the ν-zomal curve $\sqrt{l\mathfrak{A}^\circ}+\&\text{c.}=0$, and the more special form $\sqrt{l\mathfrak{A}}+\&\text{c.}=0$. As regards the last-mentioned curve, $\sqrt{l\mathfrak{A}}+\&\text{c.}=0$, the point A to which the equation $\mathfrak{A}=0$ belongs, is a focus of the curve, viz. in the case $\nu=3$, it is an ordinary focus, and in the case $\nu>3$, it is a special kind of focus, which, if the term were required, might be called a foco-focus; the Memoir contains an explanation of the general theory of the foci of plane curves. For $\nu=3$, the equation $\sqrt{l\mathfrak{A}}+\sqrt{m\mathfrak{B}}+\sqrt{n\mathfrak{C}}=0$ is really equivalent to the apparently more general form $\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}=0$. In fact, this last is in general a bicircular quartic, and, in regard to it, the before-mentioned theorem of the variable zomal becomes Mr Casey's theorem, that "the bicircular quartic (and, as a particular case thereof, the circular cubic) is the envelope of a variable circle having its centre on a given conic and cutting at right angles a given circle." This theorem is a sufficient basis for the complete theory of the trizomal curve $\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}=0$; and it is thereby very easily seen that the curve $\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}=0$ can be represented by an equation $\sqrt{l'\mathfrak{A}'}+\sqrt{m'\mathfrak{B}'}+\sqrt{n'\mathfrak{C}'}=0$. But for $\nu>3$ this is not so, and the curve $\sqrt{l\mathfrak{A}}+\&\text{c.}=0$ is only a particular form of the curve $\sqrt{l\mathfrak{A}^\circ}+\&\text{c.}=0$; and the discussion of this general form is scarcely more difficult than that of the special form $\sqrt{l\mathfrak{A}}+\&\text{c.}=0$, included therein. The investigations in relation to the theory of foci, and in particular to that of the foci of the circular cubic and bicircular quartic, precede in the Memoir the theories of the trizomal curve $\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}=0$, and the tetrazomal curve $\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}+\sqrt{p\mathfrak{D}^\circ}=0$, to which the concluding portions relate. I have accordingly divided the Memoir into four parts, viz. these are—Part I., On Polyzomal Curves in general; Part II., Subsidiary

Investigations; Part III., On the Theory of Foci; and Part IV., On the Trizomal and Tetrazomal Curves where the zomals are circles. There is, however, some necessary intermixture of the theories treated of, and the arrangement will appear more in detail from the headings of the several articles. The paragraphs are numbered continuously through the Memoir. There are four Annexes, relating to questions which it seemed to me more convenient to treat of thus separately.

It is right that I should explain the very great extent to which, in the composition of the present Memoir, I am indebted to Mr Casey's researches. His Paper "On the Equations and Properties (1) of the System of Circles touching three circles in a plane; (2) of the System of Spheres touching four spheres in space; (3) of the System of Circles touching three circles on a sphere; (4) on the System of Conics inscribed in a conic and touching three inscribed conics in a plane," was read to the Royal Irish Academy, April 9, 1866, and is published in their "Proceedings." The fundamental theorem for the equation of the pairs of circles touching three given circles was, previous to the publication of the paper, mentioned to me by Dr Salmon, and I communicated it to Professor Cremona, suggesting to him the problem solved in his letter of March 3, 1866, as mentioned in my paper, "Investigations in connexion with Casey's Equation," *Quarterly Math. Journ.* vol. VIII. 1867, pp. 334—341, [395], and as also appears, Annex No. IV of the present Memoir.

In connexion with this theorem, I communicated to Mr Casey, in March or April 1867, the theorem No. 164 of the present Memoir, that for any three given circles, centres A, B, C, the equation $\overline{BC}\sqrt{\mathfrak{A}^\circ} + \overline{CA}\sqrt{\mathfrak{B}^\circ} + \overline{AB}\sqrt{\mathfrak{C}^\circ} = 0$ (where $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, denote the mutual distances of the points A, B, C) belongs to a Cartesian. Mr Casey, in a letter to me dated 30th April, 1867, informed me of his own mode of viewing the question as follows:—"The general equation of the second order $(a, b, c, f, g, h\between\alpha, \beta, \gamma)^2 = 0$, where α, β, γ are circles, is a bicircular quartic. If we take the equation $(a, b, c, f, g, h\between\lambda, \mu, \nu)^2 = 0$ in tangential coordinates (that is, when λ, μ, ν are perpendiculars let fall from the centres of α, β, γ on any line), it denotes a conic; denoting this conic by F, and the circle which cuts α, β, γ orthogonally by J, I proved that, if a variable circle moves with its centre on F, and if it cuts J orthogonally, its envelope will be the bicircular quartic whose equation is that written down above;" and among other consequences, he mentions that the foci of F are the double foci of the quartic, and the points in which J cuts F single foci of the quartic, and also the theorem which I had sent him as to the Cartesian, and he refers to his Memoir on Bicircular Quartics as then nearly finished. An Abstract of the Memoir as read before the Royal Irish Academy, 10th February, 1867, and published in their *Proceedings*, pp. 44, 45, contains the theorems mentioned in the letter of 30th April, and some other theorems. It is not necessary that I should particularly explain in what manner the present Memoir has been, in the course of writing it, added to or altered in consequence of the information which I have thus had of Mr Casey's researches; it is enough to say that I have freely availed myself of such information, and that there is no question as to Mr Casey's priority in anything which there may be in common in his memoir on Bicircular Quartics and in the present Memoir.

PART I. (Nos. 1 to 55).—ON POLYZOMAL CURVES IN GENERAL.

Article Nos. 1 to 4. *Definition and Preliminary Remarks.*

1. As already mentioned, U, V, &c. denote rational and integral functions $(*\between x, y, z)^r$, all of the same degree r in the coordinates (x, y, z), and the equation

$$\sqrt{U} + \sqrt{V} + \&c. = 0$$

then belongs to a polyzomal curve, viz., if the number of the zomes $\sqrt{U}$, $\sqrt{V}$, &c. is $= \nu$, then we have a ν-zomal curve. The radicals, or any of them, may contain rational factors, or be of the form $P\sqrt{Q}$; but in speaking of the curve as a ν-zomal, it is assumed that any two terms, such as $P\sqrt{Q} + P'\sqrt{Q}$, involving the same radical $\sqrt{Q}$, are united into a single term, so that the number of distinct radicals is always $= \nu$; in particular (r being even), it is assumed that there is only one rational term P. But the ordinary case, and that which is almost exclusively attended to, is that in which the radicals $\sqrt{U}$, $\sqrt{V}$, &c. are distinct irreducible radicals without rational factors.

2. The curves $U = V = 0$, &c. are said to be the zomal curves, or simply the zomals of the polyzomal curve $\sqrt{U} + \sqrt{V} + \&c. = 0$; more strictly, the term zomal would be applied to the functions U, V, &c. It is to be noticed, that although the form $\sqrt{U} + \sqrt{V} + \&c. = 0$ is equally general with the form $\sqrt{lU} + \sqrt{mV} + \&c. = 0$ (in fact, in the former case, the functions U, V, &c. are considered as implicitly containing the constant factors l, m, &c., which are expressed in the latter case), yet it is frequently convenient to express these factors, and thus write the equation in the form $\sqrt{lU} + \sqrt{mV} + \&c.$ For instance, in speaking of any given curves $U = 0$, $V = 0$, &c., we are apt, disregarding the constant factors which they may involve, to consider U, V, &c. as given functions; but in this case the general equation of the polyzomal with the zomals $U = 0$, $V = 0$, &c., is of course $\sqrt{lU} + \sqrt{mV} + \&c. = 0$.

3. Anticipating in regard to the cases $\nu = 1$, $\nu = 2$, the remark which will be presently made in regard to the ν-zomal, that $\sqrt{U} + \sqrt{V} + \&c. = 0$ is the curve represented by the rationalised form of this equation, the monozomal curve $\sqrt{U} = 0$ is merely the curve $U = 0$, viz., this is any curve whatever $U = 0$ of the order r; and similarly, the bizomal curve $\sqrt{U} + \sqrt{V} = 0$ is merely the curve $U - V = 0$, viz. this is any curve whatever $\Omega = 0$, of the order r; the zomal curves $U = 0$, $V = 0$, taken separately, are not curves standing in any special relation to the curve in question $\Omega = 0$, but $U = 0$ may be any curve whatever of the order r, and then $V = 0$ is a curve of the same order r, in involution with the two curves $\Omega = 0$, $U = 0$; we may, in fact, write the equation $\Omega = 0$ under the bizomal form $\sqrt{U} + \sqrt{\Omega + U} = 0$. In the case r even, we may, however, notice the bizomal curve $P + \sqrt{U} = 0$ (P a rational function of the degree $\frac{1}{2}r$); the rational equation is here $\Omega = U - P^2 = 0$, that is $U = \Omega + P^2$, viz., P is any curve whatever of the order $\frac{1}{2}r$, and $U = 0$ is a curve of the order r, touching the given curve $\Omega = 0$ at each of its $\frac{1}{2}r^2$ intersections with the curve $P = 0$. I further

remark that the order of the ν-zomal curve $\sqrt{V}+\&c.=0$ is $2^{\nu-2}r$; this is right in the case of the bizomal curve $\sqrt{U}+\sqrt{V}=0$, the order being $=r$, but it fails for the monozomal curve $\sqrt{U}=0$, the order being in this case r, instead of $\frac{1}{2}r$, as given by the formula. The two unimportant and somewhat exceptional cases $\nu=1$, $\nu=2$, are thus disposed of, and in all that follows (except in so far as this is in fact applicable to the cases just referred to), ν may be taken to be $=3$ at least.

4. It is to be throughout understood that by the curve $\sqrt{U}+\sqrt{V}+\&c.=0$ is meant the curve represented by the rationalised equation

$$\text{Norm}\,(\sqrt{U}+\sqrt{V}+\&c.)=0,$$

viz. the Norm is obtained by attributing to all but one of the zomes $\sqrt{U}$, $\sqrt{V}$, &c., each of the two signs $+$, $-$, and multiplying together the several resulting values of the polyzome; in the case of a ν-zomal curve, the number of factors is thus $=2^{\nu-1}r$ (whence, as each factor is of the degree $\frac{1}{2}r$, the order of the curve is $2^{\nu-1}\cdot\frac{1}{2}r$, $=2^{\nu-2}r$, as mentioned above). I expressly mention that, as regards the polyzomal curve, we are not in any wise concerned with the signs of the radicals, which signs are and remain essentially indeterminate; the equation $\sqrt{U}+\sqrt{V}+\&c.=0$, is a mere symbol for the rationalised equation, Norm $(\sqrt{U}+\sqrt{V}+\&c.)=0$.

Article Nos. 5 to 12. *The Branches of a Polyzomal Curve.*

5. But we may in a different point of view attend to the signs of the radicals; if for all values of the coordinates we take the symbol $\sqrt{\overline{\overline{\ }}}$, and consider $\sqrt{\overline{\overline{U}}}$, $\sqrt{\overline{\overline{V}}}$, &c. as signifying determinately, say the *positive* values of $\sqrt{U}$, $\sqrt{V}$, &c.; then each of the several equations $\pm\sqrt{\overline{\overline{U}}}\pm\sqrt{\overline{\overline{V}}}+\&c.=0$, or, fixing at pleasure one of the signs, suppose that prefixed to $\sqrt{\overline{\overline{U}}}$, then each of the several equations $\sqrt{\overline{\overline{U}}}\pm\sqrt{\overline{\overline{V}}}\pm\&c.=0$, will belong to a *branch* of the polyzomal curve: a ν-zomal curve has thus $2^{\nu-1}$ branches corresponding to the $2^{\nu-1}$ values respectively of the polyzome. The separation of the branches depends on the precise fixation of the significations of $\sqrt{\overline{\overline{U}}}$, $\sqrt{\overline{\overline{V}}}$, &c., and in regard hereto some further explanation is necessary.

6. When U is real and positive, $\sqrt{\overline{\overline{U}}}$ may be taken to be, in the ordinary sense, the positive value of $\sqrt{U}$, and so when U is real and negative, $\sqrt{\overline{\overline{U}}}$ may be taken to be $=i$ into the positive value of $\sqrt{-U}$; and the like as regards $\sqrt{\overline{\overline{V}}}$, &c. The functions U, V, &c. are assumed to be real functions of the coordinates; hence, for any real values of the coordinates, U, V, &c. are real positive or negative quantities, and the significations of $\sqrt{\overline{\overline{U}}}$, $\sqrt{\overline{\overline{V}}}$, &c. are completely determined.

7. But the coordinates may be imaginary. In this case the functions U, V, &c. will for any given values of the coordinates acquire each of them a determinate, in general imaginary, value. If for all real values whatever of α, β, we select once for

all one of the two opposite values of $\sqrt{\alpha+\beta i}$, *calling* it the *positive* value, and representing it by $\surd\overline{\alpha+\beta i}$, then, for any particular values of the coordinates, U being $=\alpha+\beta i$, the value of $\surd\overline{U}$ may be taken to be $=\surd\overline{\alpha+\beta i}$; and the like as regards $\surd\overline{V}$, &c. $\surd\overline{U}$, $\surd\overline{V}$, &c. have thus each of them a determinate signification for any values whatever, real or imaginary, of the coordinates. The coordinates of a given point on the curve $\sqrt{U}+\sqrt{V}+\&c.=0$, will in general satisfy only one of the equations $\surd\overline{U}\pm\surd\overline{V}\pm\&c.=0$; that is, the point will belong to one (but in general only one) of the $2^{\nu-1}$ branches of the curve; the entire series of points the coordinates of which satisfy any one of the $2^{\nu-1}$ equations, will constitute the branch corresponding to that equation.

8. The signification to be attached to the expression $\surd\overline{\alpha+\beta i}$ should agree with that previously attached to the like symbol in the case of a positive or negative real quantity; and it should, as far as possible, be subject to the condition of continuity, viz., as $\alpha+\beta i$ passes continuously to $\alpha'+\beta' i$, so $\sqrt{\alpha+\beta i}$ should pass continuously to $\sqrt{\alpha'+\beta' i}$; but (as is known) it is not possible to satisfy universally this condition of continuity; viz., if for facility of explanation we consider (α, β) as the coordinates of a point in a plane, and imagine this point to describe a closed curve surrounding the origin or point $(0, 0)$, then it is not possible so to define $\sqrt{\alpha+\beta i}$ that this quantity, varying continuously as the point moves along the curve, shall, when the point has made a complete circuit, resume its original value. The signification to be attached to $\sqrt{\alpha+\beta i}$ is thus in some measure arbitrary, and it would appear that the division of the curve into branches is affected by a corresponding arbitrariness, but this arbitrariness relates only to the imaginary branches of the curve: the notion of a real branch is perfectly definite.

9. It would seem that a branch may be impossible for any series whatever of points real or imaginary. Thus, in the bizomal curve $\sqrt{U}+\sqrt{V}=0$, the branch $\surd\overline{U}+\surd\overline{V}=0$ is impossible. In fact, for any point whatever, real or imaginary, of the curve, we have $U=V$, and therefore $\surd\overline{U}=\surd\overline{V}$; the point thus belongs to the other branch $\surd\overline{U}-\surd\overline{V}=0$, not to the branch $\surd\overline{U}+\surd\overline{V}=0$; the only points belonging to the last-mentioned branch are the isolated points for which simultaneously $\surd\overline{U}=0$, $\surd\overline{V}=0$; viz., the points of intersection of the two curves $U=0$, $V=0$.

10. It is not clear to me whether the case is the same in regard to the branch $\surd\overline{U}+\surd\overline{V}+\surd\overline{W}=0$ of a trizomal curve. In fact, for each point of the curve $\sqrt{U}+\sqrt{V}+\sqrt{W}=0$ we have $(U-V-W)^2=4VW$, and therefore, $U-V-W=\pm 2\surd\overline{V}\surd\overline{W}$; there may very well be points for which the sign is $+$; that is, points for which $U=V+W+2\surd\overline{V}\surd\overline{W}$, and for these points we have $\pm\surd\overline{U}=\surd\overline{V}+\surd\overline{W}$; for real values of the coordinates the sign on the left hand must be $+$ (for otherwise the two sides

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would have opposite signs), but there is no apparent reason, or at least no obviously apparent reason, why this should be so for imaginary values of the coordinates, and if the sign be in fact $-$, then the point will belong to the branch $\sqrt{U}+\sqrt{V}+\sqrt{W}=0$.

11. But the branch in question is clearly impossible for any series of real points; so that, leaving it an open question whether the epithet "impossible" is to be understood to mean impossible for any series of real points (that is, as a mere synonym of imaginary), or whether it is to mean impossible for any series of points, real or imaginary, whatever, I say that in a ν-zomal curve some of the branches are or may be impossible, and that there is at least one impossible branch, viz., the branch $\sqrt{U}+\sqrt{V}+\&c.=0$.

12. For the purpose of referring to any branch of a polyzomal curve it will be convenient to consider $\sqrt{U}$ as signifying determinately $+\sqrt{U}$, or else $-\sqrt{U}$; and the like as regards $\sqrt{V}$, &c., but without any identity or relation between the signs prefixed to the $\sqrt{U}$, $\sqrt{V}$, &c., respectively; the equation $\sqrt{U}+\sqrt{V}+\&c.=0$, so understood, will denote determinately some one (that is, any one at pleasure) of the equations $\sqrt{U}\pm\sqrt{V}\pm\&c.=0$, and it will thus be the equation of some one (that is, any one at pleasure) of the branches of the polyzomal curve — all risk of ambiguity which might otherwise exist will be removed if we speak either of the *curve* $\sqrt{U}+\sqrt{V}$, &c. $=0$, or else of the *branch* $\sqrt{U}+\sqrt{V}+\&c.=0$. Observe that by the foregoing convention, when only one branch is considered, we avoid the necessity of any employment of the sign $\pm$, or of the sign $-$; but when two or more branches are considered in connection with each other, it is necessary to employ the sign $-$ with one or more of the radicals $\sqrt{U}$, $\sqrt{V}$, &c.; thus in the trizomal curve $\sqrt{U}+\sqrt{V}+\sqrt{W}=0$, we may have to consider the branches $\sqrt{U}+\sqrt{V}+\sqrt{W}=0$, $\sqrt{U}+\sqrt{V}-\sqrt{W}=0$; viz., either of these equations apart from the other denotes any one branch at pleasure of the curve, but when the branch represented by the one equation is fixed, then the branch represented by the other equation is also fixed.

Article Nos. 13 to 17. *The Points common to Two Branches of a Polyzomal Curve.*

13. I consider the points which are situate simultaneously on two branches of the ν-zomal curve $\sqrt{U}+\sqrt{V}+\&c.=0$. The equations of the two branches may be taken to be

$$\sqrt{U}+\&c.+(\sqrt{W}+\&c.)=0,$$

$$\sqrt{U}+\&c.-(\sqrt{W}+\&c.)=0,$$

viz., fixing the significations of $\sqrt{U}$, $\sqrt{V}$, $\sqrt{W}$, &c. in such wise that in the equation of one branch these shall each of them have the sign $+$, we may take $\sqrt{U}$, &c. to be those radicals which, in the equation of the other branch, have the sign $+$, and

$\sqrt{W}$, &c. to be those radicals which have the sign $-$. The foregoing equations break up into the more simple equations

$$\sqrt{U} + \&c. = 0, \quad \sqrt{W} + \&c. = 0,$$

which are the equations of certain *branches* of the *curves* $\sqrt{U} + \&c. = 0$, and $\sqrt{W} + \&c. = 0$, respectively, and conversely each of the intersections of these two curves is a point situate simultaneously on some two branches of the original ν-zomal curve $\sqrt{U} + \sqrt{V} + \&c. = 0$. Hence, partitioning in any manner the ν-zome $\sqrt{U} + \sqrt{V} + \&c.$ into an α-zome, $\sqrt{U} = \&c.$ and a β-zome $\sqrt{W} + \&c.$ $(\alpha + \beta = \nu)$, and writing down the equations

$$\sqrt{U} + \&c. = 0, \quad \sqrt{W} + \&c. = 0$$

of an α-zomal curve and a β-zomal curve respectively, each of the intersections of these two curves is a point situate simultaneously on two branches of the ν-zomal curve; and the points situate simultaneously on two branches of the ν-zomal curve are the points of intersection of the several pairs of an α-zomal curve and a β-zomal curve, which can be formed by any bipartition of the ν-zome.

14. There are two cases to be considered:—First, when the parts are $1, \nu - 1$ ($\nu - 1$ is > 1, except in the case $\nu = 2$, which may be excluded from consideration), or say when the ν-zome is partitioned into a *zome* and *antizome*. Secondly, when the parts α, β, are each > 1 (this implies $\nu = 4$ at least), or say when the ν-zome is partitioned into a pair of *complementary parazomes*.

15. To fix the ideas, take the tetrazomal curve $\sqrt{U} + \sqrt{V} + \sqrt{W} + \sqrt{T} = 0$, and consider first a point for which $\sqrt{U} = 0$, $\sqrt{V} + \sqrt{W} + \sqrt{T} = 0$. The Norm is the product of $(2^3 =)$ 8 factors; selecting hereout the factors

$$\sqrt{U} + \sqrt{V} + \sqrt{W} + \sqrt{T},$$

$$\sqrt{U} - \sqrt{V} - \sqrt{W} - \sqrt{T},$$

let the product of these

$$= U - (\sqrt{V} + \sqrt{W} + \sqrt{T})^2$$

be called F, and the product of the remaining six factors be called G; the rationalised equation of the curve is therefore $FG = 0$. The derived equation is $GdF + FdG = 0$; at the point in question $\sqrt{U} = 0$, $\sqrt{V} + \sqrt{W} + \sqrt{T} = 0$; G and dG are each of them finite (that is, they neither vanish nor become infinite), but we have

$$F = 0,\ dF = dU - (\sqrt{V} + \sqrt{W} + \sqrt{T})(dV \div \sqrt{V} + dW \div \sqrt{W} + dT \div \sqrt{T}),\ = dU,$$

and the derived equation is thus $GdU = 0$, or simply $dU = 0$. It thus appears that the point in question is an ordinary point on the tetrazomal curve; and, further, that the tetrazomal curve is at this point touched by the zomal curve $U = 0$. And similarly, each of the points of intersection of the two curves $\sqrt{U} = 0$, $\sqrt{V} + \sqrt{W} + \sqrt{T} = 0$, is an ordinary point on the tetrazomal curve; and the tetrazomal curve is at each of these points touched by the zomal curve $U = 0$.

16. Consider, secondly, a point for which $\sqrt{U}+\sqrt{V}=0$, $\sqrt{W}+\sqrt{T}=0$; to form the Norm, taking in this case the two factors

$$\sqrt{U}+\sqrt{V}+\sqrt{W}+\sqrt{T},$$

$$\sqrt{U}+\sqrt{V}-\sqrt{W}-\sqrt{T},$$

let their product

$$=(\sqrt{U}+\sqrt{V})^2-(\sqrt{W}+\sqrt{T})^2$$

be called F, and the product of the remaining six factors be called G; the rationalised equation is $FG=0$, and the derived equation is $FdG+GdF=0$. At the point in question G and dG are each of them finite (that is, they neither vanish nor become infinite), but we have

$$F=0,\ dF=(\sqrt{U}+\sqrt{V})(dU\div\sqrt{U}+dV\div\sqrt{V})-(\sqrt{W}+\sqrt{T})(dW\div\sqrt{W}+dT\div\sqrt{T}),=0,$$

that is, the derived equation becomes identically $0=0$; the point in question is thus a singular point, and it is easy to see that it is in fact a node, or ordinary double point, on the tetrazomal curve. And similarly, each of the points of intersection of the two curves $\sqrt{U}+\sqrt{V}=0$, $\sqrt{W}+\sqrt{T}=0$ is a node on the tetrazomal curve.

17. The proofs in the foregoing two examples respectively are quite general, and we may, in regard to a ν-zomal curve, enunciate the results as follows, viz., in a ν-zomal curve, the points situate simultaneously on two branches are either the intersections of a zomal curve and its antizomal curve, or else they are the intersections of a pair of complementary parazomal curves. In the former case, the points in question are ordinary points on the ν-zomal, but they are points of contact of the ν-zomal with the zomal; it may be added, that the intersections of the zomal and antizomal, each reckoned twice, are *all* the intersections of the ν-zomal and zomal. In the latter case, the points in question are nodes of the ν-zomal; it may be added, that the ν-zomal has not, *in general*, any nodes other than the points which are thus the intersections of a pair of complementary parazomals, and that it has not *in general* any cusps.

Article Nos. 18 to 21. *Singularities of a ν-zomal Curve.*

18. It has been already shown that the order of the ν-zomal curve is $=2^{\nu-2}r$. Considering the case where ν is $=3$ at least, the curve, as we have just seen, has contacts with each of the zomal curves, and it has also nodes. I proceed to determine the number of these contacts and nodes respectively.

19. Consider first the zomal curve $U=0$, and its antizomal $\sqrt{V}+\sqrt{W}+\&c.=0$, these are curves of the orders r and $2^{\nu-3}r$ respectively, and they intersect therefore in $2^{\nu-3}r^2$ points. Hence the ν-zomal touches the zomal in $2^{\nu-3}r^2$ points, and reckoning each of these twice, the number of intersections is $=2^{\nu-2}r^2$, viz., these are all the intersections of the ν-zomal with the zomal $U=0$. The number of contacts of the ν-zomal with the several zomals $U=0$, $V=0$, &c., is of course $=2^{\nu-3}r^2\nu$.

20. Considering next a pair of complementary parazomal curves, an α-zomal and a β-zomal respectively $(\alpha+\beta=\nu)$, these are of the orders $2^{\alpha-2}r$ and $2^{\beta-2}r$ respectively, and they intersect therefore in $2^{\alpha+\beta-4}r^2=2^{\nu-4}r^2$ points, nodes of the ν-zomal. This number is independent of the particular partition (α, β), and the ν-zomal has thus this same number, $2^{\nu-4}r^2$, of nodes in respect of each pair of complementary parazomals; hence the total number of nodes is $=2^{\nu-4}r^2$ into the number of pairs of complementary parazomals. For the partition (α, β) the number of pairs is $=[\nu]^\nu \div [\alpha]^\alpha [\beta]^\beta$, or when $\alpha=\beta$, which of course implies ν even, it is one-half of this; extending the summation from $\alpha=2$ to $\alpha=\nu-2$, each pair is obtained twice, and the number of pairs is thus $=\frac{1}{2}\Sigma\{[\nu]^\nu \div [\alpha]^\alpha [\beta]^\beta\}$; the sum extended from $\alpha=0$ to $\alpha=\nu$ is $(1+1)^\nu$, $=2^\nu$, but we thus include the terms 1, ν, ν, 1, which are together $=2\nu+2$, hence the correct value of the sum is $=2^\nu-2\nu-2$, and the number of pairs is the half of this $=2^{\nu-1}-\nu-1$. Hence the number of nodes of the ν-zomal curve is $=(2^{\nu-1}-\nu-1)\,2^{\nu-4}r^2$.

21. The ν-zomal is thus a curve of the order $2^{\nu-2}r$, with $(2^{\nu-1}-\nu-1)\,2^{\nu-4}r^2$ nodes, but without cusps; the class is therefore

$$=2^{\nu-3}r\,[(\nu+1)\,r-2],$$

and the deficiency is

$$=2^{\nu-4}r\,[(\nu+1)\,r-6]+1.$$

These are the general expressions, but even when the zomal curves $U=0$, $V=0$, &c., are given, then writing the equation of the ν-zomal under the form $\sqrt{lU}+\sqrt{mV}+$ &c. $=0$, the constants $l : m :$ &c., may be so determined as to give rise to nodes or cusps which do not occur in the general case; the formulæ will also undergo modification in the particular cases next referred to.

Article Nos. 22 to 27. *Special Case where all the Zomals have a Common Point or Points.*

22. Consider the case where the zomals $U=0$, $V=0$ have all of them any number, say k, of common intersections—these may be referred to simply as the common points. Each common point is a $2^{\nu-2}$-tuple point on the ν-zomal curve; it is on each zomal an ordinary point, and on each antizomal a $2^{\nu-3}$-tuple point, and on any α-zomal parazomal a $2^{\alpha-2}$-tuple point. Hence, considering first the intersections of any zomal with its antizomal, the common point reckons as $2^{\nu-3}$ intersections, and the k common points reckon as $2^{\nu-3}k$ intersections; the number of the remaining intersections is therefore $=2^{\nu-3}(r^2-k)$, and the zomal touches the ν-zomal in each of these points. The intersections of the zomal with the ν-zomal are the k-common points, each of them a $2^{\nu-2}$-tuple point on the ν-zomal, and therefore reckoning together as $2^{\nu-2}k$ intersections; and the $2^{\nu-3}(r^2-k)$ points of contact, each reckoning twice, and therefore together as $2^{\nu-2}(r^2-k)$ intersections $(2^{\nu-2}k+2^{\nu-2}(r^2-k)=2^{\nu-2}r^2,\ =r\,.\,2^{\nu-2}r)$; the total number of contacts with the zomals $U=0$, $V=0$, &c., is thus $=2^{\nu-3}(r^2-k)\,\nu$.

23. Secondly, considering any pair of complementary parazomals, an α-zomal and a β-zomal, each of the common points, being a $2^{\alpha-2}$-tuple point and a $2^{\beta-2}$-tuple point on the two curves respectively, counts as $2^{\alpha+\beta-4}$, $=2^{\nu-4}$ intersections, and the k common points count as $2^{\nu-4}k$ intersections; the number of the remaining intersections is therefore $=2^{\nu-4}(r^2-k)$, each of which is a node on the ν-zomal curve; and we have thus in all $2^{\nu-4}(2^{\nu-1}-\nu-1)(r^2-k)$ nodes.

24. There are, besides, the k common points, each of them a $2^{\nu-2}$-tuple point on the ν-zomal, and therefore each reckoning as $\frac{1}{2}2^{\nu-2}(2^{\nu-2}-1)$, $=2^{2\nu-5}-2^{\nu-3}$ double points, or together as $(2^{2\nu-5}-2^{\nu-3})k$ double points. Reserving the term node for the above-mentioned nodes or proper double points, and considering, therefore, the double points (dps.) as made up of the nodes and of the $2^{\nu-2}$-tuple points, the total number of dps. is thus

$$2^{\nu-4}(2^{\nu-1}-\nu-1)(r^2-k)+(2^{2\nu-5}-2^{\nu-3})k,$$

$$=2^{\nu-4}(2^{\nu-1}-\nu-1)\ r^2+\{(\nu+1)\ 2^{\nu-4}-2^{\nu-3}\}\ k;$$

or finally this is

$$=2^{\nu-4}\{(2^{\nu-1}-\nu-1)\ r^2+\ (\nu-1)\};$$

so that there is a gain $=2^{\nu-4}(\nu-1)k$ in the number of dps. arising from the k common points. There is, of course, in the class a diminution equal to twice this number, or $2^{\nu-3}(\nu-1)k$; and in the deficiency a diminution equal to this number, or $2^{\nu-4}(\nu-1)k$.

25. The zomal curves $U=0$, $V=0$, &c., may all of them pass through the same ν^2 points; we have then $k=r^2$, and the expression for the number of dps. is $=(2^{2\nu-5}-2^{\nu-3})r^2$, viz., this is $=\frac{1}{2}2^{\nu-2}(2^{\nu-2}-1)r^2$. But in this case the dps. are nothing else than the r^2 common points, each of them a $2^{\nu-2}$-tuple point, the ν-zomal curve in fact breaking up into a system of $2^{\nu-2}$ curves of the order r, each passing through the r^2 common points. This is easily verified, for if $\Theta=0$, $\Phi=0$ are some two curves of the order r, then, in the present case, the zomal curves are curves in involution with these curves; that is, they are curves of the form $l\Theta+l'\Phi=0$, $m\Theta+m'\Phi=0$, &c., and the equation of the ν-zomal curve is

$$\sqrt{l\Theta+l'\Phi}+\sqrt{m\Theta+m'\Phi}+\&c.=0.$$

The rationalised equation is obviously an equation of the degree $2^{\nu-2}$ in Θ, Φ, giving therefore a constant value for the ratio $\Theta:\Phi$; calling this q, or writing $\Theta=q\Phi$, we have

$$\sqrt{lq+l'}+\sqrt{mq+m'}+\&c.=0,$$

viz., the rationalised equation is an equation of the degree $2^{\nu-2}$ in q, and gives therefore $2^{\nu-2}$ values of q. And the ν-zomal curve thus breaks up into a system of $2^{\nu-2}$ curves each of the form $\Theta-q\Phi=0$, that is, each of them in involution with the curves $\Theta=0$, $\Phi=0$. The equation in q may have a multiple root or roots, and the system of curves so contain repetitions of the same curve or curves; an instance of this (in relation to the trizomal curve) will present itself in the sequel; but I do not at present stop to consider the question.

26. A more important case is when the zomal curves are each of them in involution with the same two given curves, one of them of the order r, the other of an inferior order. Let $\Theta=0$ be a curve of the order r, $\Phi=0$ a curve of an inferior order $r-s$; $L=0$, $M=0$, &c., curves of the order s; then the case in question is when the zomal curves are of the form $\Theta+L\Phi=0$, $\Theta+M\Phi=0$, &c., the equation of the ν-zomal is

$$\sqrt{l(\Theta+L\Phi)}+\sqrt{m(\Theta+M\Phi)}+\&\text{c.}=0,$$

where l, m, &c. are constants. This is the most convenient form for the equation, and by considering the functions L, M, &c. as containing implicitly the factors l^{-1}, m^{-1}, &c. respectively, we may take it to include the form $\sqrt{l\Theta+L\Phi}+\sqrt{m\Theta+M\Phi}+\&\text{c.}=0$, which last has the advantage of being immediately applicable to the case where any one or more of the constants l, m, &c. may be $=0$.

27. In the case now under consideration we have the $r(r-s)$ points of intersection of the curves $\Theta=0$, $\Phi=0$ as common points of all the zomals. Hence, putting in the foregoing formula $k=r(r-s)$, we have a ν-zomal curve of the order $2^{\nu-2}r$, having with each zomal $2^{\nu-2}rs$ contacts, or with all the zomals $2^{\nu-2}rs\nu$ contacts, having a node at each of the $2^{\nu-4}rs$ intersections (not being common points $\Theta=0$, $\Phi=0$) of each pair of complementary parazomals; that is, together $2^{\nu-4}(2^{\nu-1}-\nu-1)\,rs$ nodes, and having, besides, at each of the $r(r-s)$ common points, a $2^{\nu-2}$-tuple point, counting as $2^{2\nu-5}-2^{\nu-3}$ dps., together as $(2^{2\nu-5}-2^{\nu-3})\,r(r-s)$ dps.; whence, taking account of the nodes, the total number of dps. is $=2^{\nu-4}r[(2^{\nu-1}-2)\,r-(\nu-1)\,s]$.

Article Nos. 28 to 37. *Depression of Order of the ν-zomal Curve from the Ideal Factor of a Branch or Branches.*

28. In the case of the $r(r-s)$ common points as thus far considered, the order of the ν-zomal curve has remained throughout $=2^{\nu-2}r$, but the order admits of depression, viz., the constants l, m, &c., and those of the functions L, M, &c., may be such that the Norm contains the factor Φ^{ω}; the ν-zomal curve then contains as part of itself $(\Phi^{\omega}=0)$ the curve $\Phi=0$ taken ω times, and this being so, if we discard the factor in question, and consider the residual curve as being the ν-zomal, the order of the ν-zomal will be $=2^{\nu-2}r-\omega(r-s)$.

29. To explain how such a factor Φ^{ω} presents itself, consider the polyzome $\sqrt{l(\Theta+L\Phi)}+\&\text{c.}$, or, what is the same thing, $\sqrt{l}\sqrt{\Theta+L\Phi}+\&\text{c.}$, belonging to any particular branch of the curve, we may, it is clear, take $\sqrt{\Theta+L\Phi}$, &c. each in a fixed signification as equivalent to $\sqrt{\overline{\overline{\Theta+L\Phi}}}$, &c., respectively, and the particular branch will then be determined by means of the significations attached to $\sqrt{l}$, $\sqrt{m}$, &c. Expanding the several radicals, the polyzome is

$$\sqrt{l}\left\{\sqrt{\overline{\Theta}}+\tfrac{1}{2}L\frac{\Phi}{\sqrt{\overline{\Theta}}}-\tfrac{1}{8}L^2\frac{\Phi^2}{\Theta\sqrt{\overline{\Theta}}}+\&\text{c.},\right\}+\&\text{c.};$$

or, what is the same thing, it is

$$\sqrt{\Theta}\,(\sqrt{l}+\&c.)+\tfrac{1}{2}\,\frac{\Phi}{\sqrt{\Theta}}\,(L\sqrt{l}+\&c.)-\tfrac{1}{8}\,\frac{\Phi^2}{\Theta\sqrt{\Theta}}\,(L^2\sqrt{l}+\&c.)+\&c.$$

which expansion may contain the factor Φ, or a higher power of Φ. For instance, if we have $\sqrt{l}+\&c.=0$, the expansion will then contain the factor Φ; and if we also have $L\sqrt{l}+\&c.=0$ (observe this implies as many equations as there are asyzygetic terms in the whole series of functions L, M, &c.; thus, if L, M, &c., are each of them of the form $aP+bQ+cR$, with the same values of P, Q, R, but with different values of the coefficients a, b, c, then it implies the three equations $a\sqrt{l}+\&c.=0$, $b\sqrt{l}+\&c.=0$, $c\sqrt{l}+\&c.=0$; and so in other cases), if I say $L\sqrt{l}+\&c.$ be also $=0$, then the expansion will contain the factor Φ^2, and so on; the most general supposition being, that the expansion contains as factor a certain power Φ^α of Φ. Imagine each of the polyzomes expanded in this manner, and let certain of the expansions contain the factors Φ^α, Φ^β, &c., respectively. The produce of the expansions is identically equal to the product of the unexpanded polyzomes—that is, it is equal to the Norm; hence, if $\alpha+\beta+\&c.=\omega$, the Norm will contain the factor Φ^ω.

30. It has been mentioned that the form $\sqrt{l(\Theta+L\Phi)}$ is considered as including the form $\sqrt{l\Theta+L\Phi}$, that is, when $l=0$, the form $\sqrt{L\Phi}$. If in the equation of the ν-zomal curve there is any such term—for instance, if the equation be $\sqrt{L\Phi}+\sqrt{m(\Theta+M\Phi)}+\&c.=0$, the radical $\sqrt{L\Phi}$ contains the factor $\Phi^{\frac{1}{2}}$; but if L contains as factor an odd or an even power of Φ, then $\sqrt{L\Phi}$ will contain the factor Φ^α where α is either an integer, or an integer $+\frac{1}{2}$. Consider the polyzome $\sqrt{L\Phi}+\sqrt{m(\Theta+M\Phi)}+\&c.$, belonging to any particular branch of the curve; the radical $\sqrt{L\Phi}$ contains, as just mentioned, the factor Φ^α, and if the remaining terms $\sqrt{m(\Theta+M\Phi)}+\&c.$, are such that the expansion contains as factor the same or any higher power of Φ, then the expansion of the polyzome $\sqrt{L\Phi}+\sqrt{m(\Theta+M\Phi)}+\&c.$, belonging to the particular branch will contain the factor Φ^α; and similarly we may have branches containing the factors Φ^α, Φ^β, &c., whence, as before, if $\omega=\alpha+\beta+\&c.$, the Norm will contain the factor Φ^ω; the only difference is, that now α, β, &c., instead of being of necessity all integers, are each of them an integer, or an integer $+\frac{1}{2}$; of course, in the latter case the integer may be zero, or the index be $=\frac{1}{2}$. It is clear that ω must be an integer, and it is, in fact, easy to see that the fractional indices occur in pairs; for observe that α being fractional, the expansion of $\sqrt{m(\Theta+M\Phi)}+\&c.$, will contain not Φ^α, but a higher power, $\Phi^{\alpha+q}$, where $\alpha+q$ is an integer; whence *each* of the polyzomes $\sqrt{L\Phi}\pm(\sqrt{m(\Theta+M\Phi)}+\&c.)$ will contain the factor Φ^α.

31. Observe that in every case the factor Φ^α presents itself as a factor of the expansion of the polyzome corresponding to a particular branch of the curve; the polyzome itself does not contain the factor Φ^α, and we cannot in anywise say that the corresponding branch contains as factor the curve $\Phi^\alpha=0$; but we may, with great propriety of expression, say that *the branch ideally contains the curve* $\Phi^\alpha=0$; and this

being so, the general theorem is, that if we have branches ideally containing the curves $\Phi^\alpha = 0$, $\Phi^\beta = 0$, &c. respectively, then the ν-zomal curve contains not ideally but actually the factor $\Phi^\omega = 0$ $(\omega = \alpha + \beta + \text{\&c.})$, the order of the ν-zomal being thus reduced from $2^{\nu-2}r$ to $2^{\nu-2}r - \omega(r-s)$; and conversely, that any such reduction in the order of the ν-zomal arises from the factors $\Phi^\alpha = 0$, $\Phi^\beta = 0$, &c., ideally contained in the several branches of the ν-zomal.

32. It is worth while to explain the notion of an ideal factor somewhat more generally; an irrational function, taking the irrationalities thereof in a determinate manner, may be such that, as well the function itself as all its differential coefficients up to the order $\alpha - 1$, vanish when a certain parameter Φ contained in the function is put $= 0$; this is only saying, in other words, that the function expanded in ascending powers of Φ contains no power lower than Φ^α; and, in this case, we say that the irrational function contains *ideally* the factor Φ^α. The rationalised expression, or Norm, in virtue of the irrational function (taken determinately as above) thus ideally containing Φ^α, will actually contain the factor Φ^α; and if any other values of the irrational function contain respectively Φ^β, &c., then the Norm will contain the factor $\Phi^{\alpha+\beta+\text{\&c.}}$.

33. A branch ideally containing $\Phi^\alpha = 0$ may for shortness be called integral or fractional, according as the index α is an integer or a fraction; by what precedes the fractional branches present themselves in pairs. If for a moment we consider integral branches only, then if the ν-zomal contain $\Phi = 0$, this can happen in one way only, there must be some one branch ideally containing $\Phi = 0$; but if the ν-zomal contain $\Phi^2 = 0$, then this may happen in two ways,—either there is a single branch ideally containing $\Phi^2 = 0$, or else there are two branches, each of them ideally containing $\Phi = 0$. And generally, if the ν-zomal contain $\Phi^\omega = 0$, then forming any partition $\omega = \alpha + \beta + \text{\&c.}$ (the parts being integral), this may arise from there being branches ideally containing $\Phi^\alpha = 0$, $\Phi^\beta = 0$, &c. respectively. The like remarks apply to the case where we attend also to fractional branches,—thus, if the ν-zomal contain $\Phi = 0$, this may arise (not only, as above mentioned, from a branch ideally containing $\Phi = 0$, but also) from a pair of branches, each ideally containing $\Phi^{\frac{1}{2}} = 0$. And so in general, if the ν-zomal contain $\Phi^\omega = 0$, the partition $\omega = \alpha + \beta + \text{\&c.}$ is to be made with the parts integral or fractional ($= \frac{1}{2}$ or integer $+ \frac{1}{2}$ as above), but with the fractional terms in pairs; and then the factor $\Phi^\omega = 0$ may arise from branches ideally containing $\Phi^\alpha = 0$, $\Phi^\beta = 0$, &c. respectively.

34. Any zomal, antizomal, or parazomal of a ν-zomal curve, $\sqrt{l(\Theta + L\Phi)} + \text{\&c.} = 0$, is a polyzomal curve (including in the term a monozomal curve) of the same form as the ν-zomal; and may in like manner contain $\Phi = 0$, or more generally, $\Phi^\omega = 0$, viz., if $\omega = \alpha + \beta + \text{\&c.}$ be any partition of ω as above, this will be the case if the zomal, antizomal, or parazomal has branches ideally containing $\Phi^\alpha = 0$, $\Phi^\beta = 0$, &c. respectively. It is to be observed that if a zomal, antizomal, or parazomal contain $\Phi = 0$, or any higher power $\Phi^\omega = 0$, this does not in anywise imply that the zomal contains even $\Phi = 0$. But if (attending only to the most simple case) a zomal and its antizomal, or a pair of complementary parazomals, each contain $\Phi = 0$ inseparably (that

is, through a single branch ideally containing $\Phi=0$), then the ν-zomal will have two branches, each ideally containing $\Phi=0$, and it will thus contain $\Phi^2=0$. In fact, if in the zomal and antizomal, or in the complementary parazomals, the branches which ideally contain $\Phi=0$ are

$$\sqrt{l(\Theta+L\Phi)}+\&c.=0,\quad \sqrt{n(\Theta+N\Phi)}+\&c.=0$$

respectively (for a zomal, the $+\&c.$ should be omitted, and the first equation be written $\sqrt{l(\Theta+L\Phi)}=0$), then in the ν-zomal there will be the two branches

$$\left(\sqrt{l(\Theta+L\Phi)}+\&c.\right)\pm\left(\sqrt{n(\Theta+N\Phi)}+\&c.\right)=0,$$

each ideally containing $\Phi=0$.

Conversely, if a ν-zomal contain $\Phi^2=0$ by reason that it has two branches each ideally containing $\Phi=0$, then either a zomal and its antizomal will each of them, or else a pair of complementary parazomals will each of them, inseparably contain $\Phi=0$.

35. Reverting to the case of the ν-zomal curve

$$\sqrt{l(\Theta+L\Phi)}+\sqrt{m(\Theta+M\Phi)}+\&c.=0,$$

which does not contain $\Phi=0$, each of the $r(r-s)$ common points $\Theta=0$, $\Phi=0$ is a $2^{\nu-2}$-tuple point on the ν-zomal; each of these counts therefore for $2^{\nu-2}$ intersections of the ν-zomal with the curve $\Phi=0$, and we have thus the complete number $2^{\nu-2}r(r-s)$ of intersections of the two curves, viz., the curve $\Phi=0$ meets the ν-zomal in the $r(r-s)$ common points, each of them a $2^{\nu-2}$-tuple point on the ν-zomal, and in no other point.

36. But if the ν-zomal contains $\Phi^{\omega}=0$, then each of the $r(r-s)$ common points is still a $2^{\nu-2}$-tuple point on the aggregate curve; the aggregate curve therefore passes $2^{\nu-2}$ times through each common point; but among these passages are included ω passages of the curve $\Phi=0$ through the common point. The residual curve—say the ν-zomal—passes therefore only $2^{\nu-2}-\omega$ times through the common point; that is, each of the $r(r-s)$ common points is a $(2^{\nu-2}-\omega)$ tuple point on the ν-zomal. The curve $\Phi=0$ meets the ν-zomal in $\{2^{\nu-2}r-\omega(r-s)\}(r-s)$ points, viz., these include the $r(r-s)$ common points, each of them a $(2^{\nu-2}-\omega)$ tuple point on the ν-zomal, and therefore counting together as $(2^{\nu-2}-\omega)r(r-s)$ intersections; there remain consequently $\omega s(r-s)$ other intersections of the curve $\Theta=0$ with the ν-zomal.

37. In the case where the ν-zomal contains the factor $\Phi^{\omega}=0$, then throughout excluding from consideration the $r(r-s)$ common points $\Theta=0$, $\Phi=0$, the *remaining* intersections of any zomal with its antizomal are points of contact of the zomal with the ν-zomal, and the *remaining* intersections of each pair of complementary parazomals are nodes of the ν-zomal, it being understood that if any zomal, antizomal, or parazomal contain a power of $\Phi=0$, such powers of $\Phi=0$ are to be discarded, and only the residual curves attended to. The number of contacts and of nodes may in any particular case be investigated without difficulty, and some instances will present themselves in the sequel, but on account of the different ways in which the factor

$\Phi^\omega = 0$ may present itself, ideally in a single branch, or in several branches, and the consequent occurrence in the latter case of powers of $\Phi = 0$ in certain of the zomals, antizomals, or parazomals, the cases to be considered would be very numerous, and there is no reason to believe that the results could be presented in any moderately concise form; I therefore abstain from entering on the question.

Article Nos. 38 and 39. *On the Trizomal Curve and the Tetrazomal Curve.*

38. The trizomal curve

$$\sqrt{U} + \sqrt{V} + \sqrt{W} = 0$$

has for its rationalised form of equation

$$U^2 + V^2 + W^2 - 2VW - 2WU - 2UV = 0;$$

or as this may also be written,

$$(1,\ 1,\ 1,\ -1,\ -1,\ -1)(U,\ V,\ W)^2 = 0;$$

and we may from this rational equation verify the general results applicable to the case in hand, viz., that the trizomal is a curve of the order $2r$, and that

$$\begin{aligned} U &= 0, \text{ at each of its } r^2 \text{ intersections with } & V - W &= 0, \\ V &= 0, \qquad\quad ,, \qquad\qquad ,, & W - U &= 0, \\ W &= 0, \qquad\quad ,, \qquad\qquad ,, & U - V &= 0, \end{aligned}$$

respectively touch the trizomal. There are not, in general, any nodes or cusps, and the order being $= 2r$, the class is $= 2r(2r - 1)$.

39. The tetrazomal curve

$$\sqrt{U} + \sqrt{V} + \sqrt{W} + \sqrt{T} = 0$$

has for its rationalised form of equation

$$(U^2 + V^2 + W^2 + T^2 - 2UV - 2UW - 2UT - 2VW - 2VT - 2WT)^2 - 64UVWT = 0,$$

and we may hereby verify the fundamental properties, viz., that the tetrazomal is a curve of the order $4r$, touched by each of the zomals $U = 0$, $V = 0$, $W = 0$, $T = 0$ in $2r^2$ points, viz., by $U = 0$ at its intersections with $\sqrt{U} + \sqrt{W} + \sqrt{T} = 0$, that is, $V^2 + W^2 + T^2 - 2VW - 2VT - 2WT = 0$; (and the like as regards the other zomals), and having $3r^2$ nodes, viz., these are the intersections of $(\sqrt{U} + \sqrt{V} = 0,\ \sqrt{W} + \sqrt{T} = 0)$, $(\sqrt{U} + \sqrt{W} = 0,\ \sqrt{V} + \sqrt{T} = 0)$, $(\sqrt{U} + \sqrt{T} = 0,\ \sqrt{V} + \sqrt{W} = 0)$, or, what is the same thing, the intersections of $(U - V = 0,\ W - T = 0)$, $(U - W = 0,\ V - T = 0)$, $(U - T = 0,\ V - W = 0)$. There are not in general any cusps, and the class is thus $= 4r(4r - 1) - 6r^2, = 10r^2 - 4r$.

Article Nos. 40 and 41. *On the Intersection of two ν-Zomals having the same Zomal Curves.*

40. Without going into any detail, I may notice the question of the intersection of two ν-zomals which have the same zomal curves—say the two trizomals $\sqrt{U}+\sqrt{V}+\sqrt{W}=0$, $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$, or two similarly related tetrazomals. For the trizomals, writing the equations under the form

$$\sqrt{U}+\sqrt{V}+\sqrt{W}=0, \quad \sqrt{l}\sqrt{U}+\sqrt{m}\sqrt{V}+\sqrt{n}\sqrt{W}=0,$$

then, when these equations are considered as existing simultaneously, we may, without loss of generality, attribute to the radicals $\sqrt{U}$, $\sqrt{V}$, $\sqrt{W}$, the same values in the two equations respectively; but doing so, we must in the second equation successively attribute to all but one of the radicals $\sqrt{l}$, $\sqrt{m}$, $\sqrt{n}$, each of its two opposite values. For the intersections of the two curves we have thus

$$\sqrt{U} : \sqrt{V} : \sqrt{W} = \sqrt{m}-\sqrt{n} : \sqrt{n}-\sqrt{l} : \sqrt{l}-\sqrt{m},$$

viz., this is one of a system of four equations, obtained from it by changes of sign, say in the radicals $\sqrt{m}$ and $\sqrt{n}$. Each of the four equations gives a set of r^2 points; we have thus the complete number, $=4r^2$, of the points of intersection of the two curves.

41. But take, in like manner, two tetrazomal curves; writing their equations in the form

$$\sqrt{U}+\quad\sqrt{V}+\quad\sqrt{W}+\quad\sqrt{T}=0,$$

$$\sqrt{l}\sqrt{U}+\sqrt{m}\sqrt{V}+\sqrt{n}\sqrt{W}+\sqrt{p}\sqrt{T}=0,$$

then $\sqrt{U}$, $\sqrt{V}$, $\sqrt{W}$, $\sqrt{T}$ may be considered as having the same values in the two equations respectively, but we must in the second equation attribute successively, say to $\sqrt{m}$, $\sqrt{n}$, $\sqrt{p}$, each of their two opposite values. For the intersections of the two curves we have

$$(\sqrt{m}-\sqrt{l})\sqrt{V}+(\sqrt{n}-\sqrt{l})\sqrt{W}+(\sqrt{p}-\sqrt{l})\sqrt{T}=0,$$

$$(\sqrt{l}-\sqrt{m})\sqrt{U}\qquad\qquad +(\sqrt{n}-\sqrt{m})\sqrt{W}+(\sqrt{p}-\sqrt{m})\sqrt{T}=0,$$

viz., this is one of a system of eight similar pairs of equations, obtained therefrom by changes of sign of the radicals $\sqrt{m}$, $\sqrt{n}$, $\sqrt{p}$. The equations represent each of them a trizomal curve, of the order $2r$; the two curves intersect therefore in $4r^2$ points, and if each of these was a point of intersection of the two tetrazomals, we should have in all $8\times 4r^2=32r^2$ intersections. But the tetrazomals are each of them a curve of the order $4r$, and they intersect therefore in only $16r^2$ points. The explanation is, that not all the $4r^2$ points, but only $2r^2$ of them are intersections of the tetrazomals. In fact, to find *all* the intersections of the two trizomals, it is necessary in their two equations to attribute opposite signs to one of the radicals $\sqrt{W}$, $\sqrt{T}$; we obtain $2r^2$ intersections from the equations as they stand, the remaining $2r^2$ intersections from the two equations after we have in the second equation reversed the sign, say of $\sqrt{T}$.

Now, from the two equations as they stand we can pass back to the two tetrazomal equations, and the first-mentioned $2r^2$ points are thus points of intersection of the two tetrazomal curves—from the two equations after such reversal of the sign of $\sqrt{T}$, we cannot pass back to the two tetrazomal equations, and the last-mentioned $2r^2$ points are thus not points of intersection of the two tetrazomal curves. The number of intersections of the two curves is thus $8 \times 2r^2, = 16r^2$, as it should be.

Article Nos. 42 to 45. *The Theorem of the Decomposition of a Tetrazomal Curve.*

42. I consider the tetrazomal curve

$$\sqrt{lU} + \sqrt{mV} + \sqrt{nW} + \sqrt{pT} = 0,$$

where the zomal curves are in involution,—that is, where we have an identical relation,

$$\mathrm{a}U + \quad \mathrm{b}V + \quad \mathrm{c}W + \quad \mathrm{d}T = 0;$$

and I proceed to show that if l, m, n, p satisfy the relation

$$\frac{l}{\mathrm{a}} + \quad \frac{m}{\mathrm{b}} + \quad \frac{n}{\mathrm{c}} + \quad \frac{p}{\mathrm{d}} = 0,$$

the curve breaks up into two trizomals. In fact, writing the equation under the form

$$(\sqrt{lU} + \sqrt{mV} + \sqrt{nW})^2 - pT = 0,$$

and substituting for T its value, in terms of U, V, W, this is

$$\begin{aligned}(l\mathrm{d} + p\mathrm{a})\, U + (m\mathrm{d} + p\mathrm{b})\, V + (n\mathrm{d} + p\mathrm{c})\, W \\ + 2\sqrt{mn\mathrm{d}}\,\sqrt{VW} + 2\sqrt{nl\mathrm{d}}\,\sqrt{WU} + 2\sqrt{lm\mathrm{d}}\,\sqrt{UV} = 0\,;\end{aligned}$$

or, considering the left-hand side as a quadric function of $(\sqrt{U}, \sqrt{V}, \sqrt{W})$, the condition for its breaking up into factors is

$$\begin{vmatrix} l\mathrm{d} + p\mathrm{a}, & \mathrm{d}\sqrt{lm}, & \mathrm{d}\sqrt{ln} \\ \mathrm{d}\sqrt{ml}, & m\mathrm{d} + p\mathrm{b}, & \mathrm{d}\sqrt{mn} \\ \mathrm{d}\sqrt{nl}\,, & \mathrm{d}\sqrt{nm}, & n\mathrm{d} + p\mathrm{c} \end{vmatrix} = 0,$$

that is

$$p^2\,(l\mathrm{bcd} + m\mathrm{cda} + n\mathrm{dab} + p\mathrm{abc}) = 0,$$

or finally, the condition is

$$\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \quad \frac{n}{\mathrm{c}} + \quad \frac{p}{\mathrm{d}} = 0.$$

43. Multiplying by $l\mathrm{d} + p\mathrm{a}$, and observing that in virtue of the relation we have

$$(l\mathrm{d} + p\mathrm{a})\,(m\mathrm{d} + p\mathrm{b}) = lm\mathrm{d}^2 - \frac{\mathrm{abd}}{\mathrm{c}}\,pn\,,$$

$$(l\mathrm{d} + p\mathrm{a})\,(n\mathrm{d} + p\mathrm{c}) = ln\mathrm{d}^2 - \frac{\mathrm{acd}}{\mathrm{b}}\,pm,$$

the equation becomes

$$\left((l\mathrm{d}+p\mathrm{a})\sqrt{U}+\mathrm{d}\sqrt{lm}\sqrt{V}+\mathrm{d}\sqrt{ln}\sqrt{W}\right)^2=\frac{\mathrm{ad}}{\mathrm{bc}}p\left(\mathrm{b}\sqrt{n}\sqrt{V}-\mathrm{c}\sqrt{m}\sqrt{W}\right)^2,$$

or as this is more conveniently written

$$\left(\left(\sqrt{l}+\frac{\mathrm{a}p}{\mathrm{d}\sqrt{l}}\right)\sqrt{U}+\sqrt{mV}+\sqrt{nW}\right)^2=\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}\left(\mathrm{b}\sqrt{nV}-\mathrm{c}\sqrt{mV}\right)^2,$$

an equation breaking up into two equations, which may be represented by

$$\sqrt{l_1U}+\sqrt{m_1V}+\sqrt{n_1W}=0,\quad \sqrt{l_2U}+\sqrt{m_2V}+\sqrt{n_2W}=0,$$

where

$$\sqrt{l_1}=\sqrt{l}+\frac{\mathrm{a}}{\mathrm{d}}\frac{p}{\sqrt{l}}\quad,\quad \sqrt{l_2}=\sqrt{l}+\frac{\mathrm{a}}{\mathrm{d}}\frac{p}{\sqrt{l}}$$

$$\sqrt{m_1}=\sqrt{m}-\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}}\,\mathrm{b}\sqrt{n}\quad,\quad \sqrt{m_2}=\sqrt{m}+\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}}\,\mathrm{b}\sqrt{n}$$

$$\sqrt{n_1}=\sqrt{n}+\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}}\,\mathrm{c}\sqrt{m}\quad,\quad \sqrt{n_2}=\sqrt{n}-\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}}\,\mathrm{c}\sqrt{m},$$

where, in the expressions for $\sqrt{l}$, &c., the signs of the radicals

$$\sqrt{l},\ \sqrt{m},\ \sqrt{n},\ \sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}},$$

may be taken determinately in any way whatever at pleasure; the only effect of an alteration of sign would in some cases be to interchange the values of $(\sqrt{l_1}, \sqrt{m_1}, \sqrt{n_1})$ with those of $(\sqrt{l_2}, \sqrt{m_2}, \sqrt{n_2})$. The tetrazomal curve thus breaks up into two trizomals.

44. It is to be noticed that we have

$$\begin{aligned}\frac{l_1}{\mathrm{a}}+\frac{m_1}{\mathrm{b}}+\frac{n_1}{\mathrm{c}}&=\frac{l}{\mathrm{a}}+\frac{\mathrm{a}p^2}{\mathrm{d}^2l}+2\frac{p}{\mathrm{d}}\\&\quad+\frac{m}{\mathrm{b}}+\frac{\mathrm{a}}{\mathrm{cd}}\frac{np}{l}\\&\quad+\frac{n}{\mathrm{c}}+\frac{\mathrm{a}}{\mathrm{bd}}\frac{mp}{l}\\&=\left(l+\frac{\mathrm{a}p}{\mathrm{d}l}\right)\left(\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}\right);\end{aligned}$$

that is

$$\frac{l_1}{\mathrm{a}}+\frac{m_1}{\mathrm{b}}+\frac{n_1}{\mathrm{c}}=0;$$

and that similarly we have

$$\frac{l_2}{\mathrm{a}}+\frac{m_2}{\mathrm{b}}+\frac{n_2}{\mathrm{c}}=0.$$

The meaning is, that, taking the trizomal curve $\sqrt{l_1 U}+\sqrt{m_1 V}+\sqrt{n_1 W}=0$, this regarded as a tetrazomal curve, $\sqrt{l_1 U}+\sqrt{m_1 V}+\sqrt{n_1 W}+\sqrt{0T}=0$, satisfies the condition $\frac{l_1}{\mathrm{a}}+\frac{m_1}{\mathrm{b}}+\frac{n_1}{\mathrm{c}}+\frac{0}{\mathrm{d}}=0$; and the like as to the trizomal curve $\sqrt{l_2 U}+\sqrt{m_2 V}+\sqrt{n_2 W}=0$.

45. The equation by which the decomposition was effected is, it is clear, one of twelve equivalent equations; four of these are

$$\left(\sqrt{l}+\frac{\mathrm{a}p}{\mathrm{d}\sqrt{l}},\ \sqrt{m},\ \sqrt{n},\ 0\right)\left(\sqrt{U},\ \sqrt{V},\ \sqrt{W},\ \sqrt{T}\right)^2=\frac{\mathrm{a}}{\mathrm{bcd}}\,\frac{p}{l}\left(\mathrm{b}\sqrt{nW}-\mathrm{c}\sqrt{mW}\right)^2,$$

$$\left(0,\ \sqrt{m}+\frac{\mathrm{b}l}{\mathrm{a}\sqrt{m}},\ \sqrt{n},\ \sqrt{p}\right)\left(\qquad ,,\qquad\right)^2=\frac{\mathrm{b}}{\mathrm{cda}}\,\frac{l}{m}\left(\mathrm{c}\sqrt{pW}-\mathrm{d}\sqrt{nT}\right)^2,$$

$$\left(\sqrt{l},\ 0,\ \sqrt{n}+\frac{\mathrm{c}m}{\mathrm{b}\sqrt{n}},\ \sqrt{p}\right)\left(\qquad ,,\qquad\right)^2=\frac{\mathrm{c}}{\mathrm{dab}}\,\frac{m}{n}\left(\mathrm{d}\sqrt{lT}-\mathrm{a}\sqrt{pU}\right)^2,$$

$$\left(\sqrt{l},\ \sqrt{m},\ 0,\ \sqrt{p}+\frac{\mathrm{d}n}{\mathrm{c}\sqrt{p}}\right)\left(\qquad ,,\qquad\right)^2=\frac{\mathrm{d}}{\mathrm{abc}}\,\frac{n}{p}\left(\mathrm{a}\sqrt{mU}-\mathrm{b}\sqrt{lV}\right)^2,$$

and the others may be deduced from these by a cyclical permutation of (U, V, W), (a, b, c), (l, m, n), leaving T, d, p unaltered.

Article Nos. 46 to 51. *Application to the Trizomal; the Theorem of the Variable Zomal.*

46. I take the last equation written under the form

$$(\mathrm{a}\sqrt{mU}-\mathrm{b}\sqrt{lV})^2=\frac{\mathrm{abc}}{\mathrm{d}n}\left(\sqrt{lpU}+\sqrt{mpV}+\left(p+\frac{\mathrm{d}n}{\mathrm{c}}\right)\sqrt{T}\right)^2,$$

which, putting therein $p=0$, is

$$(\mathrm{a}\sqrt{mU}-\mathrm{b}\sqrt{lV})^2=\frac{\mathrm{abd}}{\mathrm{c}}\,nT,$$

which is in fact the trizomal curve,

$$\mathrm{a}\sqrt{mU}-\mathrm{b}\sqrt{lV}+\sqrt{\frac{\mathrm{abd}}{\mathrm{c}}\,nT}=0,$$

viz., the trizomal curve $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$,—if a, b, c be any quantities connected by the equation

$$\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0,$$

(the ratios a, b, c thus involving a single arbitrary parameter); and if we take T a function such that $\mathrm{a}U+\mathrm{b}V+\mathrm{c}W+\mathrm{d}T=0$; that is, $T=0$, any one of the series of curves $\mathrm{a}U+\mathrm{b}V+\mathrm{c}W=0$, in involution with the given curves $U=0$, $V=0$, $W=0$,—has its equation expressible in the form

$$\mathrm{a}\sqrt{mU}-\mathrm{b}\sqrt{lV}+\sqrt{\frac{\mathrm{abd}}{\mathrm{c}}nT}=0;$$

that is, we have the curve $T=0$ (the equation whereof contains a variable parameter) as a zomal of the given trizomal curve $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$; and we have thus from the theorem of the decomposition of a tetrazomal deduced the theorem of the variable zomal of a trizomal. The analytical investigation is somewhat simplified by assuming $p=0$ *ab initio*, and it may be as well to repeat it in this form.

47. Starting, then, with the trizomal curve

$$\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0,$$

and writing

$$\mathrm{a}U+\ \mathrm{b}V+\ \mathrm{c}W+\mathrm{d}T=0$$

as the definition of T, the coefficients being connected by

$$\frac{l}{\mathrm{a}}\ +\ \frac{m}{\mathrm{b}}\ +\ \frac{n}{\mathrm{c}}=0,$$

the equation gives

$$lU+\ mV+2\sqrt{lmUV}-nW=0;$$

or substituting in this equation for W its value in terms of U, V, T, we have

$$(\mathrm{a}n+\mathrm{c}l)\,U+(\mathrm{b}n+\mathrm{c}m)\,V+2\mathrm{c}\sqrt{lmUV}+\mathrm{d}nT=0,$$

which by the given relation between a, b, c, is converted into

$$-\frac{\mathrm{ac}}{\mathrm{b}}mU-\frac{\mathrm{bc}}{\mathrm{a}}lV+2\mathrm{c}\sqrt{lmUV}+\mathrm{d}nT=0;$$

that is

$$\mathrm{a}^2mU+\mathrm{b}^2lV-2\mathrm{ab}\sqrt{lmUV}\ =\frac{\mathrm{abd}}{\mathrm{c}}nT,$$

viz., this is

$$(\mathrm{a}\sqrt{mU}-\mathrm{b}\sqrt{lV})^2\qquad=\frac{\mathrm{abd}}{\mathrm{c}}nT,$$

or finally

$$\mathrm{a}\sqrt{mU}-\mathrm{b}\sqrt{lV}+\sqrt{\frac{\mathrm{abd}}{\mathrm{c}}nT}=0.$$

48. The result just obtained of course implies that when as above

$$\mathrm{a}U+\mathrm{b}V+\mathrm{c}W+\mathrm{d}T=0,\quad \frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0,$$

the trizomal curve $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$ can be expressed by means of any three of the four zomals U, V, W, T, and we may at once write down the four forms

$$\left(\begin{array}{cccc} \cdot\;, & \sqrt{\dfrac{n}{\mathrm{c}^2}}\,, & -\sqrt{\dfrac{m}{\mathrm{b}^2}}\,, & -\sqrt{\dfrac{l\mathrm{d}}{\mathrm{abc}}} \\ -\sqrt{\dfrac{n}{\mathrm{c}^2}}\,, & \cdot\;, & \sqrt{\dfrac{l}{\mathrm{a}^2}}\,, & -\sqrt{\dfrac{m\mathrm{d}}{\mathrm{abc}}} \\ \sqrt{\dfrac{m}{\mathrm{b}^2}}\,, & -\sqrt{\dfrac{l}{\mathrm{a}^2}}\,, & \cdot\;, & -\sqrt{\dfrac{n\mathrm{d}}{\mathrm{abc}}} \\ \sqrt{\dfrac{l\mathrm{d}}{\mathrm{abc}}}\,, & \sqrt{\dfrac{m\mathrm{d}}{\mathrm{abc}}}\,, & \sqrt{\dfrac{n\mathrm{d}}{\mathrm{abc}}}\,, & \end{array}\right)\left(\sqrt{U},\ \sqrt{V},\ \sqrt{W},\ \sqrt{T}\right)=0,$$

the last of which is the original equation $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$. It may be added that if the first equation be represented by $\sqrt{m_1V}+\sqrt{n_1W}+\sqrt{p_1T}=0$,—that is, if we have

$$\sqrt{m_1}=\sqrt{\frac{n}{\mathrm{c}^2}},\ \sqrt{n_1}=-\sqrt{\frac{m}{\mathrm{b}^2}},\ \sqrt{p_1}=\sqrt{\frac{l\mathrm{d}}{\mathrm{abc}}},$$

and therefore

$$\frac{m_1}{\mathrm{b}}+\frac{n_1}{\mathrm{c}}+\frac{p_1}{\mathrm{d}}=\frac{l}{\mathrm{bc}}\left(\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}\right),=0;$$

or if the second equation be represented by $\sqrt{l_2U}+\sqrt{n_2W}+\sqrt{p_2T}=0$,—that is, if we have

$$\sqrt{l_2}=-\sqrt{\frac{n}{\mathrm{c}^2}},\ \ \sqrt{n_2}=\sqrt{\frac{l}{\mathrm{a}^2}},\ \ \sqrt{p_2}=\sqrt{\frac{m\mathrm{d}}{\mathrm{abc}}},$$

and therefore

$$\frac{l_2}{\mathrm{a}}+\frac{n_2}{\mathrm{c}}+\frac{p_2}{\mathrm{d}}=0;$$

or if the third equation be represented by $\sqrt{l_3U}+\sqrt{m_3V}+\sqrt{p_3T}=0$,—that is, if we have

$$\sqrt{l_3}=\sqrt{\frac{m}{\mathrm{b}^2}},\ \sqrt{m_3}=-\sqrt{\frac{l}{\mathrm{a}^2}},\ \ \sqrt{p_3}=\sqrt{\frac{m\mathrm{d}}{\mathrm{abc}}},$$

and therefore

$$\frac{l_3}{\mathrm{a}}+\frac{m_3}{\mathrm{b}}+\frac{p_3}{\mathrm{d}}=0,$$

then the equation of the trizomal may also be expressed in the forms

$$\left(\begin{array}{cccc} \cdot\ , & \sqrt{m_1}\ , & \sqrt{n_1}\ , & \sqrt{p_1} \\ -\sqrt{m_1}\ , & \cdot\ , & \sqrt{\dfrac{p_1\mathrm{bc}}{\mathrm{ad}}}\ , & -\sqrt{\dfrac{n_1\mathrm{bd}}{\mathrm{ac}}} \\ -\sqrt{n_1}\ , & -\sqrt{\dfrac{p_1\mathrm{bc}}{\mathrm{ad}}}\ , & \cdot\ , & \sqrt{\dfrac{m_1\mathrm{cd}}{\mathrm{ab}}} \\ -\sqrt{p_1}\ , & \sqrt{\dfrac{n_1\mathrm{bd}}{\mathrm{ac}}}\ , & -\sqrt{\dfrac{m_1\mathrm{cd}}{\mathrm{ab}}}\ , & \cdot \end{array}\right)(\sqrt{U},\ \sqrt{V},\ \sqrt{W},\ \sqrt{T})=0,$$

$$\left(\begin{array}{cccc} \cdot\ , & -\sqrt{l_2}\ , & -\sqrt{\dfrac{p_2\mathrm{ac}}{\mathrm{bd}}}\ , & \sqrt{\dfrac{n_2\mathrm{ad}}{\mathrm{bc}}} \\ \sqrt{l_2}\ , & \cdot\ , & \sqrt{n_2}\ , & \sqrt{p_2} \\ \sqrt{\dfrac{p_2\mathrm{ac}}{\mathrm{bd}}}\ , & -\sqrt{n_2}\ , & \cdot\ , & -\sqrt{\dfrac{l_2\mathrm{cd}}{\mathrm{ab}}} \\ \sqrt{\dfrac{n_2\mathrm{ad}}{\mathrm{bc}}}\ , & -\sqrt{p_2}\ , & \sqrt{\dfrac{l_2\mathrm{cd}}{\mathrm{ab}}}\ , & \cdot \end{array}\right)(\sqrt{U},\ \sqrt{V},\ \sqrt{W},\ \sqrt{T})=0,$$

and

$$\left(\begin{array}{cccc} \cdot\ , & \sqrt{\dfrac{p_3\mathrm{ab}}{\mathrm{cd}}}\ , & -\sqrt{l_3}\ , & -\sqrt{\dfrac{m_3\mathrm{ad}}{\mathrm{bc}}} \\ -\sqrt{\dfrac{p_3\mathrm{ab}}{\mathrm{cd}}}\ , & \cdot\ , & -\sqrt{m_3}\ , & -\sqrt{\dfrac{l_3\mathrm{bd}}{\mathrm{ac}}} \\ \sqrt{l_3}\ , & \sqrt{m_3}\ , & \cdot\ , & \sqrt{p_3} \\ \sqrt{\dfrac{m_3\mathrm{ad}}{\mathrm{bc}}}\ , & -\sqrt{\dfrac{l_3\mathrm{bd}}{\mathrm{ac}}}\ , & -\sqrt{p_3}\ , & \cdot \end{array}\right)(\sqrt{U},\ \sqrt{V},\ \sqrt{W},\ \sqrt{T})=0.$$

49. These equations may, however, be expressed in a much more elegant form. Write

$$\mathrm{a}'=\frac{a}{(\beta\gamma\delta)},\quad \mathrm{b}'=-\frac{b}{(\gamma\delta\alpha)},\quad \mathrm{c}'=\frac{c}{(\delta\alpha\beta)},\quad \mathrm{d}'=\frac{-d}{(\alpha\beta\gamma)},$$

where, for shortness, $(\beta\gamma\delta)=(\beta-\gamma)(\gamma-\delta)(\delta-\beta)$, &c.; (α, β, γ) being arbitrary quantities: or, what is the same thing,

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d}=\mathrm{a}'(\beta\gamma\delta) : -\mathrm{b}'(\gamma\delta\alpha) : \mathrm{c}'(\delta\alpha\beta) : -\mathrm{d}'(\alpha\beta\gamma).$$

Assume

$$l : m : n \quad =\rho\mathrm{a}'(\beta-\gamma)^2 : \sigma\mathrm{b}'(\gamma-\alpha)^2 : \tau\mathrm{c}'(\alpha-\beta)^2;$$

then the equation $\dfrac{l}{\mathrm{a}}+\dfrac{m}{\mathrm{b}}+\dfrac{n}{\mathrm{c}}=0$ takes the form

$$\rho(\beta-\gamma)(\alpha-\delta)+\sigma(\gamma-\alpha)(\beta-\delta)+\tau(\alpha-\beta)(\gamma-\delta),$$

and the four forms of the equation are found to be

$$\left(\begin{array}{cccc} \cdot\ , & \sqrt{\tau}\,(\delta-\gamma), & \sqrt{\sigma}\,(\beta-\delta), & \sqrt{\rho}\,(\gamma-\beta) \\ \sqrt{\tau}\,(\gamma-\delta), & \cdot\ , & \sqrt{\rho}\,(\delta-\alpha), & \sqrt{\sigma}\,(\alpha-\gamma) \\ \sqrt{\sigma}\,(\delta-\beta), & \sqrt{\rho}\,(\alpha-\delta), & \cdot\ , & \sqrt{\tau}\,(\beta-\alpha) \\ \sqrt{\rho}\,(\beta-\gamma), & \sqrt{\sigma}\,(\gamma-\alpha), & \sqrt{\tau}\,(\alpha-\beta), & \end{array}\right)(\sqrt{\mathrm{a}'U},\ \sqrt{\mathrm{b}'V},\ \sqrt{\mathrm{c}'W},\ \sqrt{\mathrm{d}'T})=0,$$

viz., these are the equivalent forms of the original equation assumed to be

$$(\beta-\gamma)\sqrt{\rho\mathrm{a}'U}+(\gamma-\alpha)\sqrt{\sigma\mathrm{b}'V}+(\alpha-\beta)\sqrt{\tau\mathrm{c}'W}=0.$$

50. I remark that the theorem of the variable zomal may be obtained as a transformation theorem—viz., comparing the equation $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$ with the equation $\sqrt{lx}+\sqrt{my}+\sqrt{nz}=0$; this last belongs to a conic touched by the three lines $x=0$, $y=0$, $z=0$; the equation of the same conic must, it is clear, be expressible in a similar form by means of any other three tangents thereof, but the equation of any tangent of the conic is $\mathrm{a}x+\mathrm{b}y+\mathrm{c}z=0$, where a, b, c are any quantities satisfying the condition $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$; whence, writing $\mathrm{a}x+\mathrm{b}y+\mathrm{c}z+\mathrm{d}w=0$, we may introduce $w=0$ along with any two of the original zomals $x=0$, $y=0$, $z=0$, or, instead of them, any three functions of the form w; and then the mere change of x, y, z, w into U, V, W, T gives the theorem. But it is as easy to conduct the analysis with (U, V, W, T) as with (x, y, z, w), and, so conducted, it is really the same analysis as that whereby the theorem is established *ante*, No. 47.

51. It is worth while to exhibit the equation of the curve

$$\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0,$$

in a form containing three new zomals. Observe that the equation $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$ is satisfied by $\mathrm{a}=l\phi\chi$, $\mathrm{b}=m\chi\theta$, $\mathrm{c}=n\theta\phi$, if only $\theta+\phi+\chi=0$; or say, if $\theta=a'-a''$, $\phi=a''-a$, $\chi=a-a''$. The equation

$$\begin{aligned}\lambda\sqrt{(a-a')(a-a'')\,lU+(a'-a'')(a'-a)\,mV+(a''-a)(a''-a')\,nW}\\ +\mu\sqrt{(b-b')(b-b'')\,lU+(b'-b'')(b'-b)\,mV+(b''-b)(b''-b')\,nW}\\ +\nu\sqrt{(c-c')(c-c'')\,lU+(c'-c'')(c'-c)\,mV+(c''-c)(c''-c')\,nW}=0\end{aligned}$$

is consequently an equation involving three zomals of the proper form; and we can determine λ, μ, ν in suchwise as to identify this with the original equation $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}$, viz., writing successively $U=0$, $V=0$, $W=0$, we find

$$\begin{aligned}(a'-a'')\lambda+(b'-b'')\mu+(c'-c'')\nu=0,\\ (a''-a\ )\lambda+(b''-b\ )\mu+(c''-c\ )\nu=0,\\ (a-a')\lambda+(b-b')\mu+(c-c')\nu=0,\end{aligned}$$

equations which are, as they should be, equivalent to two equations only, and which give

$$\lambda : \mu : \nu = \begin{vmatrix} 1, & 1, & 1 \\ b, & b', & b'' \\ c, & c', & c'' \end{vmatrix} : \begin{vmatrix} 1, & 1, & 1 \\ c, & c', & c'' \\ a, & a', & a'' \end{vmatrix} : \begin{vmatrix} 1, & 1, & 1 \\ a, & a', & a'' \\ b, & b', & b'' \end{vmatrix},$$

and the equation, with these values of λ, μ, ν substituted therein, is in fact the equation of the trizomal curve $\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0$ in terms of three new zomals. It is easy to return to the forms involving one new zomal and any two of the original three zomals.

Article No. 52. *Remark as to the Tetrazomal Curve.*

52. I return for a moment to the case of the tetrazomal curve, in order to show that there is not, in regard to it in general, any theorem such as that of the variable zomal. Considering the form $\sqrt{lx} + \sqrt{my} + \sqrt{nz} + \sqrt{pw} = 0$ (the coordinates x, y, z, w are of course connected by a linear equation, but nothing turns upon this), the curve is here a quartic touched twice by each of the lines $x = 0$, $y = 0$, $z = 0$, $w = 0$ (viz., each of these is a double tangent of the curve), and having besides the three nodes $(x = y,\ z = w)$, $(x = z,\ y = w)$, $(x = w,\ y = z)$. But a quartic curve with three nodes, or trinodal quartic, has only four double tangents—that is, besides the lines $x = 0$, $y = 0$, $z = 0$, $w = 0$, there is no line $\alpha x + \beta y + \gamma z + \delta w = 0$ which is a double tangent of the curve; and writing U, V, W, T in place of x, y, z, w, then if U, V, W, T are connected by a linear equation (and, *à fortiori*, if they are not so connected), there is not any curve $\alpha U + \beta V + \gamma W + \delta T = 0$ which is related to the curve in the same way with the lines $U = 0$, $V = 0$, $W = 0$, $T = 0$; or say there is not (besides the curves $U = 0$, $V = 0$, $W = 0$, $T = 0$), any other zomal $\alpha U + \beta V + \gamma W + \delta T = 0$, of the tetrazomal curve. The proof does not show that for special forms of U, V, W, T there may not be zomals, not of the above form $\alpha U + \beta V + \gamma W + \delta T = 0$, but belonging to a separate system. An instance of this will be mentioned in the sequel.

Article Nos. 53 to 56. *The Theorem of the Variable Zomal of a Trizomal Curve resumed.*

53. I resume the foregoing theorem of the variable zomal of the trizomal curve $\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0$. The variable zomal $T = 0$ is the curve $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W = 0$, where a, b, c are connected by the equation $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$; that is, it belongs to a single series of curves selected in a certain manner out of the double series $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W = 0$ (a double series, as containing the *two* variable parameters a : b : c). These are the whole series of curves in involution with the given curves $U = 0$, $V = 0$, $W = 0$, or being such that the Jacobian of any three of them is identical with the Jacobian of the three given curves; in particular, the Jacobian of any one of the curves $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W = 0$,

and of two of the three given curves, is identical with the Jacobian of the three given curves. I call to mind that, by the Jacobian of the curves $U=0$, $V=0$, $W=0$, is meant the curve

$$J(U, V, W) = \frac{d(U, V, W)}{d(x, y, z)} = \begin{vmatrix} d_xU, & d_yU, & d_zU \\ d_xV, & d_yV, & d_zV \\ d_xW, & d_yW, & d_zW \end{vmatrix} = 0,$$

viz., the curve obtained by equating to zero the Jacobian or functional determinant of the functions U, V, W. Some properties of the Jacobian, which are material as to what follows, are mentioned in the Annex No. I.

For the complete statement of the theorem of the variable zomal, it would be necessary to interpret geometrically the condition $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$, thereby showing how the single series of the variable zomal is selected out of the double series of the curves $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W = 0$ in involution with the given curves. Such a geometrical interpretation of the condition may be sought for as follows, but it is only in a particular case, as afterwards mentioned, that a convenient geometrical interpretation is thereby obtained.

54. Consider the fixed line $\Omega = px + qy + rz = 0$, and let it be proposed to find the locus of the $(r-1)^2$ poles of the line $\Omega = 0$ in regard to the series of curves $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W = 0$, where $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$. Take (x, y, z) as the coordinates of any one of the poles in question, then in order that (x, y, z) may belong to one of the $(r-1)^2$ poles of the line $\Omega = px + qy + rz = 0$ in regard to the curve $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W = 0$, we must have

$$d_x(\mathrm{a}U + \mathrm{b}V + \mathrm{c}W) : d_y(\mathrm{a}U + \mathrm{b}V + \mathrm{c}W) : d_z(\mathrm{a}U + \mathrm{b}V + \mathrm{c}W) = p : q : r;$$

or, what is the same thing,

$$= d_x\Omega : d_y\Omega : d_z\Omega;$$

and these equations give without difficulty

$$\mathrm{a} : \mathrm{b} : \mathrm{c} = J(V, W, \Omega) : J(W, U, \Omega) : J(U, V, \Omega),$$

whence, substituting in the equation $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$, we have

$$\frac{l}{J(V, W, \Omega)} + \frac{m}{J(W, U, \Omega)} + \frac{n}{J(U, V, \Omega)} = 0$$

as the locus of the $(r-1)^2$ poles in question. Each of the Jacobians is a function of the order $2r-2$, and the order of the locus is thus $=4r-4$. As the given curves $U=0$, $V=0$, $W=0$ belong to the single series of curves, it is clear that the locus passes through the $3(r-1)^2$ points which are the $(r-1)^2$ poles of the fixed line in regard to the curves $U=0$, $V=0$, $W=0$ respectively.

55. In the case where the given trizomal is

$$\sqrt{l(\Theta+L\Phi)}+\sqrt{m(\Theta+M\Phi)}+\sqrt{n(\Theta+N\Phi)}=0,$$

$s=r-1$, that is, where the zomals $\Theta+L\Phi=0$, $\Theta+M\Phi=0$, $\Theta+N\Phi=0$ are each of them curves of the order r, passing through the r intersections of the line $\Phi=0$ with the curve $\Theta=0$, then, taking this line $\Phi=0$ for the fixed line $\Omega=0$, we have

$$J(V,\ W,\ \Omega)=J(\Theta+M\Phi,\ \Theta+N\Phi,\ \Phi)=\Phi\{M,\ N\},$$

if, for shortness, $\{M, N\}=J(M-N, \Theta, \Phi)+\Phi J(M, N, \Phi)$, and the like as to the other two Jacobians, so that, attaching the analogous significations to $\{N, L\}$ and $\{L, M\}$, the equation of the locus is

$$\frac{l}{\{M,\ N\}}+\frac{m}{\{N,\ L\}}+\frac{n}{\{L,\ M\}}=0,$$

where observe that each of the curves $\{M, N\}=0$, $\{N, L\}=0$, $\{L, M\}=0$ is a curve of the order $2r-3$; the order of the locus is thus $=4r-6$, and (as before) this locus passes through the $3(r-1)^2$ points which are the $(r-1)^2$ poles of the line $\Phi=0$ in regard to the curves $\Theta+L\Phi=0$, $\Theta+M\Phi=0$, $\Theta+N\Phi=0$ respectively.

56. In the case $r=2$, the trizomal is

$$\sqrt{l(\Theta+L\Phi)}+\sqrt{m(\Theta+M\Phi)}+\sqrt{n(\Theta+N\Phi)}=0,$$

where the zomals are the conics $\Theta+L\Phi=0$, $\Theta+M\Phi=0$, $\Theta+N\Phi=0$, each passing through the same two points $\Theta=0$, $\Phi=0$; the locus of the pole of the line $\Phi=0$, in regard to the variable zomal, is the conic

$$\frac{l}{\{M,\ N\}}+\frac{m}{\{N,\ L\}}+\frac{n}{\{L,\ M\}}=0,$$

viz., $\{M, N\}=0$, $\{N, L\}=0$, $\{L, M\}=0$, are here the lines passing through the poles of the line $\Phi=0$ in regard to the second and third, the third and first, and the first and second of the given conics respectively: treating l, m, n as arbitrary, the locus is clearly *any* conic through the poles of the line $\Phi=0$ in regard to the three conics respectively. The Jacobian of the three given conics is a conic related in a special manner to the three given conics, and which might be called the Jacobian conic thereof, and it would be easy to give a complete enunciation of the theorem for the case in hand. (See as to this, Annex No. I, above referred to.) But if, in accordance with the plan adopted in the remainder of the memoir, we at once assume that the points $\Theta=0$, $\Phi=0$ are the circular points at infinity, then the theorem can be enunciated under a more simple form—viz., if $\mathfrak{A}^\circ=0$, $\mathfrak{B}^\circ=0$, $\mathfrak{C}^\circ=0$ are the equations of any three circles, then in the trizomal

$$\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}=0,$$

the variable zomal is any circle whatever of the series of circles cutting at right angles the orthotomic circle of the three given circles, and having its centre on a certain conic which passes through the centres of the given circles. Moreover, if the coefficients l, m, n are not given in the first instance, but are regarded as arbitrary, then the last-mentioned conic is any conic whatever through the three centres, and there belongs to such conic and the series of zomals derived therefrom as above, a trizomal curve $\sqrt{l\mathfrak{A}^\circ} + \sqrt{m\mathfrak{B}^\circ} + \sqrt{n\mathfrak{C}^\circ} = 0$. This is obviously the theorem, that if a variable circle has its centre on a given conic, and cuts at right angles a given circle, then the envelope of the variable circle is a trizomal curve $\sqrt{l\mathfrak{A}^\circ} + \sqrt{m\mathfrak{B}^\circ} + \sqrt{n\mathfrak{C}^\circ}$, where $\mathfrak{A}^\circ = 0$, $\mathfrak{B}^\circ = 0$, $\mathfrak{C}^\circ = 0$ are any three circles, positions of the variable circle, and l, m, n are constant quantities depending on the selected three circles.

Part II. (Nos. 57 to 104). Subsidiary Investigations.

Article Nos. 57 and 58. *Preliminary Remarks.*

57. We have just been led to consider the conics which pass through two given points. There is no real loss of generality in taking these to be the circular points at infinity, or say the points I, J—viz., every theorem which in anywise explicitly or implicitly relates to these two points, may, without the necessity of any change in the statement thereof, be understood as a theorem relating instead to any two points P, Q. I call to mind that a circle is a conic passing through the two points I, J, and that lines at right angles to each other are lines harmonically related to the pair of lines from their intersection to the points I, J respectively, so that when (I, J) are replaced by any two given points whatever, the expression a circle must be understood to mean a conic passing through the two given points; and in speaking of lines at right angles to each other, it must be understood that we mean lines harmonically related to the pair of lines from their intersection to the two given points respectively. For instance, the theorem that the Jacobian of any three circles is their orthotomic circle, will mean that the Jacobian of any three conics which each of them passes through the two given points is the orthotomic conic through the same two points, that is, the conic such that at each of its intersections with any one of the three conics, the two tangents are harmonically related to the pair of lines from this intersection to the two given points respectively. Such extended interpretation of any theorem is applicable even to the theorems which involve distances or angles—viz., the terms "distance" and "angle" have a determinate signification when interpreted in reference (not to the circular points at infinity, but instead thereof) to any two given points whatever (see as to this my "Sixth Memoir on Quantics," Nos. 220, *et seq.*). *Phil. Trans.*, vol. CXLIX. (1859), pp. 61—90; see p. 86; [158]. And this being so, the theorem can, without change in the statement thereof, be understood as referring to the two given points.

58. I say then that any theorem (referring explicitly or implicitly) to the circular points at infinity I, J, may be understood as a theorem referring instead to any two given points. We might of course give the theorems in the first instance in terms explicitly referring to the two given points—(viz., instead of a circle, speak of a conic through the two given points, and so in other instances); but, as just explained, this is not really more general, and the theorems would be given in a less concise and

familiar form. It would not, on the face of the investigations, be apparent that in treating of the polyzomal curves

$$\sqrt{l(\Theta+L\Phi)}+\sqrt{m(\Theta+M\Phi)}+\&c.=0,$$

($\Theta=0$ a conic, $\Phi=0$ a line, as above), that we were really treating of the curves the zomals whereof are circles, and therein of the theories of foci and focofoci as about to be explained. And for these reasons I shall consider the two points $\Theta=0$, $\Phi=0$, to be the circular points at infinity I, J, and in the investigations, &c., make use of the terms circle, right angles, &c., which, in their ordinary significations, have implicit reference to these two points.

The present Part does not explicitly relate to the theory of polyzomal curves, but contains a series of researches, partly analytical and partly geometrical, which will be made use of in the following Parts III. and IV. of the Memoir.

Article Nos. 59 to 62. *The Circular Points at Infinity; Rectangular and Circular Coordinates.*

59. The coordinates made use of (except in the cases where the general trilinear coordinates (x, y, z), or any other coordinates, are explicitly referred to), will be either the ordinary rectangular coordinates x, y, or else, as we may term them, the circular coordinates ξ, η $(=x+iy,\ x-iy$ respectively, $i=\sqrt{-1}$ as usual), but in either case I shall introduce for homogeneity the coordinate z, it being understood that this coordinate is in fact $=1$, and that it may be retained or replaced by this its value, in different investigations or stages of the same investigation, as may for the time being be most convenient. In more concise terms, we may say that the coordinates are either the rectangular coordinates x, y, and $z\ (=1)$, or else the circular coordinates ξ, η, and $z\,(=1)$. The equation of the line infinity is $z=0$; the points I, J are given by the equations $(x+iy=0,\ z=0)$ and $(x-iy=0,\ z=0)$, or, what is the same thing, by the equations $(\xi=0,\ z=0)$ and $(\eta=0,\ z=0)$ respectively; or in the rectangular coordinates the coordinates of these points are $(-i,\ 1,\ 0)$ and $(i,\ 1,\ 0)$ respectively, and in the circular coordinates they are $(1,\ 0,\ 0)$ and $(0,\ 1,\ 0)$ respectively. It is, of course, only for points at infinity that the coordinate z is $=0$ (and observe that for any such point the x and y or ξ and η coordinates may be regarded as finite); for every point whatever not at infinity the coordinate z is, as stated above, $=1$.

60. Consider a point A, whose coordinates (rectangular) are $(a, a', 1)$ and (circular) $(\alpha, \alpha', 1)$, viz., $\alpha=a+a'i$, $\alpha'=a-a'i$; then the equations of the lines through A to the points I, J, are

$$x-az+i(y-a'z)=0,\quad x-az-i(y-a'z)=0$$

respectively, or they are

$$\xi-\alpha z=0\qquad,\quad \eta-\alpha' z=0$$

respectively. These equations, if (a, a') or (α, α') are arbitrary, will, it is clear, be the equations of any two lines through the points I, J, respectively.

61. We have from either of the equations in (x, y, z)

$$(x-az)^2+(y-a'z)^2=0,$$

that is, the distance from each other of any two points $(x, y, 1)$, and $(a, a', 1)$ in a line through I or J is $=0$. And in particular, if $z=0$, then $x^2+y^2=0$; that is, the distance of the point $(a, a', 1)$ from I or J is in each case $=0$.

62. Consider for a moment any three points P, Q, A; the perpendicular distance of P from QA is $=2$ triangle $PQA \div$ distance QA; if Q be any point on the line through A to either of the points I, J, and in particular if Q be either of the points I, J, then the triangle PQA is finite, but the distance QA is $=0$: that is, the perpendicular distance of P from the line through A to either of the points I, J, that is, from any line through either of these points, is $=\infty$. But, as just stated, the triangle PQA is finite, or say the triangles PIA, PJA are each finite; viz., the coordinates (rectangular) of P, A being $(x, y, z=1)$, $(a, a', 1)$ or (circular) $(\xi, \eta, z=1)$, $(\alpha, \alpha', 1)$, the expressions for the doubles of these triangles respectively are

$$\begin{vmatrix} x, & y, & z \\ -i, & 1, & 0 \\ a, & a', & 1 \end{vmatrix}, \quad \begin{vmatrix} x, & y, & z \\ i, & 1, & 0 \\ a, & a', & 1 \end{vmatrix}$$

that is, they are (rectangular coordinates) $x-az+i(y-a'z)$, $x-az-i(y-a'z)$, or (circular coordinates) $\xi-\alpha z$, $\eta-\alpha' z$.

Representing the double areas by PIA, PJA, respectively, and the squared distance of the points A, P, by $\mathfrak{A}$, we have—

$$\begin{aligned}\mathfrak{A} &= (x-az)^2+(y-a'z)^2 \\ &= (\xi-\alpha z)(\eta-\alpha' z), = PIA\,.\,PJA.\end{aligned}$$

Article No. 63. *Antipoints; Definition and Fundamental Properties.*

63. Two pairs of points (A, B) and (A_1, B_1) which are such that the lines AB, A_1B_1 bisect each other at right angles in a point O in such wise that $OA=OB=i\,OA_1=i\,OB_1$, are said to be antipoints, each of the other. In rectangular coordinates, taking the coordinates of $(AB,)$ to be $(a, 0, 1)$ and $(-a, 0, 1)$, those of (A_1, B_1) will be $(0, ai, 1)$ and $(0, -ai, 1)$ respectively, whence joining the points (A, B) with the points (I, J), the points A_1, B_1 are given as the intersections of the lines AI and BJ, and of the lines AJ and BI respectively. Or, what is the same thing, in any quadrilateral wherein I, J are opposite angles, the remaining pairs (A, B) and (A_1, B_1) are antipoints each of the other.

64. In circular coordinates, if the coordinates of A are $(\alpha, \alpha', 1)$, and those of B are $(\beta, \beta', 1)$, then the equations of

$$\begin{aligned} AI, AJ &\text{ are } \xi-\alpha z=0, \quad \eta-\alpha' z=0, \\ BI, BJ &\text{ ,, } \xi-\beta z=0, \quad \eta-\beta' z=0, \end{aligned}$$

whence the equations of

$$A_1I,\ A_1J \text{ are } \xi - \alpha z = 0,\quad \eta - \beta' z = 0,$$
$$B_1I,\ B_1J \text{ ,, } \xi - \beta z = 0,\quad \eta - \alpha' z = 0.$$

65. Considering any point P the coordinates of which are ξ, η, z $(=1)$, let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{A}_1$, $\mathfrak{B}_1$ be its squared distances from the points A, B, A_1, B_1 respectively; then by what precedes

$$\mathfrak{A} = (\xi - \alpha z)(\eta - \alpha' z),$$
$$\mathfrak{B} = (\xi - \beta z)(\eta - \beta' z),$$
$$\mathfrak{A}_1 = (\xi - \alpha z)(\eta - \beta' z),$$
$$\mathfrak{B}_1 = (\xi - \beta z)(\eta - \alpha' z),$$

and thence

$$\mathfrak{A} \,.\, \mathfrak{B} = \mathfrak{A}_1 \,.\, \mathfrak{B}_1;$$

that is, the product of the squared distances of a point P from any two points A, B, is equal to the product of the squared distances of the same point P from the two antipoints A_1, B_1. This theorem, which was, I believe, first given by me in the *Educational Times* (see reprint, vol. VI. 1866, p. 81), is an important one in the theory of foci. It is to be further noticed that we have

$$\mathfrak{A} + \mathfrak{B} - \mathfrak{A}_1 - \mathfrak{B}_1 = (\alpha - \beta)(\alpha' - \beta') z^2, = Kz^2 = K,$$

if $K, = (\alpha - \alpha')(\beta - \beta')$, be the squared distance of the points A, B, $= -$ squared distance of points A_1, B_1.

Article No. 66. *Antipoints of a Circle.*

66. A similar notion to that of two pairs of antipoints is as follows, viz., if from the centre of a circle perpendicular to its plane and in opposite senses, we measure off two distances each $= i$ into the radius, the extremities of these distances are antipoints of the circle. It is clear that the antipoints of the circle and the extremities of any diameter thereof are (in the plane of these four points) pairs of antipoints. It is to be added that each antipoint is the centre of a sphere radius zero, or say of a cone sphere, passing through the circle: the circle is thus the intersection of the two cone spheres having their centres at the two antipoints respectively.

Article No. 67. *Antipoints in relation to a Pair of Orthotomic Circles.*

67. It is a well-known property that if any circle pass through the points (A, B), and any other circle through the antipoints (A_1, B_1), then these two circles cut at right angles. Conversely if a circle pass through the points A, B, then all the orthotomic circles which have their centres on the line AB pass through the antipoints A_1, B_1. In particular, if on AB as diameter we describe a circle and on A_1B_1 as diameter a circle, then these two circles—being, it is clear, concentric circles with their radii in the ratio $1 : i$, and as concentric circles touching each other at the points (I, J)—cut each other at right angles; or say they are concentric orthotomic circles.

Article Nos. 68 to 71. *Forms of the Equation of a Circle.*

68. In rectangular coordinates the equation of a circle, coordinates of centre $(a, a', 1)$ and radius $= a''$, is

$$\mathfrak{A}^\circ = (x - az)^2 + (y - a'z)^2 - a''^2 z^2 = 0\,;$$

and in circular coordinates, the coordinates of the centre being $(\alpha, \alpha', 1)$, and radius $= a''$ as before, the equation is

$$\mathfrak{A}^\circ = (\xi - \alpha z) \quad (\eta - \alpha' z) - a''^2 z^2 = 0.$$

69. I observe in passing, that the origin being at the centre and the radius being $= 1$, then writing also $z = 1$, the equation of the circle is $\xi\eta = 1$, that is the circular coordinates of any point of the circle, expressed by means of a variable parameter θ, are $\left(\theta, \dfrac{1}{\theta}, 1\right)$.

70. Consider a current point P, the coordinates of which (rectangular) are $x, y, z (= 1)$, and (circular) are $\xi, \eta, z (= 1)$, then the foregoing expression

$$\begin{aligned}\mathfrak{A}^\circ &= (x - az)^2 + (y - a'z)^2 - a''^2 z^2 \\ &= (\xi - \alpha z)(\eta - \alpha' z) - a''^2 z^2\end{aligned}$$

denotes, it is clear, the square of the tangential distance of the point P from the circle $\mathfrak{A}^\circ = 0$.

71. But there is another interpretation of this same function $\mathfrak{A}^\circ$, viz., writing therein $z = 1$, and then

$$\mathfrak{A}^\circ = (x - a)^2 + (y - a')^2 + (a'' i)^2,$$

we see that $\mathfrak{A}^\circ$ is the squared distance of P from either of the antipoints of the circle (points lying, it will be recollected, out of the plane of the circle), and we have thus the theorem that the square of the tangential distance of any point P from the circle is equal to the square of its distance from either antipoint of the circle.

Article Nos. 72 to 77. *On a System of Sixteen Points.*

72. Take (A, B, C, D) any four concyclic points, and let the antipoints of

$$\begin{array}{llll}(B, C), & (A, D) & \text{be} & (B_1, C_1), \quad (A_1, D_1), \\ (C, A), & (B, D) & ,, & (C_2, A_2), \quad (B_2, D_2), \\ (A, B), & (C, D) & ,, & (A_3, B_3), \quad (C_3, D_3),\end{array}$$

then each of the three new sets (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), (A_3, B_3, C_3, D_3) will be a set of four concyclic points.

73. Let O be the centre of the circle through (A, B, C, D), say of the circle O, and then, the lines BC, AD meeting in R, the lines CA, BD in S, and the lines AD, CD in T, let each of these points be made the centre of a circle orthotomic to O, viz., let these new circles be called the circles R, S, T respectively.

As regards the circle R, since its centre lies in BC, the circle passes through (B_1, C_1); and since the centre lies in AD, the circle passes through (A_1, D_1), that is, the four points (A_1, B_1, C_1, D_1) lie in the circle R. Similarly (A_2, B_2, C_2, D_2) lie in the circle S, and (A_3, B_3, C_3, D_3) in the circle T.

74. The points R, S, T are conjugate points in relation to the circle O; that is, ST, TR, RS are the polars of R, S, T respectively in regard to this circle; and they are, consequently, at right angles to the lines OR, OS, OT respectively; viz., the four centres O, R, S, T are such that the line joining any two of them cuts at right angles the line joining the other two of them, and we see that the relation between the four sets is in fact a symmetrical one; this is most easily seen by consideration of the circular points at infinity I, J, the four sets of points may be arranged thus:

$$\begin{array}{cccc} A, & A_3, & A_2, & A_1, \\ B_3, & B, & B_1, & B_2, \\ C_2, & C_1, & C, & C_3, \\ D_1, & D_2, & D_3, & D, \end{array}$$

in such wise that any four of them in the same vertical line pass through I, and any four in the same horizontal line pass through J; and this being so, starting for instance with (A_3, B_3, C_3, D_3) we have antipoints

$$\begin{array}{llll} \text{of } (B_3, C_3), & (A_3, D_3) & \text{are } (B_2, C_2), & (A_2, D_2), \\ \text{,, } (C_3, A_3), & (B_3, D_3) & \text{,, } (C_1, A_1), & (B_1, D_1), \\ \text{,, } (A_3, B_3), & (C_3, D_3) & \text{,, } (A, B), & (C, D), \end{array}$$

and similarly if we start from (A_1, B_1, C_1, D_1) or (A_2, B_2, C_2, D_2).

75. I return for a moment to the construction of (A_1, B_1, C_1, D_1); these are points on the circle R, and (B_1, C_1) are the antipoints of (B, C); that is, they are the intersections of the circle R by the line at right angles to BC from its middle point, or, what is the same thing, by the perpendicular on BC from O. Similarly (A_1, D_1) are the antipoints of (A, D); that is, they are the intersections of the circle R by the perpendicular on AD from O. And the like as to (A_2, B_2, C_2, D_2) and (A_3, B_3, C_3, D_3) respectively.

76. Hence, starting with the points A, B, C, D on the circle O, and constructing as above the circles P, Q, R, and constructing also the perpendiculars from O on the six chords AB, AC, &c.,

$$\begin{array}{lllll} \text{the perpendiculars on } BC, AD & \text{meet circle } R & \text{in} & (B_1, C_1), & (A_1, D_1), \\ \text{,, } \quad CA, BD & \text{,, } \quad \text{,, } S & \text{,,} & (C_2, A_2), & (B_2, D_2), \\ \text{,, } \quad AB, CD & \text{,, } \quad \text{,, } T & \text{,,} & (A_3, B_3), & (C_3, D_3), \end{array}$$

so that the whole system is given by means of the circles P, Q, R, and the six perpendiculars.

77. If to fix the ideas (A, B, C, D) are real points taken in order on the real circle O, then the points R, S, T are each of them real; but R and T lie outside, S inside the circle O. The circles R and T are consequently real, but the circle S imaginary, viz., its radius is $=i$ into a real quantity; the imaginary points (A_1, B_1, C_1, D_1) are thus given as the intersections of a real circle by a pair of real lines, and the like as to the imaginary points (A_3, B_3, C_3, D_3); but the imaginary points (A_2, B_2, C_2, D_2) are only given as the intersections of an imaginary circle (centre real and radius a pure imaginary) by a pair of real lines. The points (C_2, A_2) *quâ* antipoints of (C, A) are easily constructed as the intersections of a real circle by a real line, and the like as to the points (B_2, D_2) *quâ* antipoints of (B, D), but the construction for the two pairs of points cannot be effected by means of the same real circle.

Article Nos. 78 to 80. *Property in regard to Four Confocal Conics.*

78. All the conics which pass through the four concyclic points A, B, C, D, have their axes in fixed directions; but three such conics are the line-pairs (BC, AD), (CA, BD), and (AB, CD), whence the directions of the axes are those of the bisectors of the angles formed by any one of these pairs of lines; hence, in particular, considering either axis of a conic through the four points, the lines AB and CD are equally inclined on opposite sides to this axis, and this leads to the theorem that the antipoints $(A_3, B_3)(C_3, D_3)$ are in a conic confocal to the given conic through (A, B, C, D); whence, also, considering any given conic whatever through (A, B, C, D), the points (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), (A_3, B_3, C_3, D_3) lie severally in three conics, each of them confocal with the given conic.

79. To prove this, consider any two confocal conics, say an ellipse and a hyperbola, and let F be one of their four intersections; join F with the common centre O, and let OT, ON be parallel to the tangent and normal respectively of the ellipse at the point F. OF, OT are in direction conjugate axes of the ellipse, and OF, ON are in direction conjugate axes of the hyperbola; and if they are also the axes in magnitude, that is, if the points T, N are the intersections of OT with the ellipse and of ON with the hyperbola respectively, then it is easy to show that $\overline{OT}^2+\overline{ON}^2=0$. And this being so, imagine on the ellipse any two points A, B such that the chord AB is parallel to OT, that is conjugate to OF; AB is bisected by OF, say in a point K, or we have parallel to OT the semichords or ordinates $KA=KB$; and we may, perpendicularly to this or parallel to ON, draw through K in the hyperbola a chord A_3B_3, which chord will be bisected in K, or we shall have $KA_3=KB_3$. Hence KA, KA_3 are in the ellipse and the hyperbola respectively ordinates conjugate to the same diameter OF, and the semi-diameters conjugate to OF being OT, ON respectively, we have $KA^2\ (=KB^2) : \overline{KA}_3{}^2\ (=KB_3{}^2) = \overline{OT}^2 : \overline{ON}^2$, this is, $\overline{KA}^2=\overline{KB}^2=-\overline{KA}_3{}^2=-\overline{KB}_3{}^2$; or (A_3, B_3) will be the antipoints of (A, B).

80. Conversely, if in the ellipse we have the two points (A, B), then drawing the diameter OF conjugate to AB, and through its extremity F, the confocal hyperbola, then the antipoints (A_3, B_3) will lie on the hyperbola. And similarly, if on the

ellipse we have the two points (C, D), then drawing the diameter OG conjugate to CD, and through its extremity G a confocal hyperbola, the antipoints (C_3, D_3) will lie on the hyperbola. Suppose (A, B, C, D) are concyclic, then, as noticed, AB and CD will be equally inclined on opposite sides to the transverse axis of the ellipse—the conjugate diameters OF, OG will therefore be equally inclined on opposite sides of the transverse axis—and the points F and G will therefore be situate symmetrically on opposite sides of the transverse axis, that is, the points F and G will respectively determine the same confocal hyperbola, and we have thus the required theorem, viz., if (A, B, C, D) are any four concyclic points on an ellipse, or say on a conic, and if (A_3, B_3) are the antipoints of (A, B), and (C_3, D_3) the antipoints of (C, D), then (A_3, B_3, C_3, D_3) will lie on a conic confocal with the given conic.

Article Nos. 81 to 85. *System of the Sixteen Points, the Axial Case.*

81. The theorems hold good when the four points A, B, C, D are in a line; the antipoints (B_1, C_1) of (B, C), &c., are in this case situate symmetrically on opposite sides of the line, so that it is evident at sight that we have (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), (A_3, B_3, C_3, D_3), each set in a circle; and that the centres R, S, T of these circles lie in the line. The construction for the general case becomes, however, indeterminate, and must therefore be varied. If in the general case we take any circle through (B, C), and any circle through (A, D), then the circle R cuts at right angles these two circles, and has, consequently, its centre R in the radical axis of the two circles; whence, when the four points are in a line, taking any circle through (B, C), or in particular the circle on BC as diameter, and any circle through (A, D), or in particular the circle on AD as diameter,—the radical axis of these two circles intersects the line in the required centre R, and the circle R is the circle with this centre cutting at right angles the two circles respectively; the circles S and T are, of course, obtained by the like construction in regard to the combinations $(C, A; B, D)$ and $(A, B; C, D)$, respectively. It may be added, that we have

$$\left.\begin{matrix} R \\ S \\ T \end{matrix}\right\} \text{centre and} \left\{\begin{matrix} \text{extremities} \\ \text{of diameter} \\ \text{of circles} \end{matrix} \begin{matrix} R \\ S \\ T \end{matrix}\right\} \text{sibiconjugate points of involutions} \left\{\begin{matrix} B, C; A, D, \\ C, A; B, D, \\ A, B; C, D, \end{matrix}\right.$$

and that (as in the general case) the circles R, S, T intersect each pair of them at right angles; and they are evidently each intersected at right angles by the line $ABCD$ (or axis of the figure), which replaces the circle O in the general case.

82. If the points A, B, C, D are taken in order on the line, then the points R, S, T are all real, viz., the point R is situate, on one side or the other, outside AD, but the points S and T are each of them situate between B and C; the circles R and T are real, but the circle S has its radius a pure imaginary quantity.

83. If one of the four points, suppose D, is at infinity on the line, then the antipoints of (A, D), of (B, D), and of (C, D) are each of them the two points (I, J).

It would at first sight appear that the only conditions for the circles R, S, T were the conditions of passing through the antipoints of (B, C), of (C, A), and of (A, B) respectively, and that these circles thus became indeterminate; but in fact the definition of the circles is then as follows, viz., R has its centre at A, and passes through the antipoints of (B, C): (whence squared radius $= AB \,.\, AC$). And similarly, S has its centre at B, and passes through antipoints of (C, A), (squared radius $= BA \,.\, BC$); and T has its centre at C, and passes through antipoints of (A, B), (squared radius $= CA \,.\, CB$); these three circles cut each other at right angles. As before, A, B, C being in order on the line, the circles R, T are real, but the circle S has its radius a pure imaginary quantity.

84. That the circles are as just mentioned appears as follows: taking the line as axis of x, and a, b, c, d for the x coordinates of the four points respectively, then the coordinates of A_1, D_1 are

$$\tfrac{1}{2}(a+d),\ \pm \tfrac{1}{2}i\,(a-d);$$

whence, m being arbitrary, the general equation of a circle through A_1, D_1 is

$$x^2+y^2-2mxz+[m(a+d)-ad]\,z^2=0,$$

writing herein

$$m=a-\frac{k^2}{d}$$

this becomes

$$x^2+y^2-2\left(a-\frac{k^2}{d}\right)xz+\left(a^2-k^2-\frac{ak^2}{d}\right)z^2=0,$$

viz., for $d=\infty$ it is

$$(x-az)^2+y^2-k^2z^2=0,$$

which is a circle having A for its centre, and its radius an arbitrary quantity k. If the circle passes through the antipoints of B, C, the coordinates of these are

$$\tfrac{1}{2}(b+c),\ \pm\tfrac{1}{2}i\,(b-c),$$

and we find

$$k^2=[\tfrac{1}{2}(b+c)-a]^2-\tfrac{1}{4}(b-c)^2=(a-b)(a-c).$$

85. Reverting to the general case of four points A, B, C, D on a line, the theorem as to the confocal conics holds good under the form that, drawing any conic whatever through (A_1, B_1, C_1, D_1), the points (A_2, B_2, C_2, D_2), and (A_3, B_3, C_3, D_3) lie in confocal conics, these conics have their centre on the line, and axes in the direction of and perpendicular to the line. When D is at infinity, the confocal conics become any three concentric circles through (B_1, C_1), (C_2, A_2), and (A_3, B_3) respectively.

Article Nos. 86 to 91. *The Involution of Four Circles.*

86. Consider any four points A, B, C, D, the centres of circles denoted by these same letters, and let $\mathfrak{A}^\circ$, $\mathfrak{B}^\circ$, $\mathfrak{C}^\circ$, $\mathfrak{D}^\circ$ signify as usual, viz., if (in orthogonal coordinates) $(a, a', 1)$ are the coordinates of the centre, and a'' the radius of the circle A, then $\mathfrak{A}^\circ$ stands for $(x-az)^2+(y-a'z)^2-a''^2z^2$, and the like for $\mathfrak{B}^\circ$, $\mathfrak{C}^\circ$, $\mathfrak{D}^\circ$. Write also

$$\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}=BCD:-CDA:DAB:-ABC,$$

where BCD, &c., are the triangles formed by the points (B, C, D), &c.; the analytical expressions are

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = \begin{vmatrix} b, & b', & 1 \\ c, & c', & 1 \\ d, & d', & 1 \end{vmatrix} : - \begin{vmatrix} c, & c', & 1 \\ d, & d', & 1 \\ a, & a', & 1 \end{vmatrix} : \begin{vmatrix} d, & d', & 1 \\ a, & a', & 1 \\ b, & b', & 1 \end{vmatrix} : - \begin{vmatrix} a, & a', & 1 \\ b, & b', & 1 \\ c, & c', & 1 \end{vmatrix}$$

so that

$$\mathrm{a} + \mathrm{b} + \mathrm{c} + \mathrm{d} = 0,$$

$$\mathrm{a}a + \mathrm{b}b + \mathrm{c}c + \mathrm{d}d = 0,$$

$$\mathrm{a}a' + \mathrm{b}b' + \mathrm{c}c' + \mathrm{d}d' = 0;$$

this being so, it is clear that we have

$$\mathrm{a}\mathfrak{A}^\circ + \mathrm{b}\mathfrak{B}^\circ + \mathrm{c}\mathfrak{C}^\circ + \mathrm{d}\mathfrak{D}^\circ =$$

$$z^2 [\mathrm{a}(a^2 + a'^2 - a''^2) + \mathrm{b}(b^2 + b'^2 - b''^2) + \mathrm{c}(c^2 + c'^2 - c''^2) + \mathrm{d}(d^2 + d'^2 - d''^2)] = Kz^2, \; = K,$$

a constant.

87. I am not aware that in the general case there is any convenient expression for this constant K; it is $=0$ when the four circles have the same orthotomic circle; in fact, taking as origin the centre of the orthotomic circle, and its radius to be $=1$, we have

$$a^2 + a'^2 - a''^2 = 1, \text{ \&c.,}$$

whence

$$K = \mathrm{a} + \mathrm{b} + \mathrm{c} + \mathrm{d} = 0;$$

that is, if the circles A, B, C, D have the same orthotomic circle, then $\mathfrak{A}^\circ, \mathfrak{B}^\circ, \mathfrak{C}^\circ, \mathfrak{D}^\circ$, a, b, c, d, signifying as above, we have

$$\mathrm{a}\mathfrak{A}^\circ + \mathrm{b}\mathfrak{B}^\circ + \mathrm{c}\mathfrak{C}^\circ + \mathrm{d}\mathfrak{D}^\circ = 0,$$

and, in particular, if the circles reduce themselves to the points A, B, C, D respectively, then (writing as usual $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ in place of $\mathfrak{A}^\circ, \mathfrak{B}^\circ, \mathfrak{C}^\circ, \mathfrak{D}^\circ$) if the four points A, B, C, D are on a circle, we have

$$\mathrm{a}\mathfrak{A} + \mathrm{b}\mathfrak{B} + \mathrm{c}\mathfrak{C} + \mathrm{d}\mathfrak{D} = 0.$$

88. This last theorem may be regarded as a particular case of the theorem

$$\mathrm{a}\mathfrak{A} + \mathrm{b}\mathfrak{B} + \mathrm{c}\mathfrak{C} + \mathrm{d}\mathfrak{D} = Kz^2 = K,$$

viz., the four circles reducing themselves to the points A, B, C, D, we can find for the constant K an expression which will of course vanish when the points are on a circle. For this purpose, let the lines BC, AD meet in R, the lines CA, BD in S, and the lines AB, CD in T; we may, to fix the ideas, consider $ABCD$ as forming a convex quadrilateral, R and T will then be the exterior centres, S the interior centre; a, b, c, d, may be taken *equal* to BCD, $-CDA$, DAB, $-ABC$, where the areas BCD, &c., are each taken positively. The expression $\mathrm{a}\mathfrak{A} + \mathrm{b}\mathfrak{B} + \mathrm{c}\mathfrak{C} + \mathrm{d}\mathfrak{D}$ has the same value, whatever is the position of the point P $(x, y, z=1)$; taking this point at R, and writing for a moment

$$RA = \alpha, \quad RB = \beta, \quad RC = \gamma, \quad RD = \delta,$$

then

$$BCD = (RCD - RBD) = \tfrac{1}{2}RD\,(RC - RB)\sin R = (\gamma - \beta)\,\delta\sin R,$$

with similar expressions for the other triangles; and we thus have

$$\mathfrak{a}\mathfrak{A} + \mathfrak{b}\mathfrak{B} + \mathfrak{c}\mathfrak{C} + \mathfrak{d}\mathfrak{D} = \tfrac{1}{2}z^2 . \sin R \left\{ \begin{array}{l} \alpha^2(\gamma - \beta)\,\delta \\ -\beta^2(\delta - \alpha)\,\gamma \\ +\gamma^2(\delta - \alpha)\,\beta \\ -\delta^2(\gamma - \beta)\,\alpha \end{array} \right\} = \tfrac{1}{2}z^2 \sin R\,(\beta\gamma - \alpha\delta)(\gamma - \beta)(\delta - \alpha),$$

that is, replacing α, β, γ, δ, by their values, and writing also $z = 1$, we have

$$\mathfrak{a}\mathfrak{A} + \mathfrak{b}\mathfrak{B} + \mathfrak{c}\mathfrak{C} + \mathfrak{d}\mathfrak{D} = \tfrac{1}{2}\sin R . (RB . RC - RA . RD)\, BC . AD,$$

where $\frac{1}{2}\sin R . BC . AD$ is in fact the area of the quadrilateral $ABCD$; we have thus

$$\begin{aligned} \mathfrak{a}\mathfrak{A} + \mathfrak{b}\mathfrak{B} + \mathfrak{c}\mathfrak{C} + \mathfrak{d}\mathfrak{D} &= (RB . RC - RA . RD)\,\square \\ &= (SC . SA - SB . SD)\,\square \\ &= (TA . TB - TC . TD)\,\square \end{aligned}$$

where it is to be observed that SA, SC being measured in opposite directions from S, must be considered, one as positive, the other as negative, and the like as regards SB, SD. This expression for the value of the constant is due to Mr Crofton. In the particular case where A, B, C, D, are on a circle, we have as before

$$\mathfrak{a}\mathfrak{A} + \mathfrak{b}\mathfrak{B} + \mathfrak{c}\mathfrak{C} + \mathfrak{d}\mathfrak{D} = 0.$$

89. If the four points A, B, C, D, are on a circle, then, taking as origin the centre of this circle and its radius as unity, the circular coordinates of the four points will be

$$\left(\alpha, \frac{1}{\alpha}, 1\right), \quad \left(\beta, \frac{1}{\beta}, 1\right), \quad \left(\gamma, \frac{1}{\gamma}, 1\right), \quad \left(\delta, \frac{1}{\delta}, 1\right),$$

the corresponding forms of $\mathfrak{A}^\circ$, &c., being

$$\mathfrak{A}^\circ = (\xi - \alpha z)\left(\eta - \frac{1}{\alpha}z\right) - \alpha''^2 z^2, \text{ \&c.}$$

the expressions for $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, observing that we have

$$\left| \begin{array}{lll} \beta, & \beta^{-1}, & 1 \\ \gamma, & \gamma^{-1}, & 1 \\ \delta, & \delta^{-1}, & 1 \end{array} \right| = \frac{1}{\beta\gamma\delta} \left| \begin{array}{lll} 1, & \beta, & \beta^2 \\ 1, & \gamma, & \gamma^2 \\ 1, & \delta, & \delta^2 \end{array} \right| = \frac{1}{\beta\gamma\delta}\,(\beta\gamma\delta), \text{ \&c.}$$

if $(\beta\gamma\delta)$, &c. denote $(\beta - \gamma)(\gamma - \delta)(\delta - \beta)$, &c., become

$$\mathfrak{a} : \mathfrak{b} : \mathfrak{c} : \mathfrak{d} = \alpha(\beta\gamma\delta) : -\beta(\gamma\delta\alpha) : \gamma(\delta\alpha\beta) : -\delta(\alpha\beta\gamma),$$

which are convenient formulæ for the case in question.

90. If the points A, B, C, D, are on a line, then taking this line for the axis of x, we may write $\mathfrak{A}^\circ = (x-az)^2 + y^2 - a''^2 z^2$, &c. It is to be remarked here that we can, without any relation whatever between the radii of the circles, satisfy the equation

$$\mathrm{a}\mathfrak{A}^\circ + \mathrm{b}\mathfrak{B}^\circ + \mathrm{c}\mathfrak{C}^\circ + \mathrm{d}\mathfrak{D}^\circ = 0;$$

in fact this will be the case if we have

$$\begin{aligned} &\mathrm{a} + \mathrm{b} + \mathrm{c} + \mathrm{d} = 0, \\ &\mathrm{a}a + \mathrm{b}b + \mathrm{c}c + \mathrm{d}d = 0, \\ &\mathrm{a}(a^2 - a''^2) + \mathrm{b}(b^2 - b''^2) + \mathrm{c}(c^2 - c''^2) + \mathrm{d}(d^2 - d''^2) = 0, \end{aligned}$$

equations which determine the ratios a : b : c : d. In the case where the circles reduce themselves to the points A, B, C, D, these equations become

$$\begin{aligned} &\mathrm{a} + \mathrm{b} + \mathrm{c} + \mathrm{d} = 0, \\ &\mathrm{a}a + \mathrm{b}b + \mathrm{c}c + \mathrm{d}d = 0, \\ &\mathrm{a}a^2 + \mathrm{b}b^2 + \mathrm{c}c^2 + \mathrm{d}d^2 = 0, \end{aligned}$$

giving

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = (bcd) : -(cda) : (dab) : -(abc);$$

if for shortness (bcd), &c. stand for $(b-c)(c-d)(d-b)$, &c.; and for these values, we have

$$\mathrm{a}\mathfrak{A} + \mathrm{b}\mathfrak{B} + \mathrm{c}\mathfrak{C} + \mathrm{d}\mathfrak{D} = 0.$$

91. A very noticeable case is when the four circles are such that the foregoing values of (a, b, c, d) also satisfy the equation

$$\mathrm{a}\mathfrak{A}^\circ + \mathrm{b}\mathfrak{B}^\circ + \mathrm{c}\mathfrak{C}^\circ + \mathrm{d}\mathfrak{D}^\circ = 0;$$

the condition for this is obviously

$$\mathrm{a}a''^2 + \mathrm{b}b''^2 + \mathrm{c}c''^2 + \mathrm{d}d''^2 = 0;$$

or, as it may also be written,

$$\frac{a''^2}{(a-b)(a-c)(a-d)} + \frac{b''^2}{(b-c)(b-d)(b-a)} + \frac{c''^2}{(c-d)(c-a)(c-b)} + \frac{d''^2}{(d-a)(d-b)(d-c)} = 0.$$

Article No. 92. *On a Locus connected with the foregoing Properties.*

92. If, as above, A, B, C, D are any four points, and $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ are the squared distances of a current point P from the four points respectively, then the locus of the foci of the conics which pass through the four points is the tetrazomal curve

$$\mathrm{a}\sqrt{\mathfrak{A}} + \mathrm{b}\sqrt{\mathfrak{B}} + \mathrm{c}\sqrt{\mathfrak{C}} + \mathrm{d}\sqrt{\mathfrak{D}} = 0.$$

In fact the sum $\mathrm{a}\mathfrak{A} + \mathrm{b}\mathfrak{B} + \mathrm{c}\mathfrak{C} + \mathrm{d}\mathfrak{D}$ has, it has been seen, a constant value for all positions of the point P; taking P to be the other focus, its squared distances are $(k-\sqrt{A})^2$, &c., whence for the first-mentioned focus we have

$$\mathrm{a}\mathfrak{A} + \mathrm{b}\mathfrak{B} + \mathrm{c}\mathfrak{C} + \mathrm{d}\mathfrak{D} = \mathrm{a}(k-\sqrt{\mathfrak{A}})^2 + \mathrm{b}(k-\sqrt{\mathfrak{B}})^2 + \mathrm{c}(k-\sqrt{\mathfrak{C}})^2 + \mathrm{d}(k-\sqrt{\mathfrak{D}})^2;$$

or recollecting that $\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}=0$, it follows that we have for the locus in question $\mathrm{a}\sqrt{\mathfrak{A}}+\mathrm{b}\sqrt{\mathfrak{B}}+\mathrm{c}\sqrt{\mathfrak{C}}+\mathrm{d}\sqrt{\mathfrak{D}}=0$; this locus will be discussed in the sequel. I remark here, that in the case where the four points are on a circle, then (as mentioned above), the axes of the several conics are in the same fixed directions; there are thus two sets of foci, those on the axis in one direction, and those on the axis in the other direction; it might therefore be anticipated, and it will appear, that in this case the tetrazomal breaks up into two trizomal curves.

Article Nos. 93 to 98. *Formulæ as to the two Sets* (A, B, C, D), *and* (A_1, B_1, C_1, D_1), *each of four Concyclic Points.*

93. Consider the four points A, B, C, D on a circle, then taking, as before, their circular coordinates to be $(\alpha, \alpha', 1)$, $(\beta, \beta', 1)$, $(\gamma, \gamma', 1)$, $(\delta, \delta', 1)$, the condition that the points may be on a circle is

$$\begin{vmatrix} 1, & \alpha, & \alpha', & \alpha\alpha' \\ 1, & \beta, & \beta', & \beta\beta' \\ 1, & \gamma, & \gamma', & \gamma\gamma' \\ 1, & \delta, & \delta', & \delta\delta' \end{vmatrix}=0,$$

viz., this equation may be written

$$\begin{aligned}&(\beta-\gamma)(\alpha-\delta) : (\gamma-\alpha)(\beta-\delta) : (\alpha-\beta)(\gamma-\delta)\\ =&(\beta'-\gamma')(\alpha'-\delta') : (\gamma'-\alpha')(\beta'-\delta') : (\alpha'-\beta')(\gamma'-\delta');\end{aligned}$$

or if, for shortness, we take

$$\begin{aligned}a&=\beta-\gamma, & f&=\alpha-\delta, & a'&=\beta'-\gamma', & f'&=\alpha'-\delta',\\ b&=\gamma-\alpha, & g&=\beta-\delta, & b'&=\gamma'-\alpha', & g'&=\beta'-\delta',\\ c&=\alpha-\beta, & h&=\gamma-\delta, & c'&=\alpha'-\beta', & h'&=\gamma'-\delta',\end{aligned}$$

and consequently

$$\begin{aligned}&af+bg+ch=0, && a'f'+b'g'+c'h'=0,\\ &a \quad =g-h, && a' \quad =g'-h',\\ &b \quad =h-f, && b' \quad =h'-f',\\ &c \quad =f-g, && c' \quad =f'-g',\\ &a+b+c=0, && a'+b'+c'=0,\end{aligned}$$

then the equation is

$$af : bg : ch = a'f' : b'g' : c'h'.$$

94. Let a, b, c, d, denote as before $(\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}=BCD:-CDA:DAB:-ABC)$, then we have

$$\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}=\begin{vmatrix}\beta, & \beta', & 1\\ \gamma, & \gamma', & 1\\ \delta, & \delta', & 1\end{vmatrix} : -\begin{vmatrix}\gamma, & \gamma', & 1\\ \delta, & \delta', & 1\\ \alpha, & \alpha', & 1\end{vmatrix} : \begin{vmatrix}\delta, & \delta', & 1\\ \alpha, & \alpha', & 1\\ \beta, & \beta', & 1\end{vmatrix} : -\begin{vmatrix}\alpha, & \alpha', & 1\\ \beta, & \beta', & 1\\ \gamma, & \gamma', & 1\end{vmatrix}$$

and we may write

$$\begin{aligned}
\mathrm{a} &= \quad . \quad, \quad ah'-a'h, \quad ag'-a'g, \quad gh'-g'h,\\
\mathrm{b} &= bh'-b'h, \quad . \quad, \quad bf'-b'f, \quad hf'-h'f,\\
\mathrm{c} &= cg'-c'g, \quad cf'-c'f, \quad . \quad, \quad fg'-f'g,\\
\mathrm{d} &= cb'-c'b, \quad ac'-a'c, \quad ba'-b'a, \quad ,
\end{aligned}$$

viz., the expressions in the same horizontal line are equal, and a, b, c, d are proportional to the expressions in the four lines respectively.

95. I say that we have

$$\frac{c'f'}{ah}\,\mathrm{a} = \frac{c'g'}{bh}\,\mathrm{b} = \frac{a'f'}{af}\,\mathrm{c} = \frac{f'g'}{ab}\,\mathrm{d},$$

viz., this will be the case if

$$\begin{aligned}
bc'\mathrm{a} &= hg'\,\mathrm{d},\\
ac'\mathrm{b} &= hf'\mathrm{d},\\
a'b\mathrm{c} &= fg'\,\mathrm{d},
\end{aligned}$$

and selecting the convenient expressions for a, b, c, d, these equations become

$$\begin{aligned}
bc'\,(gh'-g'h) &= g'h\,(cb'-c'b),\\
ac'\,(hf'-h'f) &= f'h\,(ac'-a'c),\\
a'b\,(fg'-f'g) &= fg'\,(ba'-b'a),
\end{aligned}$$

viz., these equations are respectively $bgc'h' = b'g'ch$, $cha'f' = c'h'af$, $afb'g' = a'f'bg$, and are consequently satisfied. It thus appears that the equation

$$\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0$$

is transformable into

$$\frac{c'f'}{ah}\,l + \frac{c'g'}{bh}\,m + \frac{a'f'}{af}\,n + \frac{f'g'}{ab}\,p = 0,$$

which is of course one of a system of similar forms.

96. Take (A_1, D_1) the antipoints of (A, D); (B_1, C_1) the antipoints of (B, C); or say that the circular coordinates of A_1, B_1, C_1, D_1 are $(\alpha, \delta', 1)$, $(\beta, \gamma', 1)$, $(\gamma, \beta', 1)$, $(\delta, \alpha', 1)$ respectively; the points A_1, B_1, C_1, D_1 are, as above mentioned, on a circle, the condition that this may be so being in fact

$$\begin{vmatrix}
1, & \alpha, & \delta', & \alpha\delta'\\
1, & \beta, & \gamma', & \beta\gamma'\\
1, & \gamma, & \beta', & \gamma\beta'\\
1, & \delta, & \alpha', & \delta\alpha'
\end{vmatrix} = 0,$$

equivalent to

$$af : bg : ch = a'f' : b'g' : c'h'.$$

97. Let (a_1, b_1, c_1, d_1) be the corresponding quantities to (a, b, c, d), viz., $\mathrm{a}_1 : \mathrm{b}_1 : \mathrm{c}_1 : \mathrm{d}_1 = B_1C_1D_1 : -C_1D_1A_1 : D_1A_1B_1 : -A_1B_1C_1$; we have

$$\mathrm{a}_1 : \mathrm{b}_1 : \mathrm{c}_1 : \mathrm{d}_1 = \begin{vmatrix} \beta, & \gamma', & 1 \\ \gamma, & \beta', & 1 \\ \delta, & \alpha', & 1 \end{vmatrix} : -\begin{vmatrix} \gamma, & \beta', & 1 \\ \delta, & \alpha', & 1 \\ \alpha, & \delta', & 1 \end{vmatrix} : \begin{vmatrix} \delta, & \alpha', & 1 \\ \alpha, & \delta', & 1 \\ \beta, & \gamma', & 1 \end{vmatrix} : -\begin{vmatrix} \alpha, & \delta', & 1 \\ \beta, & \gamma', & 1 \\ \gamma, & \beta', & 1 \end{vmatrix}$$

giving rise to a similar set of forms

$$\begin{aligned}
\mathrm{a}_1 &= \quad . \quad , -ac' + ha', \quad a'g + b'a, \quad -c'g - b'h, \\
\mathrm{b}_1 &= -c'b - g'h, \quad . \quad , -f'b - g'f, \quad -f'h + c'f, \\
\mathrm{c}_1 &= b'c + h'g, \quad -f'c + h'f, \quad . \quad , f'g + g'f, \\
\mathrm{d}_1 &= g'c + h'b, \quad -h'a + a'c, \quad -a'b - g'a, \quad . \quad ,
\end{aligned}$$

and leading to

$$\frac{cf}{a'c'}\mathrm{a}_1 = -\frac{cg}{c'g'}\mathrm{b}_1 = \frac{af}{a'f'}\mathrm{c}_1 = -\frac{fg}{a'g'}\mathrm{d}_1,$$

so that the equation

$$\frac{l_1}{\mathrm{a}_1} + \frac{m_1}{\mathrm{b}_1} + \frac{n_1}{\mathrm{c}_1} + \frac{p_1}{\mathrm{d}_1} = 0,$$

is transformable into

$$\frac{cf}{a'c'}l_1 - \frac{cg}{c'g'}m_1 + \frac{af}{a'f'}n_1 - \frac{fg}{a'g'}p_1 = 0.$$

98. Let A, B, C, D, be, as above, points on a circle; (A_1, D_1) and (B_1, C_1) the antipoints of (A, D), (B, C) respectively. Write

$$\begin{aligned}
\mathfrak{A} &= (\xi - \alpha z)(\eta - \alpha' z), & \mathfrak{A}_1 &= (\xi - \alpha z)(\eta - \delta' z), \\
\mathfrak{B} &= (\xi - \beta z)(\eta - \beta' z), & \mathfrak{B}_1 &= (\xi - \beta z)(\eta - \gamma' z), \\
\mathfrak{C} &= (\xi - \gamma z)(\eta - \gamma' z), & \mathfrak{C}_1 &= (\xi - \gamma z)(\eta - \beta' z), \\
\mathfrak{D} &= (\xi - \delta z)(\eta - \delta' z), & \mathfrak{D}_1 &= (\xi - \delta z)(\eta - \alpha' z);
\end{aligned}$$

then we have identically

$$\begin{aligned}
(\delta-\alpha)(\delta'-\alpha')\mathfrak{B} &= (\beta-\delta)(\beta'-\delta')\mathfrak{A} + (\beta-\alpha)(\beta'-\alpha')\mathfrak{D} - (\beta-\delta)(\beta'-\alpha')\mathfrak{A}_1 - (\beta-\alpha)(\beta'-\delta')\mathfrak{D}_1, \\
(\delta-\alpha)(\delta'-\alpha')\mathfrak{C} &= (\gamma-\delta)(\gamma'-\delta')\mathfrak{A} + (\gamma-\alpha)(\gamma'-\alpha')\mathfrak{D} - (\gamma-\delta)(\gamma'-\alpha')\mathfrak{A}_1 - (\gamma-\alpha)(\gamma'-\delta')\mathfrak{D}_1, \\
(\delta-\alpha)(\delta'-\alpha')\mathfrak{B}_1 &= (\beta-\delta)(\gamma'-\delta')\mathfrak{A} + (\beta-\alpha)(\gamma'-\alpha')\mathfrak{D} - (\beta-\delta)(\gamma'-\alpha')\mathfrak{A}_1 - (\beta-\alpha)(\gamma'-\delta')\mathfrak{D}_1, \\
(\delta-\alpha)(\delta'-\alpha')\mathfrak{C}_1 &= (\gamma-\delta)(\beta'-\delta')\mathfrak{A} + (\gamma-\alpha)(\beta'-\alpha')\mathfrak{D} - (\gamma-\delta)(\beta'-\alpha')\mathfrak{A}_1 - (\gamma-\alpha)(\beta'-\delta')\mathfrak{D}_1,
\end{aligned}$$

or, in the foregoing notation,

$$\begin{aligned}
ff'\mathfrak{B} &= gg'\mathfrak{A} + cc'\mathfrak{D} + gc'\mathfrak{A}_1 + cg'\mathfrak{D}_1, \\
ff'\mathfrak{C} &= hh'\mathfrak{A} + bb'\mathfrak{D} - hb'\mathfrak{A}_1 - bh'\mathfrak{D}_1, \\
ff'\mathfrak{B}_1 &= gh'\mathfrak{A} - cb'\mathfrak{D} - gb'\mathfrak{A}_1 + ch'\mathfrak{D}_1, \\
ff'\mathfrak{C}_1 &= hg'\mathfrak{A} - bc'\mathfrak{D} + hc'\mathfrak{A}_1 - bg'\mathfrak{D}_1.
\end{aligned}$$

Article Nos. 99 to 104. *Further Properties in relation to the same Sets* (A, B, C, D) *and* (A_1, B_1, C_1, D_1).

99. It is to be shown that in virtue of these equations, and if moreover $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0$, then it is possible to find l_1, m_1, n_1, p_1, such that we have identically

$$-l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}-p\mathfrak{D}+l_1\mathfrak{A}_1-m_1\mathfrak{B}_1-n_1\mathfrak{C}_1+p_1\mathfrak{D}_1=0.$$

This equation will in fact be identically true if only

$$\begin{array}{llll} -ff'l+gg'm+hh'n & . & -gh'm_1-g'hn_1 & =0, \\ cc'm+bb'n-ff'p & . & +cb'm_1+bc'n_1 & =0, \\ gc'm-hb'n & +ff'l_1 & +gb'm_1-hc'n_1 & =0, \\ cg'm-bh'n & . & +ch'm_1+bg'n_1+ff'p_1 & =0. \end{array}$$

From the first and second equations eliminating m_1 or n_1, the other of these quantities disappears of itself, and we thus obtain two equations which must be equivalent to a single one, viz., we have

$$bc'ff'l+c'g'afm+bha'f'n+g'hff'p=0,$$
$$b'cff'l+cga'f'm+b'h'afn+gh'ff'p=0;$$

which equations may also be written

$$\frac{c'f'}{ah}l+\frac{c'g'}{bh}m+\frac{a'f'}{af}n+\frac{f'g'}{ab}p=0,$$

$$\frac{cf}{a'h'}l+\frac{cg}{b'h'}m+\frac{af}{a'f'}n+\frac{fg}{a'b'}p=0;$$

and it thus appears that the equations are equivalent to each other, and to the assumed relation

$$\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0.$$

100. Similarly, from the third and fourth equations eliminating m or n, the other of these quantities disappears of itself, and we find

$$cg'ff'l_1-cga'f'm_1+afc'g'n_1-c'gff'p_1=0,$$
$$bh'ff'l_1-afb'h'm_1+bha'f'n_1-b'hff'p_1=0,$$

equations which may be written

$$\frac{cf}{a'c'}l-\frac{cg}{c'g'}m+\frac{af}{a'f'}n-\frac{fg}{g'a'}p=0,$$

$$\frac{f'h'}{ah}l-\frac{b'h'}{bg}m+\frac{a'f'}{af}n-\frac{b'g'}{ab}p=0,$$

where we see that the two equations are equivalent to each other and to the equation

$$\frac{l_1}{\mathrm{a}_1}+\frac{m_1}{\mathrm{b}_1}+\frac{n_1}{\mathrm{c}_1}+\frac{p_1}{\mathrm{d}_1}=0.$$

It thus appears that the quantities l_1, m_1, n_1, p_1, must satisfy this last equation. It is to be observed that the first and second equations being, as we have seen, equivalent to a single equation, either of the quantities m_1, n_1, may be assumed at pleasure, but the other is then determined; the third and fourth equations then give l_1, p_1; and the quantities l_1, m_1, n_1, p_1, so obtained, satisfy *identically* the equation $\frac{l_1}{\mathrm{a}_1}+\frac{m_1}{\mathrm{b}_1}+\frac{n_1}{\mathrm{c}_1}+\frac{p_1}{\mathrm{d}_1}=0$.

101. Now writing

$$ff'l_1=-g(c'm+b'm_1)+h(b'n+c'n_1),$$
$$ff'p_1=-c(g'm-h'm_1)+b(h'n-g'n_1),$$

and

$$ff'p=\quad c(c'm+b'm_1)+b(b'n+c'n_1),$$
$$ff'l=\quad g(g'm-h'm_1)+h(h'n-g'n_1),$$

we find

$$\begin{aligned}f^2f'^2(l_1p_1-lp)&=-(bg+ch)[(c'm+b'm_1)(h'n-g'n_1)+(g'm-h'm_1)(b'n+c'n_1)],\\&=(bg+ch)\ (b'g'+c'h')\ (m_1n_1-mn),\\&=aa'ff'(m_1n_1-mn),\end{aligned}$$

that is

$$ff'(l_1p_1-lp)=aa'(m_1n_1-mn)$$

viz., this equation is satisfied identically by the values of l_1, m_1, n_1, p_1 determined as above.

102. Hence if $m_1n_1=mn$, we have also $l_1p_1=lp$, and we can determine m_1, n_1, so that m_1n_1 shall $=mn$, viz., in the first or second of the four equations (these two being equivalent to each other, as already mentioned), writing $m_1=\theta n$, and therefore $n_1=\frac{1}{\theta}m$, we have

$$-ff'l+gg'm+hh'n-gh'n\theta-g'hm\frac{1}{\theta}=0,$$

$$cc'm+bb'n-ff'p+cb'n\theta+bc'm\frac{1}{\theta}=0,$$

which are, in fact, the same quadric equation in θ, viz., we have

$$\frac{-ff'l+gg'm+hh'n}{cc'm+bb'n-ff'p}=-\frac{gh'}{cb'}=-\frac{g'h}{bc'}$$

The final result is that there are two sets of values of l_1, m_1, n_1, p_1, each satisfying the identity

$$-l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}-p\mathfrak{D}+l_1\mathfrak{A}_1-m_1\mathfrak{B}_1-n_1\mathfrak{C}_1+p_1\mathfrak{D}_1=0,$$

and for each of which we have

$$\frac{l_1}{\mathrm{a}_1}+\frac{m_1}{\mathrm{b}_1}+\frac{n_1}{\mathrm{c}_1}+\frac{p_1}{\mathrm{d}_1}=0,\quad l_1p_1=lp,\ m_1n_1=mn.$$

103. Consider, in particular, the case where $p=0$; the relation

$$\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0,$$

here becomes

$$l=-\frac{ag'}{bf}m-\frac{a'h}{c'f}n.$$

The equation in θ is

$$(cc'm+bb'n)\,\theta+cb'n\theta^2+bc'm=0,$$

viz., this is

$$(c\theta+c'm)(b'n\theta+b)=0,$$

giving

$$\theta=-\frac{b}{c}\ ,\quad m_1=-\frac{bn}{c}\ ,\quad n_1=-\frac{cm}{b},$$

or else

$$\theta=-\frac{c'm}{b'n},\quad m_1=-\frac{c'm}{b'},\quad n_1=-\frac{b'n}{c'}.$$

Since in the present case $l_1p_1=0$, we have either $l_1=0$, or else $p_1=0$, and as might be anticipated, the two values of θ correspond to these two cases respectively, viz., proceeding to find the values of l_1, p_1, the completed systems are

$$\theta=-\frac{b}{c}\ ,\quad l_1=\frac{a}{bcf'}\left(cc'm-bb'n\right),\quad m_1=-\frac{bn}{c},\quad n_1=-\frac{cm}{b},\quad p_1=0,$$

$$\theta=-\frac{c'm}{b'n},\quad l\grave{}_1=0\qquad\qquad ,\quad m\grave{}_1=-\frac{c'm}{b'},\quad n\grave{}_1=-\frac{b'n}{c'},\quad p\grave{}_1=\frac{a'}{b'c'f}\left(cc'm-bb'n\right),$$

so that for the first system we have

$$\frac{l_1}{\mathrm{a}_1}+\frac{m_1}{\mathrm{b}_1}+\frac{n_1}{\mathrm{c}_1}=0,\quad m_1n_1=mn,\quad -l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}=-l_1\mathfrak{A}_1+m_1\mathfrak{B}_1+n_1\mathfrak{C}_1,$$

and for the second system

$$\frac{m\grave{}_1}{\mathrm{b}_1}+\frac{n\grave{}_1}{\mathrm{c}_1}+\frac{p\grave{}_1}{\mathrm{d}_1}=0,\quad m\grave{}_1n\grave{}_1=mn,\quad -l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}=-p\grave{}_1\mathfrak{D}_1+m\grave{}_1\mathfrak{B}_1+n\grave{}_1\mathfrak{C}_1.$$

104. The whole of the foregoing investigation would have assumed a more simple form if the circular coordinates had been taken with reference to the centre of the circle $ABCD$ as origin, and the radius of this circle been put $=1$; we should then have $\alpha'=\frac{1}{\alpha}$, &c., and consequently

$$a'=-\frac{1}{\beta\gamma}a,\quad b'=-\frac{1}{\gamma\alpha}b,\quad c'=-\frac{1}{\alpha\beta}c,\quad f'=-\frac{1}{\alpha\delta}f,\quad g'=-\frac{1}{\beta\delta}g,\quad h'=-\frac{1}{\gamma\delta}h;$$

but the symmetrical relation of the circles $ABCD$ and $A_1B_1C_1D_1$ would not have been so clearly shown.

I will however give the investigation in this simplified form, for the identity $-l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}=-l_1\mathfrak{A}+m_1\mathfrak{B}+n_1\mathfrak{C}$; viz., in this case we have

$$\frac{l}{\alpha}=-\frac{m}{\beta}\frac{(\beta-\gamma)(\beta-\delta)}{(\gamma-\alpha)(\alpha-\delta)}-\frac{n}{\gamma}\frac{(\beta-\gamma)(\gamma-\delta)}{(\alpha-\beta)(\alpha-\gamma)},$$

and the identity to be satisfied is

$$\begin{aligned}-l\,(\xi-\alpha z)\left(\eta-\frac{1}{\alpha}z\right)&=-l_1\,(\xi-\alpha z)\left(\eta-\frac{1}{\delta}z\right)\\ +m\,(\xi-\beta z)\left(\eta-\frac{1}{\beta}z\right)&\quad+m_1\,(\xi-\beta z)\left(\eta-\frac{1}{\gamma}z\right)\\ +n\,(\xi-\gamma z)\left(\eta-\frac{1}{\gamma}z\right)&\quad+n_1\,(\xi-\gamma z)\left(\eta-\frac{1}{\beta}z\right);\end{aligned}$$

writing $\xi=\alpha z$, $\eta=\frac{1}{\beta}z$, we find m_1, and writing $\xi=\alpha z$, $\eta=\frac{1}{\gamma}z$, we find n_1, and it is then easy to obtain the value of l_1, viz., the results are

$$\frac{l_1}{\delta}=\frac{m}{\beta}\frac{(\alpha-\beta)(\beta-\gamma)}{(\gamma-\alpha)(\alpha-\delta)}+\frac{n}{\gamma}\frac{(\beta-\gamma)(\gamma-\alpha)}{(\alpha-\beta)(\alpha-\delta)},\quad m_1=-n\frac{\gamma-\alpha}{\alpha-\beta},\quad n_1=-m\frac{\alpha-\beta}{\gamma-\alpha},$$

and therefore $m_1n_1=mn$; it may be added that we have

$$-\frac{l_1}{\delta}=\frac{\beta-\gamma}{\alpha-\delta}\left(\frac{m_1}{\gamma}+\frac{n_1}{\delta}\right),$$

viz., this is the form assumed by the equation $\frac{l_1}{a_1}+\frac{m_1}{b_1}+\frac{n_1}{c_1}=0$.

Part III. (Nos. 105 to 157). On the Theory of Foci.

Article Nos. 105 to 110. *Explanation of the General Theory.*

105. If from a focus of a conic we draw two tangents to the curve, these pass respectively through the two circular points at infinity, and we have thence the generalised definition of a focus as established by Plücker, viz., in any curve a focus is a point such that the lines joining it with the two circular points at infinity are respectively tangents to the curve; or, what is the same thing, if from each of the circular points at infinity, say from the points I, J, tangents are drawn to the curve, the intersections of each tangent from the one point with each tangent from the other point are the foci of the curve. A curve of the class n has thus in general n^2 foci. It is to be added that, as in the conic the line joining the points of contact of the two tangents from a focus is the directrix corresponding to that focus, so in general the line joining the points of contact of the tangents from the focus through the points I, J respectively is the directrix corresponding to the focus in question.

106. A circular point at infinity I or J, may be an ordinary or a singular point on the curve, and the tangent at this point then counts, or, in the case of a multiple point, the tangents at this point count a certain number of times, say q times, among the tangents which can be drawn to the curve from the point; the number of the remaining tangents is thus $= n - q$. In particular, if the circular point at infinity be an ordinary point, then the tangent counts twice, or we have $q = 2$; if it be a node, each of the tangents counts twice, or $q = 4$; if it be a cusp, the tangent counts three times, or $q = 3$. Similarly, if the other circular point an infinity be an ordinary or a singular point on the curve, the tangent or tangents there count a certain number of times, say q' times, among the tangents to the curve from this point; the number of the remaining tangents is thus $= n - q'$. And if as usual we disregard the tangents at the two points I, J respectively, and attend only to the remaining tangents, the number of the foci is $= (n - q)(n - q')$.

107. Among the tangents from the point I or J there may be a tangent which, either from its being a multiple tangent (that is, a tangent having ordinary contact at two or more distinct points), or from being an osculating tangent at one or more points, counts a certain number of times, say r, among the tangents from the point in question. Similarly, if among the tangents from the other point J or I, there is a tangent which counts r' times, then the foci are made up as follows, viz. we have

Intersections of the two singular tangents counting as .	$r'r$ foci.
Intersections of the first singular tangent with each of the ordinary tangents from the other circular point at infinity, as	$(n - q' - r')\,r$ „
Do. for second singular tangent, ...	$(n - q - r)\,r'$ „
Intersections of the ordinary tangents	$(n - q - r)(n - q' - r')$ „
Giving together the	$(n - q)(n - q')$ foci:

and the like observation applies to the more general case where the tangents from each of the points I, J include more than one singular tangent.

108. There is yet another case to be considered; the line infinity may be an ordinary or a singular tangent to the curve: assuming that it counts s times among the tangents from either of the circular points at infinity, the numbers of the remaining tangents are $n - q - s$, $n - q' - s$ from the two points I, J respectively, and the number of foci is $= (n - q - s)(n - q' - s)$.

109. In the case of a real curve the two points I, J are related in the same manner to the curve, and we have therefore $q = q'$; the singular tangents (if any) from the two points respectively being the same as well in character as in number. Writing $n - q - s = n - q' - s$, $= p$, and not for the present attending to the case of singular tangents, I shall assume that the number of tangents to the curve from each of the two points is $= p$; the number of foci is thus $= p^2$; and to each focus there corresponds a directrix, viz., this is the line through the points of contact of the tangents from the focus to the two points I, J respectively.

110. Consider any two foci A, B not *in lineâ* with either of the points I, J, then joining these with the points I, J, and taking A_1, B_1 the intersections of AI, BJ and of AJ, BI (A_1, B_1 being therefore by a foregoing definition the antipoints of (A, B)), then A_1, B_1 are, it is clear, foci of the curve. We may out of the p^2 foci select, and that in $1.2..p$ different ways, a system of p foci such that no two of them lie *in lineâ* with either of the points I, J; and this being so, taking the antipoints of each of the $\frac{1}{2}p(p-1)$ pairs out of the p foci, we have, inclusively of the p foci, in all $p+2.\frac{1}{2}p(p-1)$, that is p^2 foci, the entire system of foci.

Article Nos. 111 to 117. *On the Foci of Conics.*

111. A conic is a curve of the class 2, and the number of foci is thus $=4$. Taking as foci any two points A, B, the remaining two foci will be the antipoints A_1, B_1. In order that a given point A may be a focus, the conic must touch the lines AI, AJ; similarly, in order that a given point B may be a focus, the conic must touch the lines BI, BJ; the equation of a conic having the given points A, B for foci contains therefore a single arbitrary parameter.

112. In the case, however, of the parabola the curve touches the line infinity; there is consequently from each of the points I, J only a single tangent to the curve, and consequently only one focus: the parabola having a given point A for its focus is a conic touching the line infinity and the lines AI, AJ, or say the three sides of the triangle AIJ; its equation contains therefore two arbitrary parameters.

113. Returning to the general conic, there are certain trizomal forms of the focal equation, not of any great interest, but which may be mentioned. Using circular coordinates, and taking $(\alpha, \alpha', 1)$ and $(\beta, \beta', 1)$ for the coordinates of the given foci A, B respectively, the conic touches the lines $\xi-\alpha z=0$, $\eta-\alpha' z=0$, $\xi-\beta z=0$, $\eta-\beta' z=0$; the equation of a conic touching the first three lines is

$$\sqrt{l(\xi-\alpha z)}+\sqrt{m(\xi-\beta z)}+\sqrt{n(\eta-\alpha' z)}=0,$$

where l, m, n are arbitrary, and it is easy to obtain, in order that the conic may touch the fourth line $\eta-\beta' z=0$, the condition

$$n=-\frac{\beta-\alpha}{\beta'-\alpha'}(m-l).$$

114. In fact, n having this value, the equation gives

$$l(\xi-\alpha z)+m(\xi-\beta z)+2\sqrt{lm(\xi-\alpha z)(\xi-\beta z)}=-\frac{\beta-\alpha}{\beta'-\alpha'}(m-l)\big(\eta-\beta' z+(\beta'-\alpha')z\big),$$

and taking over the term

$$\frac{\beta-\alpha}{\beta'-\alpha'}(m-l)(\beta'-\alpha')z,\quad =(\beta-\alpha)(m-l)z,$$

this gives

$$l(\xi-\beta z)+m(\xi-\alpha z)+2\sqrt{lm(\xi-\alpha z)(\xi-\beta z)}=-\frac{\beta-\alpha}{\beta'-\alpha'}(m-l)(\eta-\beta' z),$$

which puts in evidence the tangent $\eta-\beta'z$. It is easy to see that the equation may be written in any one of the four forms

$$\sqrt{l}\,(\xi-\alpha z)+\sqrt{m}\,(\xi-\beta z)+\sqrt{-\frac{\beta-\alpha}{\beta'-\alpha'}(m-l)(\eta-\alpha'z)}=0,$$

$$\sqrt{m\,(\xi-\alpha z)}+\sqrt{l\,(\xi-\beta z)}+\sqrt{-\frac{\beta-\alpha}{\beta'-\alpha'}(m-l)(\eta-\beta'z)}=0,$$

$$\sqrt{l\,(\eta-\alpha'z)}+\sqrt{m(\eta-\beta'z)}+\sqrt{-\frac{\beta'-\alpha'}{\beta-\alpha}(m-l)(\xi-\alpha z)}=0,$$

$$\sqrt{m\,(\eta-\alpha'z)}+\sqrt{l\,(\eta-\beta'z)}+\sqrt{-\frac{\beta'-\alpha'}{\beta-\alpha}(m-l)(\xi-\beta z)}=0,$$

viz., in forms containing any three of the four radicals $\sqrt{\xi-\alpha z}$, $\sqrt{\xi-\beta z}$, $\sqrt{\eta-\alpha'z}$, $\sqrt{\eta-\beta'z}$. The conic is thus expressed as a trizomal curve, the zomals being each a line, viz., they are any three out of the four focal tangents; the order of the curve, as deduced from the general expression $2^{\nu-2}r$, is $=2$; so that there is here no depression of order.

115. But the ordinary form of the focal equation is a more interesting one; viz., $\mathfrak{A}$, $\mathfrak{B}$ being as usual the squared distances of the current point from the two given foci respectively, say

$$\mathfrak{A}=(\xi-\alpha z)(\eta-\alpha'z),$$

$$\mathfrak{B}=(\xi-\beta z)(\eta-\beta'z),$$

then $2a$ being an arbitrary parameter, the equation is

$$2az+\sqrt{\mathfrak{A}}+\sqrt{\mathfrak{B}}=0,$$

viz., the equation is here that of a trizomal curve, the zomals being curves of the second order, that is, the zomals are $(z^2=0)$ the line infinity twice, and the line-pairs AI, AJ and BI, BJ respectively: the general expression $2^{\nu-2}r$ gives therefore the order $=4$; but in the present case there are two branches, viz., the branches

$$2az+\sqrt{\mathfrak{A}}-\sqrt{\mathfrak{B}}=0,\quad 2az-\sqrt{\mathfrak{A}}+\sqrt{\mathfrak{B}}=0,$$

each ideally containing $(z=0)$ the line infinity; the curve contains therefore $(z^2=0)$ the line infinity twice, and omitting this factor the order is $=2$, as it should be.

116. To express the equation by means of the other two foci A_1, B_1, writing the equation under the form

$$\mathfrak{A}+\mathfrak{B}+2\sqrt{\mathfrak{A}\mathfrak{B}}-4a^2z^2=0,$$

and then if $\mathfrak{A}_1$, $\mathfrak{B}_1$ are the squared distances of the current point from A_1, B_1 respectively, we have (*ante*, No. 65),

$$\mathfrak{A}\mathfrak{B}=\mathfrak{A}_1\mathfrak{B}_1,$$

$$\mathfrak{A}+\mathfrak{B}-\mathfrak{A}_1-\mathfrak{B}_1=kz^2,$$

110. Consider any two foci A, B not *in lineâ* with either of the points I, J, then joining these with the points I, J, and taking A_1, B_1 the intersections of AI, BJ and of AJ, BI (A_1, B_1 being therefore by a foregoing definition the antipoints of (A, B)), then A_1, B_1 are, it is clear, foci of the curve. We may out of the p^2 foci select, and that in $1.2..p$ different ways, a system of p foci such that no two of them lie *in lineâ* with either of the points I, J; and this being so, taking the antipoints of each of the $\frac{1}{2}p(p-1)$ pairs out of the p foci, we have, inclusively of the p foci, in all $p+2.\frac{1}{2}p(p-1)$, that is p^2 foci, the entire system of foci.

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112. In the case, however, of the parabola the curve touches the line infinity; there is consequently from each of the points I, J only a single tangent to the curve, and consequently only one focus: the parabola having a given point A for its focus is a conic touching the line infinity and the lines AI, AJ, or say the three sides of the triangle AIJ; its equation contains therefore two arbitrary parameters.

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$$\sqrt{l(\xi-\alpha z)}+\sqrt{m(\xi-\beta z)}+\sqrt{n(\eta-\alpha' z)}=0,$$

where l, m, n are arbitrary, and it is easy to obtain, in order that the conic may touch the fourth line $\eta-\beta' z=0$, the condition

$$n=-\frac{\beta-\alpha}{\beta'-\alpha'}(m-l).$$

114. In fact, n having this value, the equation gives

$$l(\xi-\alpha z)+m(\xi-\beta z)+2\sqrt{lm(\xi-\alpha z)(\xi-\beta z)}=-\frac{\beta-\alpha}{\beta'-\alpha'}(m-l)\left(\eta-\beta' z+(\beta'-\alpha')z\right),$$

and taking over the term

$$\frac{\beta-\alpha}{\beta'-\alpha'}(m-l)(\beta'-\alpha')z, \quad =(\beta-\alpha)(m-l)z,$$

this gives

$$l(\xi-\beta z)+m(\xi-\alpha z)+2\sqrt{lm(\xi-\alpha z)(\xi-\beta z)}=-\frac{\beta-\alpha}{\beta'-\alpha'}(m-l)(\eta-\beta' z),$$

viz., of a circle having its centre on the major axis at a distance $= ae \sin\theta$ from the centre, and its radius $= b\cos\theta$. (I notice, in passing, that this gives in practice a very convenient graphical construction of the ellipse.) It may be remarked that for $\theta = \pm\sin^{-1}e$, the circle becomes

$$\left(x \pm (a - \frac{b^2}{a})\right)^2 + y^2 = \frac{b^4}{a^2},$$

viz., this is the circle of curvature at one or other of the extremities of the major axis; as θ passes from 0 to $\pm\sin^{-1}e$ we have a series of real circles, which, by their continued intersection, generate the ellipse; as θ increases from $\theta = \pm\sin^{-1}e$ to $\pm 90°$, the circles continue real, but the consecutive circles no longer intersect in any real point,—and ultimately for $\theta = \pm 90°$, the circles become evanescent at the two foci respectively.

121. In the case $q > 1$, we have a real representation of

$$(x - qae)^2 + y^2 + b^2(q^2 - 1),$$

as the squared distance of the point (x, y) from a point $(X, 0, Z)$ out of the plane of the figure, viz., putting this $= (x - X)^2 + y^2 + Z^2$, we have

$$qae = X, \quad Z^2 = b^2(q^2 - 1),$$

whence

$$Z^2 = b^2\left(\frac{X^2}{a^2e^2} - 1\right),$$

or, what is the same thing,

$$\frac{X^2}{a^2 - b^2} - \frac{Z^2}{b^2} = 1;$$

that is, the locus is the focal hyperbola, viz., a hyperbola in the plane of zx, having its vertices at the foci, and its foci at the vertices of the ellipse.

122. If instead of the form first considered, we start from the trizomal form

$$2bz + \sqrt{x^2 + (y - aeiz)^2} + \sqrt{x^2 + (y + aeiz)^2} = 0,$$

then we have the zomal or circle of double contact under the form

$$x^2 + (y - qaei)^2 = a^2(1 - q^2);$$

or putting herein $q = -i\tan\phi$, this is

$$x^2 + (y - ae\tan\phi)^2 = a^2\sec^2\phi;$$

so that we have the ellipse as the envelope of a variable circle having its centre on the minor axis of the ellipse, distance from the centre $= ae\tan\phi$, and radius $= a\sec\phi$. This is, in fact, Gergonne's theorem, according to which the ellipse is the secondary caustic or orthogonal trajectory of rays issuing from a point and refracted at a right line into a rarer medium. It is to be remarked that for $\tan\phi = \pm\frac{ae}{b}$, the equation of the circle is

$$x^2 + \left(y \pm (b - \frac{a^2}{b})\right)^2 = \frac{a^4}{b^2},$$

where k is the squared distance of the foci A, B, $=4a^2e^2$ suppose: whence putting $a^2(1-e^2)=b^2$, the equation becomes

$$\mathfrak{A}_1+\mathfrak{B}_1+2\sqrt{\mathfrak{A}_1\mathfrak{B}_1}-4b^2z^2=0,$$

that is

$$\sqrt{\mathfrak{A}_1}+\sqrt{\mathfrak{B}_1}+2bz=0,$$

which is the required new form. It is hardly necessary to remark that the equation $2az+\sqrt{\mathfrak{A}}+\sqrt{\mathfrak{B}}=0$, putting therein $z=1$, and expressing $\mathfrak{A}$, $\mathfrak{B}$ in rectangular coordinates measured along the axes, is the ordinary focal equation $2a=\sqrt{(x-ae)^2+y^2}+\sqrt{(x+ae)^2+y^2}$.

117. I remark that the equation $2az+\sqrt{\mathfrak{A}}+\sqrt{\mathfrak{B}}=0$ gives rise to $4a^2z^2+\mathfrak{A}-\mathfrak{B}+4az\sqrt{\mathfrak{A}}=0$, but here $\mathfrak{A}-\mathfrak{B}=-4aexz$, so that the equation contains $z=0$, and omitting this it becomes $(az-ex)+\sqrt{\mathfrak{A}}=0$, a bizomal form, being a curve of the order $=2$, as it should be; this is in fact the ordinary equation in regard to a focus and its directrix.

Article Nos. 118 to 123. *Theorem of the Variable Zomal as applied to a Conic.*

118. The equation $2kz+\sqrt{\mathfrak{A}^\circ}+\sqrt{\mathfrak{B}^\circ}=0$ is in like manner that of a conic; in fact, this would be a curve of the order $=4$, but there are as before the two branches $2kz+\sqrt{\mathfrak{A}^\circ}-\sqrt{\mathfrak{B}^\circ}=0$, $2kz-\sqrt{\mathfrak{A}^\circ}+\sqrt{\mathfrak{B}^\circ}=0$, each ideally containing $(z=0)$ the line infinity, and the order is thus reduced to be $=2$. Each of the circles $\mathfrak{A}^\circ=0$, $\mathfrak{B}^\circ=0$ is a circle having double contact with the conic (this of course implies that the centre of the circle is on an axis of the conic). We may if we please start from the form $2kz+\sqrt{\mathfrak{A}}+\sqrt{\mathfrak{B}}=0$, and then by means of the theorem of the variable zomal introduce into the equation one, two, or three such circles.

119. It is in this point of view that I will consider the question, viz., adapting the formula to the case of the ellipse, and starting from the form

$$2az+\sqrt{(x-aez)^2+y^2}+\sqrt{(x+aez)^2+y^2}=0,$$

the equation of the variable zomal or circle of double contact may be taken to be

$$\frac{4a^2z^2}{-2}+\frac{(x-aez)^2+y^2}{1-q}+\frac{(x+aez)^2+y^2}{1+q}=0,$$

where q is an arbitrary parameter; writing for greater simplicity $z=1$, and reducing, the equation is

$$(x-qae)^2+y^2=b^2(1-q^2).$$

120. If $q<1$, then writing $q=\sin\theta$, we obtain the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

as the envelope of the variable circle

$$(x-ae\sin\theta)^2+y^2=b^2\cos^2\theta,$$

125. Hence taking the points on the curve to be the circular points at infinity, we have the sixteen foci lying in fours upon four different circles—that is, we have four tetrads of concyclic foci. Let any one of these tetrads be A, B, C, D, then if

$$\begin{array}{llll} \text{Antipoints of} & (B, C)(A, D) & \text{are} & (B_1, C_1), \quad (A_1, D_1), \\ \quad ,, & (C, A)(B, D) & ,, & (C_2, A_2), \quad (B_2, D_2), \\ \quad ,, & (A, B)(C, D) & ,, & (A_3, B_3), \quad (C_3, D_3), \end{array}$$

the four tetrads of concyclic foci are

$$\begin{array}{cccc} A, & B, & C, & D; \\ A_1, & B_1, & C_1, & D_1; \\ A_2, & B_2, & C_2, & D_2; \\ A_3, & B_3, & C_3, & D_3. \end{array}$$

It is to be observed that if A, B, C, D are any four points on a circle, then if, as above, we pair these in any manner, and take the antipoints of each pair, the four antipoints lie on a circle, and thus the original system A, B, C, D, of four points on a circle, leads to the remaining three systems of four points on a circle. The theory is in fact that already discussed *ante*, No. 72 *et seq.*

126. The preceding theory applies without alteration to the bicircular quartic, viz., the quartic curve which has a node at each of the circular points at infinity. The class is here $=8$, but among the tangents from a node each of the two tangents at the node is to be reckoned twice, and the number of the remaining tangents is $=4$: the number of foci is $=16$. And, by the general theorem that in a binodal quartic the pencils of tangents from the two nodes respectively are homologous, the sixteen foci are related to each other precisely in the manner of the foci of the circular cubic. The latter is in fact a particular case of the former, viz., the bicircular quartic may break up into the line infinity, and a circular cubic.

Article Nos. 127 to 129. *Centre of the Circular Cubic, and Nodo-Foci, &c. of the Bicircular Quartic.*

127. The tangents at I, J have not been recognised as tangents from I, J, giving by their intersection a focus, but it is necessary in the theory to pay attention to the tangents in question. It is clear that these tangents are in fact asymptotes—viz., in the case of the circular cubic they are the two imaginary asymptotes of the curve, and in the case of a bicircular quartic, the two pairs of imaginary parallel asymptotes; but it is convenient to speak of them as the tangents at I, J.

128. In the case of a circular cubic, the tangents at I and J meet in a point which I call the centre of the curve, viz., this is the intersection of the two imaginary asymptotes.

129. In the case of a bicircular quartic, the two tangents at I and the two tangents at J meet in four points, which (although not recognising them as foci) I call the nodo-foci; these lie in pairs on two lines, diagonals of the quadrilateral formed by the four tangents (the third diagonal is of course the line IJ), which diagonals I call the "nodal axes;" and the point of intersection of the two nodal axes is the "centre" of the curve. The nodo-foci are four points, two of them real, the other two imaginary, viz., they are two pairs of antipoints, the lines through the two pairs respectively being, of course, the nodal axes; these are consequently real lines bisecting each other at right angles in the centre (with the relation $1 : i$ between the distances). The centre may also be defined as the intersection of the harmonic of IJ in regard to the tangents at I, and the harmonic of this same line in regard to the tangents at J. Speaking of the tangents as asymptotes, the nodo-foci are the angles of the rhombus formed by the two pairs of parallel asymptotes; the nodal axes are the diagonals of this rhombus, and the centre is the point of intersection of the two diagonals; as such it is also the intersection of the two lines drawn parallel to and midway between the lines forming each pair of parallel asymptotes.

Article No. 130. *Circular Cubic and Bicircular Quartic; the Axial or Symmetrical Case.*

130. In a circular cubic or bicircular quartic, the pencil of the tangents from I and that of the tangents through J, considered as corresponding to each other in some one of the four arrangements, may be such that the line IJ considered as belonging to the two pencils respectively shall correspond to itself, and when this is so, the four foci, A, B, C, D, which are the intersections of the corresponding tangents in question, will lie in a line (viz., the conic which exists in the general case will break up into a line-pair consisting of the line IJ and another line). The line in question may be called the focal axis; it will presently be shown that in the case of the circular cubic it passes through the centre, and that in the case of the bicircular quartic it not only passes through the centre, but coincides with one or other of the nodal axes, viz., with that passing through the real or the imaginary nodo-foci; that is, the curve may have on the focal axis two real or else two imaginary nodo-foci. The focal axis contains, as has been mentioned, four foci—the remaining twelve foci are situate symmetrically, six on each side of the focal axis, the arrangement of the sixteen foci being as mentioned *ante*, No. 81 *et seq.*; the focal axis is in fact an axis of symmetry of the curve, and if preferred it may be named the axis of symmetry, transverse axis, or simply the axis. And the curve (circular cubic, or bicircular quartic) is in this case a "symmetrical" or "axial" curve.

Article Nos. 131 to 140. *Circular Cubic and Bicircular Quartic: Singular Forms.*

131. The circular cubic may have a node or a cusp. If this were at one of the points I, J the curve would be imaginary, and I do not attend to the case; and for the same reason, for the bicircular quartic I do not attend to the case where *one* of

the points I, J is a cusp. There remain then for the circular cubic and for the bicircular quartic the cases where there is a node or a cusp at a real point of the curve; and for the bicircular quartic the case where each of the points I, J is a cusp—in general the curve has no other node or cusp, but it may besides have a node or cusp at a real point thereof.

132. I consider first the case of the bicircular quartic where each of the points I, J is a cusp. The curve is in this case of necessity symmetrical [1]—it is in fact a Cartesian; viz., the Cartesian may be taken by definition to be a quartic curve having a cusp at each of the circular points at infinity. But in this case, as distinguished from the general case of the bicircular quartic, there is an essential degeneration of all the focal properties, and it is necessary to explain what these become. The centre is evidently the intersection of the cuspidal tangents; the nodo-foci (so far as they can be said to exist) coalesce with the centre, and they do not in so coalescing determine any definite directions for the nodal axes; that is, there are no nodal axes, and the only theorem in regard to the focal axis or axis of symmetry is, that it passes through the centre. Of the four tangents through the point I, one has come to coincide with the line IJ; and similarly, of the four tangents through the point J one has come to coincide with the line JI: there remain only three tangents through I and three tangents through J, and these by their intersections determine nine foci—viz., three foci A, B, C on the axis, and besides (B_1, C_1) the antipoints of (B, C): (C_2, A_2) the antipoints of (C, A) and (A_3, B_3) the antipoints of (A, B).

133. The remaining seven foci have disappeared, viz., we may consider that one of them has gone off to infinity on the focal axis, and that three pairs of foci have come to coincide with the points I, J respectively. The circle O (as in the general case of a symmetrical quartic) has become a line, the focal axis; the circles R, S, T (contrary to what might at first sight appear) continue to be determinate circles, viz., these have their centres at A, B, C respectively, and pass through the points (B_1, C_1), (C_2, A_2), and (A_3, B_3) respectively, see *ante*, No. 83. But on each of these circles we have not more than two proper foci, and it is only on the axis as representing the circle O that we have three proper foci, the axial foci A, B, C: in regard hereto it is to be remarked that the equation of the curve can be expressed not only by means of these three foci in the form $\sqrt{l\mathfrak{A}} + \sqrt{m\mathfrak{B}} + \sqrt{n\mathfrak{C}} = 0$; but by means of any two of them in the form $\sqrt{l\mathfrak{A}} + \sqrt{m\mathfrak{B}} + K = 0$, where K is a constant, or, what is the same thing (z being introduced for homogeneity in the expressions of $\mathfrak{A}$ and $\mathfrak{B}$ respectively), in the form $\sqrt{l\mathfrak{A}} + \sqrt{m\mathfrak{B}} + Kz^2 = 0$.

134. Using for the moment the expression "twisted" as opposed to symmetrical—

[1] It will appear, *post* Nos. 161—164, that if starting with three given points as the foci of a bicircular quartic, we impose the condition that the nodes at I, J shall be each of them a cusp, then either the quartic will be the circle through the three points taken twice, in which case the assumed focal property of the given three points disappears altogether, or else the three points must be *in lineâ*, and thus the curve be symmetrical, that is, a Cartesian.

(viz., the curve is twisted when there is not any axis of symmetry, but the foci lie only on circles)—then the classification is

Circular Cubics, twisted,

„ „ symmetrical,

Bicircular Quartics, twisted,

„ „ symmetrical, $\begin{cases}\text{Ordinary,}\\ \text{Bicuspidal = Cartesian,}\end{cases}$

and each of these kinds may be general, nodal, or cuspidal—viz., for the two last mentioned kinds there may be a node or a cusp at a real point of the curve.

135. In the case of a node, say the point N; first if the curve (circular cubic or bicircular quartic) be twisted—then of the four foci A, B, C, D we have two, suppose B and C, coinciding with N; and the sixteen foci are as follows, viz.

B, C, A, D are N, N, A, D;

B_1, C_1, A_1, D_1 „ N, N, Antipoints of (A, D);

C_2, A_2, B_2, D_2 „ Antipoints of (N, A), Antipoints of (N, D);

A_3, B_3, C_3, D_3 „ Do. do.

viz., we have the points (A, D) each once, the node N four times, the antipoints of (A, D) once, and the antipoints of (N, A) and of (N, D), each pair twice. But properly there are only four foci, viz., the points A, D and their antipoints. The circle O subsists as in the general case, and so does the circle $R(BC, AD)$, viz., this has for centre the intersection of the line AD by the tangent at N to the circle O, and it passes through the point N, of course cutting the circle O at right angles: the circles S and T each reduce themselves each to the point N considered as an evanescent circle, or what is the same thing to the line-pair NI, NJ.

136. The case is nearly the same if the curve be symmetrical, but in the case of the bicircular quartic excluding the Cartesian: viz., we have on the axis the foci B, C coinciding at N, and the other two foci A, D; the sixteen foci are as above—and the circle R is determined by the proper construction as applied to the case in hand, viz., the centre R is the intersection of the axis by the radical axis of the point N (considered as an evanescent circle) and the circle on AD as diameter; that is $\overline{RN}^2 = RA \,.\, RD$. And the circles S and T reduce themselves each to the point N considered as an evanescent circle.

137. Next if we have a cusp, say the point K: first if the curve (circular cubic or bicircular quartic) be twisted—then of the four foci A, B, C, D, three, suppose A, B, C, coincide with K; and the sixteen foci are as follows, viz.,

B, C, A, D are K, K, K, D,

B_1, C_1, A_1, D_1 „ K, K, Antipoints of (K, D),

C_2, A_2, B_2, D_2 „ Do. do.

A_3, B_3, C_3, D_3 „ Do. do.

viz., we have the point D once, the point K nine times, and the antipoints of K, D three times. But properly the point D is the only focus. The circle O is, it would appear, *any* circle through K, D, but possibly the particular circle which touches the cuspidal tangent may be a better representative of the circle O of the general case—the circles R, S, T reduce themselves each to the point K considered as an evanescent point.

138. The like is the case if the curve be symmetrical, but in the case of the bicircular quartic excluding the Cartesian; the circle O is here the axis, which is in fact the cuspidal tangent.

139. For the Cartesian, if there is a node N; then of the three foci A, B, C, two, suppose B and C, coincide with N; the nine foci are A once, N four times, and the antipoints of N, A twice: but properly the point A is the only focus. And if there be a cusp K; then all the three foci A, B, C coincide with K; and the nine foci are K nine times; but in fact there is no proper focus.

140. A circular cubic cannot have two nodes unless it break up into a line and circle; and similarly a bicircular quartic cannot have two nodes (exclusive of course of the points I, J) unless it break up into two circles; the last-mentioned case will be considered in the sequel in reference to the problem of tactions.

Article No. 141. *As to the Analytical Theory for the Circular Cubic and the Bicircular Quartic respectively.*

141. It may be remarked in regard to the analytical theory about to be given, that although the investigation is very similar for the circular cubic and for the bicircular quartic, yet the former cannot be deduced from the latter case. In fact if for the bicircular quartic, using a form somewhat more general than that which is ultimately adopted, we suppose that for the two nodes respectively ($\xi=0$, $z=0$) and ($\eta=0$, $z=0$), then if $l\xi+mz=0$, $l'\xi+m'z=0$, $n\eta+pz=0$, $n'\eta+p'z=0$ are the tangents at the two nodes respectively, the equation will be

$$(l\xi+mz)(l'\xi+m'z)(n\eta+pz)(n'\eta+p'z)+ez^2\xi\eta+z^3(a\xi+b\eta)+cz^4=0,$$

and if (in order to make this equation divisible by z, and the curve so to break up into the line $z=0$ and a cubic) we write $l=0$ or $n=0$, then the curve will indeed break up as required, but we shall have, not the general cubic through the two points ($\xi=0$, $z=0$), ($\eta=0$, $z=0$), but in each case a nodal cubic, viz., if $l=0$ there will be a node at the point ($\eta=0$, $z=0$), and if $n=0$ a node at the point ($\xi=0$, $z=0$).

Article Nos. 142 to 144. *Analytical Theory for the Circular Cubic.*

142. I consider then the two cases separately; and first the circular cubic. The equation may be taken to be

$$\xi\eta(p\xi+q\eta)+ez\xi\eta+z^2(a\xi+b\eta+cz)=0,$$

or, what is the same thing,

$$\xi\eta\,(p\xi + q\eta + ez) + z^2(a\xi + b\eta + cz) = 0,$$

viz. (ξ, η, z) being any coordinates whatever, this is the general equation of a cubic passing through the points $(\xi = 0,\ z = 0)$, $(\eta = 0,\ z = 0)$, and at these points touched by the lines $\xi = 0$, $\eta = 0$ respectively. And if $(\xi, \eta, z = 1)$ be circular coordinates, then we have the general equation of a circular cubic having the lines $\xi = 0$, $\eta = 0$ for its asymptotes, or say the point $\xi = 0$, $\eta = 0$ for its centre; the equation of the remaining asymptote is evidently $p\xi + q\eta + ez = 0$; to make the curve real we must have (p, q) and (a, b) conjugate imaginaries, e and c real.

143. Taking in any case the points I, J to be the points $\xi = 0$, $z = 0$ and $\eta = 0$, $z = 0$ respectively, for the equation of a tangent from I write $p\xi = \theta z$; then we have

$$\theta\eta\,(\theta z + q\eta + ez) + z\,(a\theta z + bp\eta + cpz) = 0,$$

that is

$$z^2(a\theta + cp) + \eta z\,(\theta^2 + e\theta + bp) + \eta^2 . q\theta = 0,$$

and the line will be a tangent if only

$$(\theta^2 + e\theta + bp)^2 - 4q\theta\,(a\theta + cp) = 0,$$

that is, the four tangents from I are the lines $p\xi = \theta z$, where θ is any root of this equation; similarly the four tangents from J are the lines $q\eta = \phi z$, where ϕ is any root of the equation

$$(\phi^2 + e\phi + aq)^2 - 4p\phi\,(b\phi + cq) = 0.$$

Writing the two equations under the forms

$$\left\{\begin{matrix} 6, \\ 3e, \\ e^2 + 2bp - 4aq\ , \\ 3ebp \quad - 6cpq, \\ 6b^2p^2, \end{matrix}\right\}\!(\theta,\ 1)^4 = 0, \qquad \left\{\begin{matrix} 6, \\ 3e, \\ e^2 + 2aq - 4bp\ , \\ 3eaq \quad - 6cpq, \\ 6a^2q^2, \end{matrix}\right\}\!(\phi,\ 1)^4 = 0,$$

the equations have the same invariants; viz., for the first equation the invariants are easily found to be

$$I = \quad 3\,(e^2 - 4bp - 4aq)^2 + 72\,(ce - 2ab)\,pq,$$

$$J = -\quad (e^2 - 4bp - 4aq)^3 - 36\,(ce - 2ab)\,pq\,(e^2 - 4bp - 4aq) - 216\,c^2p^2q^2,$$

and then by symmetry the other equation has the same invariants. The absolute invariant $I^3 \div J^2$ has therefore the same value in the two equations; that is, the equations are linearly transformable the one into the other, which is the before-mentioned theorem that the two pencils are homographic.

144. The two equations will be satisfied by $\theta = \phi$, if only $bp = aq$; that is, if $p = \dfrac{a}{k}$, $q = \dfrac{b}{k}$; putting for convenience $\dfrac{e}{k}$ in place of e, the equation of the curve is then

$$\xi\eta\,(a\xi + b\eta + ez) + kz^2(a\xi + b\eta + cz) = 0.$$

In this case the pencils of tangents are $a\xi = k\theta z$, $b\eta = k\theta z$, where θ is determined by a quartic equation, or taking the corresponding lines (which by their intersections determine the foci A, B, C, D) to be $(a\xi = k\theta_1 z,\ b\eta = k\theta_1 z)$, &c., these four points lie in the line $a\xi - b\eta = 0$, which is a line through the centre of the curve, or point $\xi = 0$, $\eta = 0$: the formulæ just obtained belong therefore to the symmetrical case of the circular cubic. Passing to rectangular coordinates, writing $z = 1$, and taking $y = 0$ for the equation of the axis, it is easy to see that the equation may be written

$$(x^2 + y^2)(x - a) + k(x - b) = 0;$$

or, changing the origin and constants,

$$xy^2 + (x - a)(x - b)(x - c) = 0.$$

Article Nos. 145 to 149. *Analytical Theory for the Bicircular Quartic.*

145. The equation for the bicircular quartic may be taken to be

$$k(\xi^2 - \alpha^2 z^2)(\eta^2 - \beta^2 z^2) + ez^2\xi\eta + z^3(a\xi + b\eta) + cz^4 = 0,$$

viz. (ξ, η, z) being any coordinates whatever, this is the equation of a quartic curve having a node at each of the points $(\xi = 0,\ z = 0)$ and $(\eta = 0,\ z = 0)$: the equations of the two tangents at the one node are $\xi - \alpha z = 0$, $\xi + \alpha z = 0$; and those of the two tangents at the other node are $\eta - \beta z = 0$, $\eta + \beta z = 0$; $\xi = 0$ is thus the harmonic of the line $z = 0$ in regard to the tangents at $(\xi = 0,\ z = 0)$, and $\eta = 0$ is the harmonic of the same line $z = 0$ in regard to the tangents at $(\eta = 0,\ z = 0)$. If $(\xi, \eta, z = 1)$ be circular coordinates, then we have the general equation of the bicircular quartic having the lines $\xi + \alpha z = 0$, $\xi - \alpha z = 0$ for one pair, and the lines $\eta - \beta z = 0$, $\eta + \beta z = 0$ for the other pair of parallel asymptotes; and therefore the point $\xi = 0$, $\eta = 0$ for centre, and the lines $\beta\xi - \alpha\eta = 0$, $\beta\xi + \alpha\eta = 0$ for nodal axes. In order that the curve may be real we must have (α, β), (a, b) conjugate imaginaries, k, e, c real. The points $(\xi = 0,\ z = 0)$ and $(\eta = 0,\ z = 0)$ are as before the points I, J. If $\alpha = 0$, the node at I becomes a cusp, and so if $\beta = 0$, the node at J becomes a cusp; the form thus includes the case of a bicuspidal or Cartesian curve.

146. To find the tangents from I, writing in the equation of the curve $\xi = \theta\alpha z$ we have

$$k\alpha^2(\theta^2 - 1)(\eta^2 - \beta^2 z^2) + e\alpha\theta\eta z + z(a\alpha\theta z + b\eta) + cz^2 = 0;$$

that is

$$\begin{aligned} &\eta^2\ .\ k\alpha^2(\theta^2 - 1), \\ &+ \eta z\ .\ e\alpha\theta + b, \\ &+ z^2\ .\ -k\alpha^2\beta^2(\theta^2 - 1) + a\alpha\theta + c = 0, \end{aligned}$$

and the condition of tangency is

$$4k(\theta^2 - 1)\{k\alpha^2\beta^2(\theta^2 - 1) - a\alpha\theta - c\} + \left(e\theta + \frac{b}{\alpha}\right)^2 = 0;$$

viz., the tangents from I are $\xi = \theta \alpha z$, where θ is any root of this equation. Similarly, if we have

$$4k(\phi^2-1)\{k\alpha^2\beta^2(\phi^2-1)-b\beta\phi-c\}+\left(e\phi+\frac{a}{\beta}\right)^2=0,$$

the tangents from J are $\eta = \phi\beta z$, where ϕ is any root of this equation.

147. The two equations may be written

$$\left(\begin{array}{l} 24k^2\alpha^2\beta^2, \\ -\ 6ka\alpha, \\ -\ 8k^2\alpha^2\beta^2 - 4kc + e^2, \\ 6a\alpha \qquad\qquad + 3e\dfrac{b}{\alpha}, \\ 24k^2\alpha^2\beta^2 + 24kc + 6\dfrac{b^2}{\alpha^2} \end{array}\right)(\theta, 1)^4 = 0, \quad \left(\begin{array}{l} 24k^2\alpha^2\beta^2, \\ -\ 6kb\beta, \\ -\ 8k^2\alpha^2\beta^2 - 4kc + e^2, \\ 6kb\beta \qquad\qquad + 3e\dfrac{a}{\beta}, \\ 24k^2\alpha^2\beta^2 + 24kc + 6\dfrac{a^2}{\beta^2} \end{array}\right)(\phi, 1)^4 = 0,$$

which equations have the same invariants; in fact for the first equation the invariants are found to be as follows, viz., if for shortness $C = -8k^2\alpha^2\beta^2 - 4kc + e^2$, then

$$I = 576k^4\alpha^4\beta^4 + 576k^3c\alpha^2\beta^2 + 144k^2(a^2\alpha^2 + b^2\beta^2) + 72kab + 3C^2,$$

$$J = C\{576k^4\alpha^4\beta^4 + 576k^3c\alpha^2\beta^2 + 144k^2(a^2\alpha^2 + b^2\beta^2) + 36kea\beta - C^2\}$$
$$- 864k^3eab\alpha^2\beta^2 - 216k^2e^2(a^2\alpha^2 + b^2\beta^2) - 216k^2a^2b^2,$$

and then by symmetry the other equation has the same invariants. The absolute invariant $I^3 \div J^2$ has thus the same value in the two equations, that is, the equations are linearly transformable the one into the other, which is the before-mentioned theorem that the pencils are homographic.

148. The equations will be satisfied by $\theta = \phi$ if only $a\alpha = b\beta$, that is, if $a, b = m\beta, m\alpha$; or by $\theta = -\phi$ if only $a\alpha = -b\beta$, that is, if $a, b = m\beta, -m\alpha$: the equation of the curve is in these two cases respectively

$$k(\xi^2 - \alpha^2z^2)(\eta^2 - \beta^2z^2) + ez^2\xi\eta + mz^3(\beta\xi + \alpha\eta) + cz^4 = 0,$$

$$k(\xi^2 - \alpha^2z^2)(\eta^2 - \beta^2z^2) + ez^2\xi\eta + mz^3(\beta\xi - \alpha\eta) + cz^4 = 0.$$

If to fix the ideas we attend to the first case, then the equation in θ is

$$\left(\begin{array}{l} 24k^2\alpha^2\beta^2, \\ -\ 6km\alpha\beta, \\ -\ 8k^2\alpha^2\beta^2 - 4kc + e^2, \\ 6km\alpha\beta \qquad + 3me, \\ 24k^2\alpha^2\beta^2 + 24kc + 6m^2 \end{array}\right)(\theta, 1)^4 = 0;$$

and we may take as corresponding tangents through the two nodes respectively $\xi = \theta\alpha z$, $\eta = \theta\beta z$; the foci A, B, C, D, which are the intersections of the pairs of lines ($\xi = \theta_1\alpha z$,

$\eta=\theta_1\beta z$), &c., lie, it is clear, in the line $\beta\xi-\alpha\eta=0$, which is one of the nodal axes of the curve. Similarly, in the second case, if θ be determined by the foregoing equation, we may take as corresponding tangents through the two nodes respectively $\xi=\theta\alpha z$, $\eta=-\theta\beta z$; the foci (A, B, C, D), which are the intersections of the pairs of lines ($\xi=\theta_1\alpha z$, $\eta=-\theta_1\beta z$), &c., lie in the line $\beta\xi+\alpha\eta=0$, which is the other of the nodal axes of the curve. In either case the foci A, B, C, D lie in a line, that is, we have the curve symmetrical; and, as we have just seen, the focal axis, or axis of symmetry, is one or other of the nodal axes.

149. In the case of the Cartesian, or when $\alpha=0$, $\beta=0$, viz., the equation $a\alpha=b\beta$ is satisfied identically, and this seems to show that the Cartesian is symmetrical; it is to be observed, however, that for $\alpha=0$, $\beta=0$ the foregoing formulæ fail, and it is proper to repeat the investigation for the special case in question. Writing $\alpha=0$, $\beta=0$, the equation of the curve is

$$k\xi^2\eta^2+ez^2\xi\eta+z^3(a\xi+b\eta)+cz^4=0,$$

and then, taking $\xi=\theta bz$ for the equation of the tangent from I, we have

$$\begin{aligned}&\eta^2\ .\ kb^2\theta^2\\ &+\eta z\ .\ b\,(e\theta+1)\\ &+z^2\ .\ ab\theta+c=0,\end{aligned}$$

and the condition of tangency is

$$4k\theta^2(ab\theta+c)-(e\theta+1)^2=0\,;$$

viz., we have here a cubic equation. Similarly, if we have $\eta=\theta az$ for the equation of a tangent from J, then

$$4k\phi^2(ab\phi+c)-(e\phi+1)^2=0.$$

Hence θ being determined by the cubic equation as above, we may take $\phi=\theta$, and consequently the equations of the corresponding tangents will be $\xi=\theta bz$, $\eta=\theta az$, viz., the foci A, B, C will be given as the intersections of the pairs of lines ($\xi=\theta_1 bz$, $\eta=\theta_1 az$), &c. The foci lie therefore in the line $a\xi-b\eta=0$; or the curve is symmetrical, the focal axis, or axis of symmetry, passing through the centre.

Article Nos. 150 to 158. *On the Property that the Points of Contact of the Tangents from a Pair of Concyclic Foci lie in a Circle.*

150. We have seen that the sixteen foci form four concyclic sets (A, B, C, D), (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), (A_3, B_3, C_3, D_3), that is, A, B, C, D are in a circle. We may, if we please, say that any one focus is concyclic—viz., it lies in a circle with three other foci; but any two foci taken at random are not concyclic; it is only a pair such as (A, B) taken out of a set of four concyclic foci which are concyclic, viz., there exist two other foci lying with them in a circle. The number of such pairs is, it is clear $=24$. Let A, B be any two concyclic foci, I say that the points of contact of the tangents AI, AJ, BI, BJ, lie in a circle.

151. Consider the case of the bicircular quartic, and take as before $(\xi = 0, z = 0)$, and $(\eta = 0, z = 0)$ for the coordinates of the points I, J respectively. Let the two tangents from the focus A be $\xi - \alpha z = 0$, $\eta - \alpha' z = 0$, say for shortness $p = 0$, $p' = 0$, then the equation of the curve is expressible in the form $pp'U = V^2$ [1], where $U = 0$, $V = 0$ are each of them a circle, viz., U and V are each of them a quadric function containing the terms z^2, $z\eta$, $z\xi$, and $\xi\eta$. Taking an indeterminate coefficient λ, the equation may be written

$$pp'(U + 2\lambda V + \lambda^2 pp') = (V + \lambda pp')^2,$$

and then λ may be so determined that $U + 2\lambda V + \lambda^2 pp' = 0$, shall be a 0-circle, or pair of lines through I and J. It is easy to see that we have thus for λ a cubic equation, that is, there are three values of λ, for each of which the function $U + 2\lambda V + \lambda^2 pp'$ assumes the form $(\xi - \beta z)(\eta - \beta' z)$, $= qq'$ suppose: taking any one of these, and changing the value of V so as that we may have V in place of $V + \lambda pp'$, the equation is $pp'qq' + V^2$, where $V = 0$ is as before a circle, the equation shows that the points of contact of the tangents $p = 0$, $p' = 0$, $q = 0$, $q' = 0$ lie in this circle $V = 0$. The circumstance that λ is determined by a cubic equation would suggest that the focus $q = 0$, $q' = 0$ is one of the three foci B, C, D concyclic with A; but this is the very thing which we wish to prove, and the investigation, though somewhat long, is an interesting one.

152. Starting from the form $pp'qq' = V^2$, then introducing as before an arbitrary coefficient λ, the equation may be written

$$pp'(qq' + 2\lambda V + \lambda^2 pp') = (V + \lambda pp')^2,$$

and we may determine λ so that $qq' + 2\lambda V + \lambda^2 pp' = 0$ shall be a pair of lines. Writing $V = H\xi\eta - L\eta z - L'\xi z + Mz^2$, and substituting for pp' and qq' their values $(\xi - \alpha z)(\eta - \alpha' z)$ and $(\xi - \beta z)(\eta - \beta' z)$, the equation in question is

$$(1 + 2\lambda H + \lambda^2)\,\xi\eta - (\beta + 2\lambda L + \lambda^2\alpha)\,\eta z - (\beta' + 2\lambda L' + \lambda^2\alpha')\,\xi z + (\beta\beta' + 2\lambda M + \lambda^2\alpha\alpha')\,z^2 = 0,$$

and the required condition is

$$(1 + 2\lambda H + \lambda^2)(\beta\beta' + 2\lambda M + \lambda^2\alpha\alpha') = (\beta + 2\lambda L + \lambda^2\alpha)(\beta' + 2\lambda L' + \lambda^2\alpha');$$

or reducing, this is

$$\begin{aligned}&(2M + 2H\beta\beta' - 2L'\beta - 2L\beta)\\ &+ \lambda\left((\alpha - \beta)(\alpha' - \beta') + 4HM - 4LL'\right)\\ &+ \lambda^2(2M + 2H\alpha\alpha' - 2L'\alpha - 2L\alpha') = 0,\end{aligned}$$

viz., λ is determined by a quadric equation. Calling its roots λ_1, and λ_2, the foregoing equation, substituting therein successively these values, becomes $(\xi - \gamma z)(\eta - \gamma' z) = 0$, and $(\xi - \delta z)(\eta - \delta' z) = 0$ respectively, say $rr' = 0$ and $ss' = 0$.

[1] This investigation is similar to that in Salmon's *Higher Plane Curves*, p. 196, in regard to the double tangents of a quartic curve.

153. We have to show that the four foci $(p=0,\ p'=0)$, $(q=0,\ q'=0)$, $(r=0,\ r'=0)$, $(s=0,\ s'=0)$ are a set of concyclic foci; that is, that the lines $p=0$, $q=0$, $r=0$, $s=0$ correspond homographically to the lines $p'=0$, $q'=0$, $r'=0$, $s'=0$; or, what is the same thing, that we have

$$\begin{vmatrix} 1, & \alpha, & \alpha', & \alpha\alpha' \\ 1, & \beta, & \beta', & \beta\beta' \\ 1, & \gamma, & \gamma', & \gamma\gamma' \\ 1, & \delta, & \delta', & \delta\delta' \end{vmatrix} = 0,$$

or, as it will be convenient to write this equation,

$$\frac{\alpha-\beta}{\alpha'-\beta'}\,\frac{\gamma-\delta}{\gamma'-\delta'} = \frac{\alpha-\delta}{\alpha'-\delta'}\,\frac{\beta-\gamma}{\beta'-\gamma'}.$$

154. We have

$$\gamma = \frac{\beta + 2\lambda_1 L + \lambda_1^2\alpha}{1 + 2H\lambda_2 + \lambda_1^2},\quad \gamma' = \frac{\beta' + 2\lambda_1 L' + \lambda_1^2\alpha'}{1 + 2H\lambda_1 + \lambda_1^2},$$

$$\delta = \frac{\beta + 2\lambda_2 L + \lambda_2^2\alpha}{1 + 2H\lambda_2 + \lambda_2^2},\quad \delta' = \frac{\beta' + 2\lambda_2 L' + \lambda_2^2\alpha'}{1 + 2H\lambda_2 + \lambda_2^2}.$$

The expressions of $\alpha-\delta$, &c., are severally fractions, the denominators of which disappear from the equation; the numerators are

$$\begin{aligned}
\text{for } \alpha-\delta, &= \alpha(1 + 2\lambda_2 H + \lambda_2^2) - (\beta + 2\lambda_2 L + \alpha\lambda_2^2), \\
&= \alpha - \beta + 2\lambda_2(\alpha H - L); \\
\text{for } \beta-\gamma, &= \beta(1 + 2\lambda_1 H + \lambda_1^2) - (\beta + 2\lambda_1 L + \alpha\lambda_1^2), \\
&= \lambda_1\{2(\beta H - L)(\alpha-\beta)\}; \\
\text{for } \gamma-\delta, &= (\beta + 2L\lambda_1 + \alpha\lambda_1^2)(1 + 2H\lambda_2 + \lambda_2^2) \\
&\quad - (\beta + 2L\lambda_2 + \alpha\lambda_2^2)(1 + 2H\lambda_1 + \lambda_1^2), \\
&= (\alpha'-\beta')\{2H^2\alpha\beta - 2HL(\alpha+\beta) + 2L^2 + \tfrac{1}{2}(\alpha-\beta)^2\};
\end{aligned}$$

and it hence easily appears that the equation to be verified is

$$\frac{2H^2\alpha\beta - 2HL(\alpha+\beta) + 2L^2 + \frac{1}{2}(\alpha-\beta)^2}{2H^2\alpha'\beta' - 2HL'(\alpha'+\beta') + 2L'^2 + \frac{1}{2}(\alpha'-\beta')^2} = \frac{\alpha-\beta+2(\alpha H - L)\lambda_2}{\alpha'-\beta'+2(\alpha' H - L')\lambda_2}\cdot\frac{2(\beta H - L) - (\alpha-\beta)\lambda_1}{2(\beta' H - L') - (\alpha'-\beta')\lambda_2}.$$

155. This is

$$\frac{B-C}{B'-C'} = \frac{A + B\lambda_1 + C\lambda_2 + D\lambda_1\lambda_2}{A' + B'\lambda_1 + C'\lambda_2 + D'\lambda_1\lambda_2},$$

if for shortness

$$\begin{aligned}
A &= \ \ 2(\alpha-\beta)(\beta H - L)\ , & A' &= \ \ 2(\alpha'-\beta')(\beta' H - L')\ , \\
B &= -\ (\alpha-\beta)^2 \ , & B' &= -\ (\alpha'-\beta')^2\ , \\
C &= \ \ 4(\alpha H - L)(\beta H - L), & C' &= \ \ 4(\alpha' H - L')(\beta' H - L'), \\
D &= -2(\alpha-\beta)(\alpha H - L)\ , & D' &= -2(\alpha'-\beta')(\alpha' H - L')
\end{aligned}$$

and the equation then is

$$AB' - A'B + CA' - C'A - (\lambda_1+\lambda_2)(BC' - B'C) + \lambda_1\lambda_2\big(CD' - C'D - (BD' - B'D)\big).$$

156. Calculating $AB' - A'B$, $CA' - C'A$, $CD' - C'D$, $BD' - B'D$, these are at once seen to divide by $\{(\alpha\beta' - \alpha'\beta)H + L(\alpha' - \beta') - L'(\alpha - \beta)\}$; we have, moreover,

$$\begin{aligned} BC' - B'C &= -4(\alpha-\beta)^2(\alpha'H - L')(\beta'H - L') + 4(\alpha'-\beta')^2(\alpha H - L)(\beta H - L), \\ &= -\{(\alpha\alpha' - \beta\beta')H - L(\alpha'-\beta') - L'(\alpha-\beta)\}\{\alpha\beta' - \alpha'\beta)H + L(\alpha'-\beta') - L'(\alpha-\beta)\}, \end{aligned}$$

viz., this also contains the same factor; and omitting it, the equation is found to be

$$\begin{aligned} &\{(\alpha-\beta)(\alpha'-\beta') - 4(\beta H - L)(\beta' H - L')\} \\ &- 2\{(\alpha\alpha' - \beta\beta')H - L(\alpha'-\beta') - L'(\alpha-\beta)\}(\lambda_1 + \lambda_2) \\ &+ \{-(\alpha-\beta)(\alpha'-\beta') + 4(\alpha H - L)(\alpha' H - L')\}\lambda_1\lambda_2 = 0; \end{aligned}$$

viz., substituting for $\lambda_1 + \lambda_2$ and $\lambda_1\lambda_2$ their values, this is

$$\begin{aligned} &\{(\alpha-\beta)(\alpha'-\beta') - 4(\beta H - L)(\beta' H - L')\}(M + H\alpha\alpha' - L\alpha' - L'\alpha) \\ &- \{(\alpha\alpha' - \beta\beta')H - L(\alpha'-\beta')\}\{(\alpha-\beta)(\alpha'-\beta') + 4HM - 4LL'\} \\ &+ \{-(\alpha-\beta)(\alpha'-\beta') + 4(\alpha H - L)(\alpha' H - L')\}\{M + H\beta\beta' - L\beta' - L'\beta\} = 0, \end{aligned}$$

which should be identically true. Multiplying by H, and writing in the form

$$\begin{aligned} &\{(\alpha-\beta)(\alpha'-\beta') - 4(\beta H - L)(\beta' H - L')\}\big(HM - LL' + (\alpha H - L)(\alpha' H - L')\big) \\ &- \{(\alpha H - L)(\alpha' H - L') - (\beta H - L)(\beta' H - L')\}\big((\alpha-\beta)(\alpha'-\beta') + 4(HM - LL')\big) \\ &+ \{-(\alpha-\beta)(\alpha'-\beta') + 4(\alpha H - L)(\alpha' H - L')\}\big(HM - LL' + (\beta H - L)(\beta' H - L')\big) = 0, \end{aligned}$$

we at once see that this is so, and the theorem is thus proved, viz., that the equation being $pp'qq' = V^2$, the foci $(p = 0,\ p' = 0)$ and $(q = 0,\ q' = 0)$ are concyclic.

157. By what precedes, λ being a root of the foregoing quadric equation, we may write

$$qq' + 2\lambda V + \lambda^2 pp' = K^2 rr',$$

where the focus $r = 0$, $r' = 0$ is concyclic with the other two foci; but from the equation of the curve $V = \sqrt{pp'qq'}$, that is we have

$$qq' + 2\lambda\sqrt{pp'qq'} + \lambda^2 pp' = Krr',$$

or, what is the same thing,

$$\lambda\sqrt{pp'} + \sqrt{qq'} + K\sqrt{rr'} = 0,$$

viz., this is a form of the equation of the curve: substituting for p, p', q, q', r, r' their values, writing also

$$\begin{aligned} \mathfrak{A} &= (\xi - \alpha z)(\eta - \alpha' z), \\ \mathfrak{B} &= (\xi - \beta z)(\eta - \beta' z), \\ \mathfrak{C} &= (\xi - \gamma z)(\eta - \gamma' z), \end{aligned}$$

and changing the constants λ, K (viz. $\lambda : 1 : K = \sqrt{l} : \sqrt{m} : \sqrt{n}$) the equation is

$$\sqrt{l\mathfrak{A}} + \sqrt{m\mathfrak{B}} + \sqrt{n\mathfrak{C}} = 0,$$

viz., we have the theorem that for a bicircular quartic if $(\xi - \alpha z = 0,\ \eta - \alpha' z = 0)$, $(\xi - \beta z = 0,\ \eta - \beta' z = 0,\ (\xi - \gamma z = 0),\ \eta - \gamma' z = 0)$ be any three concyclic foci, then the equation is as just mentioned; that is, the curve is a trizomal curve, the zomals being the three given foci regarded as 0-circles. The same theorem holds in regard to the circular cubic, and a similar demonstration would apply to this case.

158. It may be noticed that we might, without proving as above that the two foci $(p=0,\ p'=0)$, $(q=0,\ q'=0)$ were concyclic, have passed at once from the form $pp'qq' = V^2$, to the form $\lambda\sqrt{pp'} + \sqrt{qq'} + K\sqrt{rr'} = 0$ (or $\sqrt{l\mathfrak{A}} = \sqrt{m\mathfrak{B}} = \sqrt{n\mathfrak{C}} = 0$), and then by the application of the theorem of the variable zomal (thereby establishing the existence of a fourth focus concyclic with the three) have shown that the original two foci were concyclic. But it seemed the more orderly course to effect the demonstration without the aid furnished by the reduction of the equation to the trizomal form.

PART IV. (Nos. 159 to 206). ON TRIZOMAL AND TETRAZOMAL CURVES WHERE THE ZOMALS ARE CIRCLES.

Article Nos. 159 to 165. *The Trizomal Curve—The Tangents at I, J, &c.*

159. I consider the trizomal

$$\sqrt{l\mathfrak{A}^\circ} + \sqrt{m\mathfrak{B}^\circ} + \sqrt{n\mathfrak{C}^\circ} = 0,$$

where A, B, C being the centres of three given circles, $\mathfrak{A}^\circ$, &c. denote as before, viz., in rectangular and in circular coordinates respectively, we have

$$\begin{aligned}
\mathfrak{A}^\circ &= (x - az)^2 + (y - a'z)^2 - a''^2 z^2, &&= (\xi - \alpha z)(\eta - \alpha' z) - a''^2 z^2,\\
\mathfrak{B}^\circ &= (x - bz)^2 + (y - b'z)^2 - b''^2 z^2, &&= (\xi - \beta z)(\eta - \beta' z) - b''^2 z^2,\\
\mathfrak{C}^\circ &= (x - cz)^2 + (y - c'z)^2 - c''^2 z^2, &&= (\xi - \gamma z)(\eta - \gamma' z) - c''^2 z^2.
\end{aligned}$$

By what precedes, the curve is of the order $=4$, touching each of the given circles twice, and having a double point, or node, at each of the points I, J; that is, it is a bicircular quartic: but if for any determinate values of the radicals $\sqrt{l}$, $\sqrt{m}$, $\sqrt{n}$, we have

$$\sqrt{l} \quad + \sqrt{m} \quad + \sqrt{n} \quad = 0,$$

then there is a branch

$$\sqrt{l\mathfrak{A}^\circ} + \sqrt{m\mathfrak{B}^\circ} + \sqrt{n\mathfrak{C}^\circ} = 0,$$

containing $(z=0)$ the line infinity; and the order is here $=3$: viz., the curve here passes through each of the points I, J and through another point at infinity (that is, there is an asymptote), and is thus a circular cubic.

160. I commence by investigating the equations of the nodal tangents at the points I, J respectively; using for this purpose the circular coordinates $(\xi,\ \eta,\ z=1)$, it is to be observed that, in the rationalised equation, for finding the tangents at $(\xi=0,\ z=0)$ we have only to attend to the terms of the second order in $(\xi,\ z)$, and

similarly for finding the tangents at $(\eta=0, z=0)$ we have only to attend to the terms of the second order in (η, z). But it is easy to see that any term involving a'', b'', or c'' will be of the third order at least in (ξ, z), and similarly of the third order at least in (η, z); hence for finding the tangents we may reject the terms in question, or, what is the same thing, we may write a'', b'', c'' each $=0$, thus reducing the three circles to their respective centres. The equation thus becomes

$$\sqrt{l(\xi-\alpha z)(\eta-\alpha' z)}+\sqrt{m(\xi-\beta z)(\eta-\beta' z)}+\sqrt{n(\xi-\gamma z)(\eta-\gamma' z)}=0.$$

For finding the tangents at $(\xi=0, z=0)$ we have in the rationalised equation to attend only to the terms of the second order in (ξ, z); and it is easy to see that any term involving α', β', γ' will be of the third order at least in (ξ, z), that is, we may reduce α', β', γ' each to zero; the irrational equation then becomes divisible by $\sqrt{\eta}$, and throwing out this factor, it is

$$\sqrt{l(\xi-\alpha z)}+\sqrt{m(\xi-\beta z)}+\sqrt{n(\xi-\gamma z)}=0,$$

viz., this equation which evidently belongs to a pair of lines passing through the point $(\xi=0, z=0)$ gives the tangents at the point in question; and similarly the tangents at the point $(\eta=0, z=0)$ are given by the equation

$$\sqrt{l(\eta-\alpha' z)}+\sqrt{m(\eta-\beta' z)}+\sqrt{n(\eta-\gamma' z)}=0.$$

161. To complete the solution, attending to the tangents at $(\xi=0, z=0)$, and putting for shortness

$$\begin{aligned}\lambda &= l-m-n,\\ \mu &= -l+m-n,\\ \nu &= -l-m+n,\\ \Delta &= l^2+m^2+n^2-2mn-2nl-2lm,\end{aligned}$$

the rationalised equation is easily found to be

$$\begin{aligned}&\xi^2\,.\,\Delta\\ &-2\xi z\,(l\lambda\alpha+m\mu\beta+n\nu\gamma)\\ &+\ z^2(l^2\alpha^2+m^2\beta^2+n^2\gamma^2-2mn\beta\gamma-2nl\gamma\alpha-2lm\alpha\beta)=0\,;\end{aligned}$$

and it is to be noticed that in the case of the circular cubic or when $\sqrt{l}+\sqrt{m}+\sqrt{n}=0$, then $\Delta=0$, so that the equation contains the factor z, and throwing this out, the equation gives a single line, which is in fact the tangent of the circular cubic.

162. Returning to the bicircular quartic, we may seek for the condition in order that the node may be a cusp: the required condition is obviously

$$\Delta(l^2\alpha^2+m^2\beta^2+n^2\gamma^2-2mn\beta\gamma-2nl\gamma\alpha-2lm\alpha\beta)-(l\lambda\alpha+m\mu\beta+n\nu\gamma)^2=0,$$

or observing that

$$\begin{aligned}\Delta-\lambda^2 &= -4mn,\ \&c.\\ \Delta+\mu\nu &= -2l\lambda,\ \&c.\end{aligned}$$

this is

$$l\alpha^2+m\beta^2+n\gamma^2+\lambda\beta\gamma+\mu\gamma\alpha+\nu\alpha\beta=0,$$

or substituting for λ, μ, ν, their values, it is

$$l(\alpha-\beta)(\alpha-\gamma)+m(\beta-\gamma)(\beta-\alpha)+n(\gamma-\alpha)(\gamma-\beta)=0,$$

or, as it is more simply written,

$$\frac{l}{\beta-\gamma}+\frac{m}{\gamma-\alpha}+\frac{n}{\alpha-\beta}=0.$$

163. If the node at $(\eta=0,\ z=0)$ be also a cusp, then we have in like manner

$$\frac{l}{\beta'-\gamma'}+\frac{m}{\gamma'-\alpha'}+\frac{n}{\alpha'-\beta'}=0.$$

Now observing that

$$(\gamma-\alpha)(\alpha'-\beta')-(\gamma'-\alpha')(\alpha-\beta),=\begin{vmatrix}\alpha, & \alpha', & 1\\ \beta, & \beta', & 1\\ \gamma, & \gamma', & 1\end{vmatrix}$$

$$=(\alpha-\beta)(\beta'-\gamma')-(\alpha'-\beta')(\beta-\gamma),$$

$$=(\beta-\gamma)(\gamma'-\alpha')-(\beta'-\gamma')(\gamma-\alpha),$$

$=\Omega$ suppose: the two equations give

$$l:m:n=\Omega(\beta-\gamma)(\beta'-\gamma'):\Omega(\gamma-\alpha)(\gamma'-\alpha'):\Omega(\alpha-\beta)(\alpha'-\beta');$$

or if Ω is not $=0$, then

$$l:m:n=\quad(\beta-\gamma)(\beta'-\gamma'):\quad(\gamma-\alpha)(\gamma'-\alpha'):\quad(\alpha-\beta)(\alpha'-\beta').$$

164. If

$$\Omega=\begin{vmatrix}\alpha, & \alpha', & 1\\ \beta, & \beta', & 1\\ \gamma, & \gamma', & 1\end{vmatrix},=0,$$

or, what is the same thing, if

$$\begin{vmatrix}a, & a', & 1\\ b, & b', & 1\\ c, & c', & 1\end{vmatrix}=0,$$

the centres A, B, C are in a line; taking it as the axis of x, we have $\alpha=\alpha'=a$, $\beta=\beta'=b$, $\gamma=\gamma'=c$; and the conditions for the cusps at I, J respectively reduce themselves to the single condition

$$\frac{l}{b-c}+\frac{m}{c-a}+\frac{n}{a-b}=0,$$

so that this condition being satisfied, the curve

$$\sqrt{l\{(x-az)^2+y^2-a''^2z^2\}}+\sqrt{m\{(x-bz)^2+y^2-b''^2z^2\}}+\sqrt{n\{(x-cz)^2+y^2-c''^2z^2\}}=0$$

is a Cartesian; viz., given any three circles with their centres on a line, there are a singly infinite series of Cartesians, each touched by the three circles respectively;

the line of centres is the axis of the curve, but the centres A, B, C are not the foci, except in the case $a''=0$, $b''=0$, $c''=0$, where the circles vanish. The condition for l, m, n is satisfied if $l : m : n=(b-c)^2 : (c-a)^2 : (a-b)^2$; these values, writing $\sqrt{l} : \sqrt{m} : \sqrt{n}=b-c : c-a : a-b$, give not only $\sqrt{l}+\sqrt{m}+\sqrt{n}=0$, but also $a\sqrt{l}+b\sqrt{m}+c\sqrt{n}=0$; these are the conditions for a branch containing ($z^2=0$) the line infinity twice; the equation

$$(b-c)\sqrt{(x-az)^2+y^2-a''^2z^2}+(c-a)\sqrt{(x-bz)^2+y^2-b''^2z^2}+(a-b)\sqrt{(x-cz)^2+y^2-c''^2z^2}=0,$$

is thus that of a conic, and if $a''=0$, $b''=0$, $c''=0$, then the curve reduces itself to $y^2=0$, the axis twice.

165. If Ω is not $=0$, then we have

$$l : m : n=(\beta-\gamma)(\beta'-\gamma') : (\gamma-\alpha)(\gamma'-\alpha') : (\alpha-\beta)(\alpha'-\beta'),$$

viz., l, m, n are as the squared distances $\overline{BC}^2$, $\overline{CA}^2$, $\overline{AB}^2$, say as $f^2 : g^2 : h^2$; or when the centres of the given circles A, B, C are not in a line, then f, g, h being the distances BC, CA, AB of these centres from each other, we have, touching each of the given circles twice, the *single* Cartesian

$$f\sqrt{\mathfrak{A}^\circ}+g\sqrt{\mathfrak{B}^\circ}+h\sqrt{\mathfrak{C}^\circ}=0,$$

which, in the particular case where the radii a'', b'', c'' are each $=0$, becomes

$$f\sqrt{\mathfrak{A}}+g\sqrt{\mathfrak{B}}+h\sqrt{\mathfrak{C}}=0,$$

viz., this is the circle through the points A, B, C, say the circle ABC, twice.

Article Nos. 166 to 169. *Investigation of the Foci of a Conic represented by an Equation in Areal Coordinates.*

166. I premise as follows: Let A, B, C be any given points, and in regard to the triangle ABC let the *areal* coordinates of a current point P be u, v, w; that is, writing PBC, &c., for the areas of these triangles, take the coordinates to be

$$u : v : w=PBC : PCA : PAB,$$

or, what is the same thing in the rectangular coordinates (x, y, $z=1$), if

$$(a, a', 1), (b, b', 1), (c, c', 1),$$

be the coordinates of A, B, C respectively, take

$$u : v : w=\begin{vmatrix} x, & y, & z \\ b, & b', & 1 \\ c, & c', & 1 \end{vmatrix} : \begin{vmatrix} x, & y, & z \\ c, & c', & 1 \\ a, & a', & 1 \end{vmatrix} : \begin{vmatrix} x, & y, & z \\ a, & a', & 1 \\ b, & b', & 1 \end{vmatrix},$$

or in the circular coordinates (ξ, η, $z=1$), if (α, α', 1), (β, β', 1), (γ, γ', 1) be the coordinates of the three points respectively, then

$$u : v : w = \begin{vmatrix} \xi, & \eta, & z \\ \beta, & \beta', & 1 \\ \gamma, & \gamma', & 1 \end{vmatrix} : \begin{vmatrix} \xi, & \eta, & z \\ \gamma, & \gamma', & 1 \\ \alpha, & \alpha', & 1 \end{vmatrix} : \begin{vmatrix} \xi, & \eta, & z \\ \alpha, & \alpha', & 1 \\ \beta, & \beta', & 1 \end{vmatrix}.$$

167. For the point I we have $(\xi, \eta, z) = (0, 1, 0)$, and hence if its areal coordinates be (u_0, v_0, w_0), we have

$$u_0 : v_0 : w_0 = \beta - \gamma : \gamma - \alpha : \alpha - \beta,$$

and hence also, (u, v, w) referring to the current point P, we find

$$v_0 w - w_0 v = (\gamma - \alpha)\,[(\alpha' - \beta')(\xi - \alpha z) - (\alpha - \beta)(\eta - \alpha' z)]$$
$$-(\alpha - \beta)\,[(\gamma' - \alpha')(\xi - \alpha z) - (\gamma - \alpha)(\eta - \alpha' z)], = \Omega(\xi - \alpha z),$$

if

$$\Omega = (\gamma - \alpha)(\alpha' - \beta') - (\alpha - \beta)(\gamma' - \alpha'), = \begin{vmatrix} \alpha, & \alpha', & 1 \\ \beta, & \beta', & 1 \\ \gamma, & \gamma', & 1 \end{vmatrix};$$

whence

$$v_0 w - w_0 v : w_0 u - w u_0 : u_0 v - u v_0 = \xi - \alpha z : \xi - \beta z : \xi - \gamma z,$$

and in precisely the same manner, if u_0', v_0', w_0' refer to the point J, then

$$u_0' : v_0' : w_0' = \beta' - \gamma' : \gamma' - \alpha' : \alpha' - \beta',$$

and

$$v_0' w - w_0' v : w_0' u - w u_0' : u_0' v - u v_0' = \eta - \alpha' z : \eta - \beta' z : \eta - \gamma' z.$$

168. Consider the conic

$$(a, b, c, f, g, h \between u, v, w)^2 = 0,$$

where u, v, w are any trilinear coordinates whatever; and take the inverse coefficients to be (A, B, C, F, G, H) $(A = bc - f^2$, &c.), then for any given point the coordinates of which are (u_0, v_0, w_0), the equation of the tangents from this point to the conic is, as is well known,

$$(A, B, C, F, G, H \between v_0 w - w_0 v,\ w_0 u - u_0 w,\ u_0 v - v_0 u)^2 = 0;$$

consequently for the conic

$$(a, b, c, f, g, h \between u, v, w)^2 = 0,$$

where (u, v, w) are areal coordinates referring, as above, to any three given points A, B, C, the equation of the pair of tangents from the point I to the conic is

$$(A, B, C, F, G, H \between \xi - \alpha z,\ \xi - \beta z,\ \xi - \gamma z)^2 = 0,$$

and that of the pair of tangents from J is

$$(A, B, C, F, G, H \between \eta - \alpha' z,\ \eta - \beta' z,\ \eta - \gamma' z)^2 = 0,$$

these two line-pairs intersecting, of course, in the foci of the conic.

169. In particular, if the conic is a conic passing through the points A, B, C, then taking its equation to be

$$lvw + mwu + nuv = 0,$$

the inverse coefficients are as (l^2, m^2, n^2, $-2mn$, $-2nl$, $-2lm$), and we have for the equations of the two line-pairs

$$\sqrt{l(\xi - \alpha z)} + \sqrt{m(\xi - \beta z)} + \sqrt{n(\xi - \gamma z)} = 0,$$

$$\sqrt{l(\eta - \alpha' z)} + \sqrt{m(\eta - \beta' z)} + \sqrt{n(\eta - \gamma' z)} = 0.$$

Article No. 170. *The Theorem of the Variable Zomal.*

170. Consider the four circles

$$\mathfrak{A}^\circ = 0,\ \mathfrak{B}^\circ = 0,\ \mathfrak{C}^\circ = 0,\ \mathfrak{D}^\circ = 0\ \left(\mathfrak{A}^\circ = (x - az)^2 + (y - a'z)^2 - a''^2 z^2, \&c.\right),$$

which have a common orthotomic circle; so that as before

$$\mathrm{a}\mathfrak{A}^\circ + \mathrm{b}\mathfrak{B}^\circ + \mathrm{c}\mathfrak{C}^\circ + \mathrm{d}\mathfrak{D}^\circ = 0,$$

where

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = BCD : -CDA : DAB : -ABC.$$

I consider the first three circles as given, and the fourth circle as a variable circle cutting at right angles the orthotomic circle of the three given circles; this being so, attending only to the ratios a : b : c, we may write

$$\mathrm{a} : \mathrm{b} : \mathrm{c} \quad = DBC : \quad DCA : DAB,$$

that is, (a, b, c) are proportional to the areal coordinates of the centre of the variable circle in regard to the triangle ABC.

171. Suppose that the centre of the variable circle is situate on a given conic, then expressing the equation of this conic in areal coordinates in regard to the triangle ABC, we have between (a, b, c) the equation obtained by substituting these values for the coordinates in the equation of the conic; that is, the equation of the variable circle is

$$\mathrm{a}\mathfrak{A}^\circ + \mathrm{b}\mathfrak{B}^\circ + \mathrm{c}\mathfrak{C}^\circ \quad = 0,$$

where (a, b, c) are connected by an equation

$$(a, b, c, f, g, h \between \mathrm{a}, \mathrm{b}, \mathrm{c})^2 = 0.$$

Hence (A, B, C, F, G, H) being the inverse coefficients, the equation of the envelope of the variable circle is

$$(A, B, C, F, G, H \between \mathfrak{A}^\circ, \mathfrak{B}^\circ, \mathfrak{C}^\circ)^2 = 0,$$

and, in particular, if the conic be a conic passing through the points A, B, C, and such that its equation in the areal coordinates (u, v, w) in regard to the triangle ABC is

$$lvw + mwu + nuv = 0,$$

then the equation of the envelope is

$$(l^2,\ m^2,\ n^2,\ -mn,\ -nl,\ -lm \between \mathfrak{A}^\circ,\quad \mathfrak{B}^\circ,\quad \mathfrak{C}^\circ)^2 = 0\,;$$

that is, it is

$$(1,\ 1,\ 1,\ -1,\ -1,\ -1 \between l\mathfrak{A}^\circ,\ m\mathfrak{B}^\circ,\ n\mathfrak{C}^\circ)^2 = 0,$$

or, what is the same thing, it is

$$\sqrt{l\mathfrak{A}^\circ} + \sqrt{m\mathfrak{B}^\circ} + \sqrt{n\mathfrak{C}^\circ} = 0.$$

172. It has been seen that the equations of the nodal tangents at the points I, J respectively are respectively

$$\sqrt{l(\xi - \alpha z)} + \sqrt{m(\xi - \beta z)} + \sqrt{n(\xi - \gamma z)} = 0,$$

$$\sqrt{l(\eta - \alpha' z)} + \sqrt{m(\eta - \beta' z)} + \sqrt{n(\eta - \gamma' z)} = 0,$$

and that these are the equations of the tangents to the conic $lvw + mwu + nuv = 0$ from the points I, J respectively. We have thus Casey's theorem for the generation of the bicircular quartic as follows:—The envelope of a variable circle which cuts at right angles the orthotomic circle of three given circles $\mathfrak{A}^\circ = 0$, $\mathfrak{B}^\circ = 0$, $\mathfrak{C}^\circ = 0$, and has its centre on the conic $lvw + mwu + nuv = 0$ which passes through the centres of the three given circles is the bicircular quartic, or trizomal

$$\sqrt{l\mathfrak{A}^\circ} + \sqrt{m\mathfrak{B}^\circ} + \sqrt{n\mathfrak{C}^\circ} = 0,$$

which has its nodo-foci coincident with the foci of the conic.

173. To complete the analytical theory, it is proper to express the equation of the orthotomic circle by means of the areal coordinates (u, v, w). Writing for shortness $a^2 + a'^2 - a''^2 = \grave{a}$, &c., and therefore

$$\mathfrak{A}^\circ = x^2 + y^2 - 2axz - 2a'yz - \grave{a}z^2, \text{ \&c.},$$

then if as before

$$u : v : w = \begin{vmatrix} x, & y, & z \\ b, & b', & 1 \\ c, & c', & 1 \end{vmatrix} : \begin{vmatrix} x, & y, & z \\ c, & c', & 1 \\ a, & a', & 1 \end{vmatrix} : \begin{vmatrix} x, & y, & z \\ a, & a', & 1 \\ b, & b', & 1 \end{vmatrix},$$

and therefore

$$x : y : z = au + bv + cw : a'u + b'v + c'w : u + v + w,$$

the equation of the orthotomic circle is

$$\begin{vmatrix} x - az, & y - a'z, & ax + a'y - \grave{a}z \\ x - bz, & y - b'z, & bx + b'y - \grave{b}z \\ x - cz, & y - c'z, & cx + c'y - \grave{c}z \end{vmatrix} = 0,$$

viz., throwing out the factor z, this is

$$u(ax + a'y - \grave{a}z) + v(bx + b'y - \grave{b}z) + w(cx + c'y - \grave{c}z) = 0,$$

or, what is the same thing, it is

$$(au+bv+cw)\,x+(a'u+b'v+c'w)\,y-(a^{\backprime}u+b^{\backprime}v+c^{\backprime}w)\,z \qquad = 0,$$

viz., it is

$$(au+bv+cw)^2\;+(a'u+b'v+c'w)^2\;-(a^{\backprime}u+b^{\backprime}v+c^{\backprime}w)(u+v+w)=0,$$

that is, substituting for $a^{\backprime}$, $b^{\backprime}$, $c^{\backprime}$ their values, it is

$$\begin{aligned} & a''^2u^2 \quad + b''^2v^2 \;+\; c''^2w^2 \\ &+ \left(b''^2+c''^2-(b-c)^2-(b'-c')^2\right) vw \\ &+ \left(c''^2+a''^2-(c-a)^2-(c'-a')^2\right) wu \\ &+ \left(a''^2+b''^2-(a-b)^2-(a'-b')^2\right) uv = 0, \end{aligned}$$

and it may be observed that using for a moment α, β, γ to denote the angles at which the three circles taken in pairs respectively intersect, then we have $2b''c''\cos\alpha = b''^2+c''^2-(b-c)^2-(b'-c')^2$, &c., and the equation of the orthotomic circle thus is

$$(1,\ 1,\ 1,\ \cos\alpha,\ \cos\beta,\ \cos\gamma\between a''u,\ b''v,\ c''w)^2=0.$$

174. We have in the foregoing enunciation of the theorem made use of the three given circles A, B, C, but it is clear that these are in fact *any* three circles in the series of the variable circle, and that the theorem may be otherwise stated thus:

The envelope of a variable circle which has its centre in a given conic, and cuts at right angles a given circle, is a bicircular quartic, such that its nodo-foci are the foci of the conic.

Article Nos. 175 to 177. *Properties depending on the relation between the Conic and Circle.*

175. I refer to the conic of the theorem simply as the conic, and to the fixed circle simply as the circle, or when any ambiguity might otherwise arise, then as the orthotomic circle. This being so, I consider the effect in regard to the trizomal curve, of the various special relations which may exist between the circle and the conic.

If the conic touch the circle, the curve has a node at the point of contact.

If the conic has with the circle a contact of the second order, the curve has a cusp at the point of contact.

If the centre of the circle lie on an axis of the conic, then the four intersections lie in pairs symmetrically in regard to this axis, or the curve has this axis as an axis of symmetry.

If the conic has double contact with the circle (this implies that the centre of the circle is situate on an axis of the conic) the curve has a node at each of the points of contact, viz., it breaks up into two circles intersecting in these two points.

The centres of the two circles respectively are the two foci of the conic, which foci lie on the axis in question. Observe that in the general case there are at each of the circular points at infinity two tangents, without any correspondence of the tangents of the one pair singly to those of the other pair, and there are thus four intersections, the four foci of the conic; in the present case, where the curve is a pair of circles, the two tangents to the same circle correspond to each other, and intersect in the two foci on the axis in question. The other two foci, or antipoints of these, are each of them the intersection of a tangent of the one circle by a tangent of the other circle.

If the conic has with the circle a contact of the third order (this implies that the circle is a circle of maximum or minimum curvature, at the extremity of an axis of the conic), then the curve has at this point a tacnode, viz., it breaks up into two circles touching each other and the conic at the point in question, and having their centres at the two foci situate on that axis of the conic respectively.

176. If the conic is a parabola, then the curve is a circular cubic having the four intersections of the parabola and circle for a set of concyclic foci, and having the focus of the parabola for *centre*. The like particular cases arise, viz.,

If the circle touch the parabola, the curve has a node at the point of contact.

If the circle has, with the parabola, a contact of the second order, the curve has a cusp at the point of contact.

If the centre of the circle is situate on the axis of the parabola, then the four intersections are situate in pairs symmetrically in regard to this axis, and the curve has this axis for an axis of symmetry.

If the circle has double contact with the parabola (which, of course, implies that the centre lies on the axis), then the curve has a node at each of the points of contact, viz., the curve breaks up into a line and circle intersecting at the two points of contact, and the circle has its centre at the focus of the parabola.

If the circle has with the parabola a contact of the third order (this implies that the circle is the circle of maximum curvature, touching the parabola at its vertex), then the curve has a tacnode, viz., it breaks up into a line and circle touching each other and the parabola at the vertex, that is, the line is the tangent to the parabola at its vertex, and the circle is the circle having the focus of the parabola for its centre, and passing through the vertex, or what is the same thing, having its radius $=\frac{1}{2}$ of the semi-latus rectum of the parabola.

177. If the conic be a circle, then the curve is a bicircular quartic such that its four nodo-foci coincide together at the centre of the circle; viz., the curve is a Cartesian having the centre of the conic for its cuspo-focus, that is, for the intersection of the cuspidal tangents of the Cartesian. The intersections of the conic with the other circle, or say with the orthotomic circle, are a pair of non-axial foci of the Cartesian; viz., the antipoints of these are two of the axial foci. The third axial focus is the centre of the orthotomic circle.

Article No. 178. *Case of Double Contact, Casey's Equation in the Problem of Tactions.*

178. In the case where the conic has double contact with the orthotomic circle, then (as we have seen) the envelope of the variable circle is a pair of circles, each touching the variable circle; or, if we start with three given circles and a conic through their centres, then the envelope is a pair of circles, each of them touching each of the three given circles; that is, we have a solution of the problem of tactions. Multiplying by 2, the equation found *ante,* No. 173, for the variable circle, and then for the moment representing it by $(\mathrm{a, b, c, f, g, h} \between u, v, w)^2 = 0$; then attributing any signs at pleasure to the radicals $\sqrt{\mathrm{a}}$, $\sqrt{\mathrm{b}}$, $\sqrt{\mathrm{c}}$, the equation of a conic through the centres of the given circles, and having double contact with the orthotomic circle, will be

$$(\mathrm{a, b, c, f, g, h} \between u, v, w)^2 - (u\sqrt{\mathrm{a}} + v\sqrt{\mathrm{b}} + w\sqrt{\mathrm{c}})^2 = 0,$$

viz., representing this equation as before by

$$lvw + mwu + nuv = 0,$$

we have

$$l : m : n = \mathrm{f} - \sqrt{\mathrm{bc}} : \mathrm{g} - \sqrt{\mathrm{ca}} : \mathrm{h} - \sqrt{\mathrm{ab}},$$

that is, substituting for a, b, c, f, g, h their values, and taking, for instance, a, b, c $= a''\sqrt{2},\ b''\sqrt{2},\ c''\sqrt{2}$, we find

$$\begin{aligned} l : m : n = \ & (b'' - c'')^2 - (b - c)^2 - (b' - c')^2 \\ : \ & (c'' - a'')^2 - (c - a)^2 - (c' - a')^2 \\ : \ & (a'' - b'')^2 - (a - b)^2 - (a' - b')^2, \end{aligned}$$

that is, l, m, n are as the squares of the tangential distances (direct) of the three circles taken in pairs, and this being so, the equation of a pair of circles touching each of the three given circles is $\sqrt{l\mathfrak{A}^\circ} + \sqrt{m\mathfrak{B}^\circ} + \sqrt{n\mathfrak{C}^\circ} = 0$. It is clear that, instead of taking the three direct tangential distances, we may take one direct tangential distance and two inverse tangential distances, viz., the tangential distances corresponding to any three centres of similitude which lie in a line; we have thus in all the equations of four pairs of circles, viz., of the eight circles which touch the three given circles. This is Casey's theorem in the problem of tactions.

Article No. 179. *The Intersections of the Conic and Orthotomic Circle are a set of four Concyclic Foci.*

179. The conic of centres intersects the orthotomic circle in four points, and for each of these the radius of the variable circle is $= 0$, that is, the points in question are a set of four concyclic foci (A, B, C, D) of the curve. Regarding the foci as given, the circle which contains them is of course the orthotomic circle; and there are a singly infinite series of curves, viz., these correspond to the singly infinite series of conics which can be drawn through the given foci. As for a given curve there are

four sets of concyclic foci, there are four different constructions for the curve, viz., the orthotomic circle may be any one of the four circles O, R, S, T, which contain the four sets of concyclic foci respectively; and the conic of centres is a conic through the corresponding set of four concyclic foci. We have thus four conics, but the foci of each of them coincide with the nodo-foci of the curve, that is, the conics are confocal; that such confocal conics exist has been shown, *ante*, Nos. 78 to 80.

Article Nos. 180 and 181. *Remark as to the Construction of the Symmetrical Curve.*

180. It is to be observed that in applying as above the theorem of the variable zomal to the construction of a symmetrical curve, the orthotomic circle made use of was one of the circles R, S, T, not the circle O, which is in this case the axis; in fact, we should then have the conic and the orthotomic circle each of them coinciding with the axis. And the variable circle, *quà* circle having its centre on the axis, cuts the axis at right angles whatever the radius may be; that is, the variable circle is no longer sufficiently determined by the theorem. The curve may nevertheless be constructed as the envelope of a variable circle having its centre on the axis; viz., writing $\mathfrak{A}^\circ = (x-az)^2 + y^2 - a''^2z^2$, &c., and starting with the form

$$\sqrt{l\mathfrak{A}^\circ} + \sqrt{m\mathfrak{B}^\circ} + \sqrt{n\mathfrak{C}^\circ} = 0,$$

then recurring to the demonstration of the theorem (*ante*, No. 47), the equation of the variable circle is $\mathrm{a}\mathfrak{A}^\circ + \mathrm{b}\mathfrak{B}^\circ + \mathrm{c}\mathfrak{C}^\circ = 0$, where a, b, c are any quantities satisfying $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$, or, what is the same thing, taking q an arbitrary parameter, and writing $\frac{l}{\mathrm{a}} = 1+q$, $\frac{m}{\mathrm{b}} = 1-q$, $\frac{n}{\mathrm{c}} = -2$, the equation of the variable circle is

$$\frac{1}{1+q}\, l\mathfrak{A}^\circ + \frac{1}{1-q}\, m\mathfrak{B}^\circ - \tfrac{1}{2} n\mathfrak{C}^\circ = 0.$$

Compare Nos. 118—123 for the like mode of construction of a conic; but it is proper to consider this in a somewhat different form.

181. Assume that the equation of the variable circle is

$$\mathfrak{D}^\circ = (x-dz)^2 + y^2 - d''^2z^2 = 0\,;$$

we have therefore identically

$$\mathrm{a}\mathfrak{A}^\circ + \mathrm{b}\mathfrak{B}^\circ + \mathrm{c}\mathfrak{C}^\circ + \mathrm{d}\mathfrak{D}^\circ = 0,$$

viz., this gives

$$\begin{aligned} &\mathrm{a} \quad + \mathrm{b} \quad + \mathrm{c} = -\mathrm{d}\,, \\ &\mathrm{a}a + \mathrm{b}b + \mathrm{c}c = -\mathrm{d}d, \\ &\mathrm{a}\,(a^2 - a''^2) + \mathrm{b}\,(b^2 - b''^2) + \mathrm{c}\,(c^2 - c''^2) = -\,\mathrm{d}\,(d^2 - d''^2), \end{aligned}$$

and from these equations we obtain a, b, c equal respectively to given multiples of d; substituting these values in the equation $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$, d divides out, and we have an equation involving the parameters of the given circles, and also d, d'', the parameters of the variable circle; viz., an equation determining d'', the radius of the variable circle, in terms of d, the coordinate of its centre. I consider in particular the case where the given circles are points; that is, where the given equation is

$$\sqrt{l\mathfrak{A}} + \sqrt{m\mathfrak{B}} + \sqrt{n\mathfrak{C}} = 0.$$

The equations here are

$$\begin{aligned} \mathrm{a} \quad + \mathrm{b} \quad + \mathrm{c} &= -\mathrm{d}, \\ \mathrm{a}a + \mathrm{b}b + \mathrm{c}c &= -\mathrm{d}d, \\ \mathrm{a}a^2 + \mathrm{b}b^2 + \mathrm{c}c^2 &= -\mathrm{d}\,(d^2 - d''^2), \end{aligned}$$

and from these we obtain

$$\begin{aligned} \mathrm{a}\,(a-b)\,(a-c) &= -\mathrm{d}\,\big((d-b)\,(d-c) - d''^2\big) \\ \mathrm{b}\,(b-c)\,(b-a) &= -\mathrm{d}\,\big((d-c)\,(d-a) - d''^2\big) \\ \mathrm{c}\,(c-a)\,(c-b) &= -\mathrm{d}\,\big((d-a)\,(d-b) - d''^2\big), \end{aligned}$$

so that the equation $\dfrac{l}{\mathrm{a}} + \dfrac{m}{\mathrm{b}} + \dfrac{n}{\mathrm{c}} = 0$ becomes

$$\frac{l\,(a-b)\,(a-c)}{(d-b)\,(d-c)-d''^2} + \frac{m\,(b-c)\,(b-a)}{(d-c)\,(d-a)-d''^2} + \frac{n\,(c-a)\,(c-b)}{(d-a)\,(d-b)-d''^2} = 0,$$

or, as this is more conveniently written,

$$\frac{l}{b-c}\,\frac{1}{(d-b)\,(d-c)-d''^2} + \frac{m}{c-a}\,\frac{1}{(d-c)\,(d-a)-d''^2} + \frac{n}{a-b}\,\frac{1}{(d-a)\,(d-b)-d''^2} = 0,$$

viz., considering d, d'' as the abscissa and ordinate of a point on a curve, and representing them by x, y respectively, the equation of this curve is

$$\frac{l}{b-c}\,\frac{1}{(x-b)\,(x-c)-y^2} + \frac{m}{c-a}\,\frac{1}{(x-c)\,(x-a)-y^2} + \frac{n}{c-a}\,\frac{1}{(x-a)\,(x-b)-y^2} = 0,$$

which is a certain quartic curve; and we have the original curve

$$\sqrt{l\mathfrak{A}} + \sqrt{m\mathfrak{B}} + \sqrt{n\mathfrak{C}} = 0,$$

as the envelope of a variable circle having for its diameter the double ordinate of this quartic curve.

Write for shortness $\dfrac{l}{b-c}$, $\dfrac{m}{c-a}$, $\dfrac{n}{a-b} = L,\ M,\ N$ respectively, then the equation of the quartic curve may be written

$$\Sigma L\ [(x-a)^2\,(x-b)\,(x-c) - y^2\,(x-a)\,(2x-b-c) + y^4] = 0,$$

viz., this is

$$\begin{aligned} \Sigma L\ [&x\,(x-a)\,(x-b)\,(x-c) \\ &- y^2\,(2x^2 - (a+b+c)\,x + (ab+ac+bc)) + y^4 \\ &- a\,(x-a)\,(x-b)\,(x-c) + y^2\,(ax+bc)] = 0, \end{aligned}$$

or what is the same thing, the equation is

$$(L+M+N)\left[x(x-a)(x-b)(x-c)-y^2\left(2x^2-(a+b+c)x+ab+ac+bc\right)+y^4\right]$$
$$-(La+Mb+Nc)(x-a)(x-b)(x-c)$$
$$+y^2\{(La+Mb+Nc)x+Lbc+Mca+Nab\}=0.$$

In the particular case where $L+M+N=0$, that is, where

$$\frac{l}{b-c}+\frac{m}{c-a}+\frac{n}{a-b}=0,$$

the quartic curve becomes a cubic, viz., putting for shortness

$$-\delta=\frac{Lbc+Mca+Nab}{La+Mb+Nc},$$

the equation of the cubic is

$$y^2=\frac{(x-a)(x-b)(x-c)}{x-\delta},$$

viz., this is a cubic curve having three real asymptotes, and a diameter at right angles to one of the asymptotes, and at the inclinations $+45°$, $-45°$ to the other two asymptotes respectively—say that it is a "rectangular" cubic. The relation $\frac{l}{b-c}+\frac{m}{c-a}+\frac{n}{a-b}=0$ implies that the curve $\sqrt{l\mathfrak{A}}+\sqrt{m\mathfrak{B}}+\sqrt{n\mathfrak{C}}=0$ is a Cartesian, and we have thus the theorem that the envelope of a variable circle having for diameter the double ordinate of a rectangular cubic is a Cartesian.

I remark that using a particular origin, and writing the equation of the rectangular cubic in the form $y^2=x^2-2mx+\alpha+\frac{2A}{x}$, the equation of the variable circle is

$$(x-d)^2+y^2=d^2-2md+\alpha+\frac{2A}{d},$$

that is

$$x^2+y^2-\alpha-2d(x-m)-\frac{2A}{d}=0,$$

where d is the variable parameter. Forming the derived equation in regard to d, we have

$$x-m=\frac{A}{d^2},$$

and thence

$$x^2+y^2-\alpha=\frac{4A}{d},$$

$$(x^2+y^2-\alpha)^2=\frac{16A^2}{d^2}=16A(x-m),$$

that is, the equation of the envelope is $(x^2+y^2-\alpha)^2=16A(x-m)=0$, which is a known form of the equation of a Cartesian.

Article Nos. 182 and 183. *Focal Formulæ for the General Curve.*

182. Considering any three circles centres A, B, C, and taking $\mathfrak{A}^\circ$, &c., to denote as usual, let the equation of the curve be

$$\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}=0\,;$$

then considering a fourth circle, centre D, a position of the variable circle, and having therefore the same orthotomic circle with the given circles, so that as before

$$\mathrm{a}\mathfrak{A}^\circ+\mathrm{b}\mathfrak{B}^\circ+\mathrm{c}\mathfrak{C}^\circ+\mathrm{d}\mathfrak{D}^\circ=0,$$

the formulæ No. 47 (changing only U, V, W, T into $\mathfrak{A}^\circ$, $\mathfrak{B}^\circ$, $\mathfrak{C}^\circ$, $\mathfrak{D}^\circ$) are at once applicable to express the equation of the curve in terms of any three of the four circles A, B, C, D.

In particular, the circles may reduce themselves to the four points A, B, C, D, a set of concyclic foci, and here, the equation being originally given in the form

$$\sqrt{l\mathfrak{A}}+\sqrt{m\mathfrak{B}}+\sqrt{n\mathfrak{C}}=0,$$

the same formulæ are applicable to express the equation in terms of any three of the four foci.

183. It is to be observed that in this case if the positions of the four foci are given by means of the circular coordinates $\left(\alpha,\ \frac{1}{\alpha},\ 1\right)$, &c., which refer to the centre of the circle $ABCD$ as origin, and with the radius of this circle taken as unity, then the values of a, b, c, d (*ante*, No. 90), are given in the form adapted to the formulæ of No. 49, viz., we have

$$\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}=\alpha\,(\beta\gamma\delta):-\beta\,(\gamma\delta\alpha):\gamma\,(\delta\alpha\beta):-\delta\,(\alpha\beta\gamma),$$

where $(\beta\gamma\delta)=(\beta-\gamma)(\gamma-\delta)(\delta-\beta)$, &c. The relation $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$, putting therein $l:m:n=\rho\alpha(\beta-\gamma)^2:\sigma\beta(\gamma-\alpha)^2:\tau\gamma(\alpha-\beta)^2$, (or, what is the same thing, taking the equation of the curve to be given in the form $(\beta-\gamma)\sqrt{\rho\alpha\mathfrak{A}}+(\gamma-\alpha)\sqrt{\sigma\beta\mathfrak{B}}+(\alpha-\beta)\sqrt{\tau\gamma\mathfrak{C}}=0$), becomes

$$\rho\,(\beta-\gamma)(\alpha-\delta)+\sigma\,(\gamma-\alpha)(\beta-\delta)+\tau\,(\alpha-\beta)(\gamma-\delta)=0,$$

viz., this equation, considering ρ, σ, τ, α, β, γ as given, determines the position of the fourth focus D, or when A, B, C, D are given, it is the relation which must exist between ρ, σ, τ; and the four forms of the equation are

$$\left(\begin{array}{llll} \cdot\ , & \sqrt{\tau}\,(\delta-\gamma), & \sqrt{\sigma}\,(\beta-\delta), & \sqrt{\rho}\,(\gamma-\beta) \\ \sqrt{\tau}\,(\gamma-\delta), & \cdot\ , & \sqrt{\rho}\,(\delta-\alpha), & \sqrt{\sigma}\,(\alpha-\gamma) \\ \sqrt{\sigma}\,(\delta-\beta), & \sqrt{\rho}\,(\alpha-\delta), & \cdot\ , & \sqrt{\tau}\,(\beta-\alpha) \\ \sqrt{\rho}\,(\beta-\gamma), & \sqrt{\sigma}\,(\gamma-\alpha), & \sqrt{\tau}\,(\alpha-\beta), & \cdot \end{array}\right)(\sqrt{\alpha\mathfrak{A}},\ \sqrt{\beta\mathfrak{B}},\ \sqrt{\gamma\mathfrak{C}},\ \sqrt{\delta\mathfrak{D}})=0,$$

viz., the curve is represented by means of any one of these four equations involving each of them three out of the four given foci A, B, C, D.

Article Nos. 184 and 185. *Case of the Circular Cubic.*

184. In the case of a circular cubic, we must have

$$\rho(\beta-\gamma)(\alpha-\delta)+\sigma(\gamma-\alpha)(\beta-\delta)+\tau(\alpha-\beta)(\gamma-\delta)=0,$$
$$\sqrt{\alpha\rho}\,(\beta-\gamma)\qquad+\sqrt{\beta\sigma}\,(\gamma-\alpha)\qquad+\sqrt{\gamma\tau}\,(\alpha-\beta)\qquad=0,$$

which, when the foci A, B, C, D are given, determine the values of $\rho:\sigma:\tau$ in order that the curve may be a circular cubic. We see at once that there are two sets of values, and consequently two circular cubics having each of them the given points A, B, C, D for a set of concyclic foci. The two systems may be written

$$\sqrt{\rho}:\sqrt{\sigma}:\sqrt{\tau}=\sqrt{\alpha\delta}-\sqrt{\beta\gamma}:\sqrt{\beta\delta}-\sqrt{\gamma\alpha}:\sqrt{\gamma\delta}-\sqrt{\alpha\beta},$$

viz., it being understood that $\sqrt{\alpha\delta}$ means $\sqrt{\alpha}\,.\sqrt{\delta}$, &c., then, according as $\sqrt{\delta}$ has one or other of its two opposite values, we have one or other of the two systems of values of $\rho:\sigma:\tau$. To verify this, observe that writing the equation under the form

$$\sqrt{\alpha\rho}:\sqrt{\beta\sigma}:\sqrt{\gamma\tau}=\alpha\sqrt{\delta}-\sqrt{\alpha\beta\gamma}:\beta\sqrt{\delta}-\sqrt{\alpha\beta\gamma}:\gamma\sqrt{\delta}-\sqrt{\alpha\beta\gamma},$$

the second equation is verified; and that writing them under the form

$$\rho:\sigma:\tau=-(\beta+\gamma)(\alpha+\delta)+M:-(\gamma+\alpha)(\beta+\delta)+M:-(\alpha+\beta)(\gamma+\delta)+M,$$

where

$$M=\beta\gamma+\alpha\delta+\gamma\alpha+\beta\delta+\alpha\beta+\gamma\delta-2\sqrt{\alpha\beta\gamma\delta},$$

the second equation is also verified.

185. If we assume for a moment $\alpha=\cos a+i\sin a=e^{ia}$, &c., viz., if a, b, c, d be the inclinations to any fixed line of the radii through A, B, C, D respectively, then we have

$$\sqrt{\alpha\delta}\pm\sqrt{\beta\gamma}=e^{\frac{1}{2}(a+b+c+d)i}\,\{e^{\frac{1}{4}(a+d-b-c)i}\pm e^{-\frac{1}{4}(a+d-b-c)i}\},$$
$$\sqrt{\alpha}\,(\beta-\gamma)=e^{\frac{1}{2}(a+b+c)i}\,\{e^{\frac{1}{2}(b-c)i}-e^{-\frac{1}{2}(b-c)i}\},$$

and thence

$$\begin{aligned}\sqrt{\alpha\rho}\,(\beta-\gamma):\sqrt{\beta\sigma}\,(\gamma-\alpha):\sqrt{\gamma\tau}\,(\alpha-\beta)=\ &\cos\tfrac{1}{4}(a+d-b-c)\sin\tfrac{1}{2}(b-c)\\ &:\cos\tfrac{1}{4}(b+d-c-a)\sin\tfrac{1}{2}(c-a)\\ &:\cos\tfrac{1}{4}(c+d-a-b)\sin\tfrac{1}{2}(a-b);\end{aligned}$$

or else

$$\begin{aligned}=\ &\sin\tfrac{1}{4}(a+d-b-c)\sin\tfrac{1}{2}(b-c)\\ &:\sin\tfrac{1}{4}(b+d-c-a)\sin\tfrac{1}{2}(c-a)\\ &:\sin\tfrac{1}{4}(c+d-a-b)\sin\tfrac{1}{2}(a-b).\end{aligned}$$

Putting in these formulæ,

$$\begin{aligned}\tfrac{1}{4}(a-b-c)&=A, &&\text{then we have} & B-C&=\tfrac{1}{2}(b-c),\\ \tfrac{1}{4}(b-c-a)&=B, &&\quad\text{,,} & C-A&=\tfrac{1}{2}(c-a),\\ \tfrac{1}{4}(c-a-b)&=C, &&\quad\text{,,} & A-B&=\tfrac{1}{2}(a-b),\end{aligned}$$

and for either set of values the verification of the relation

$$\sqrt{\alpha\rho}\,(\beta-\gamma)+\sqrt{\beta\sigma}\,(\gamma-\alpha)+\sqrt{\gamma\tau}\,(\alpha-\beta)=0,$$

will depend on the two identical equations

$$\sin A \sin (B-C)+\sin B \sin (C-A)+\sin C \sin (A-B)=0,$$

$$\cos A \sin (B-C)+\cos B \sin (C-A)+\cos C \sin (A-B)=0:$$

although the foregoing solution for the case of a circular cubic is the most elegant one, I will presently return to the question and give the solution in a different form.

Article No. 186. *Focal Formulæ for the Symmetrical Curve.*

186. In the symmetrical case, where the foci A, B, C, D are on a line, then if, as usual, a, b, c, d denote the distances from a fixed point, we have the expressions of (a, b, c, d) in a form adapted to the formulæ of No. 49, viz.,

$$\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}=(b-c)(c-d)(d-b):-(c-d)(d-a)(a-c):(d-a)(a-b)(b-d):-(a-b)(b-c)(c-a),$$

so that, assuming

$$l:m:n=\rho\,(b-c)^2:\sigma\,(c-a)^2:\tau\,(a-b)^2,$$

the equation

$$\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0,$$

becomes

$$\rho\,(b-c)\,(a-d)+\sigma\,(c-a)\,(b-d)+\tau\,(a-b)\,(c-d)=0,$$

and the equation of the curve may be presented under any one of the four forms

$$\left(\begin{array}{cccc} . & \sqrt{\tau}\,(d-c), & \sqrt{\sigma}\,(b-d), & \sqrt{\rho}\,(c-b) \\ \sqrt{\tau}\,(c-d), & . & \sqrt{\rho}\,(d-a), & \sqrt{\sigma}\,(a-c) \\ \sqrt{\sigma}\,(d-b), & \sqrt{\rho}\,(a-d), & . & \sqrt{\tau}\,(b-a) \\ \sqrt{\rho}\,(b-c), & \sqrt{\sigma}\,(c-a), & \sqrt{\tau}\,(a-b), & . \end{array}\right|(\sqrt{\mathfrak{A}},\ \sqrt{\mathfrak{B}},\ \sqrt{\mathfrak{C}},\ \sqrt{\mathfrak{D}})=0.$$

Article No. 187. *Case of the Symmetrical Circular Cubic.*

187. For a circular cubic we must have

$$\rho\,(b-c)\,(a-d)+\sigma\,(c-a)\,(b-d)+\tau\,(a-b)\,(c-d)=0,$$

$$\sqrt{\rho}\,(b-c)\qquad+\sqrt{\sigma}\,(c-a)\qquad+\sqrt{\tau}\,(a-b)\qquad=0.$$

These equations give $\sqrt{\rho}:\sqrt{\sigma}:\sqrt{\tau}=1:1:1$ (values which obviously satisfy the two equations), or else

$$\sqrt{\rho}:\sqrt{\sigma}:\sqrt{\rho}=a+d-b-c:b+d-c-a:c+d-a-b.$$

In fact, these values obviously satisfy the second equation; and to see that they satisfy the first equation, we have only to write them under the form

$$\rho : \sigma : \tau = M - 4(b+c)(a+d) : M - 4(c+a)(b+d) : M - 4(a+b)(c+d),$$

where $M = (a+b+c+d)^2$. The first set gives for the curve

$$(b-c)\sqrt{\mathfrak{A}} + (c-a)\sqrt{\mathfrak{B}} + (a-b)\sqrt{\mathfrak{C}} = 0,$$

but this contains the line $z=0$ not once only, but twice; it in fact is $(y^2=0)$, the axis taken twice; the only proper cubic with the foci A, B, C, D *in lineâ* is therefore

$$(b-c)(a+d-b-c)\sqrt{\mathfrak{A}} + (c-a)(b+d-c-a)\sqrt{\mathfrak{B}} + (a-b)(c+d-a-b)\sqrt{\mathfrak{C}} = 0,$$

the equation of which is, of course, expressible in each of the other three forms.

Article Nos. 188 to 192. *Case of the General Circular Cubit.*

188. Returning to the general case of the circular cubic, the lines BC, AD meet in R, and if we denote by a_1, b_1, c_1, d_1, the distances from R of the four points respectively, so that $b_1c_1 = a_1d_1 = \text{rad.}\,{}^2R$, then observing that a, b, c, d are proportional to the triangles BCD, CDA, DAB, ABC, with signs such that $\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}=0$, we find

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = -d_1(b_1-c_1) : c_1(a_1-d_1) : -b_1(a_1-d_1) : a_1(b_1-c_1);$$

and this being so, the equations $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$, $\sqrt{l}+\sqrt{m}+\sqrt{n}=0$, give two systems of values of $\sqrt{l} : \sqrt{m} : \sqrt{n}$, viz., these are

$$\sqrt{l} : \sqrt{m} : \sqrt{n} = b_1-c_1 : c_1-a_1 : \quad a_1-b_1,$$

and

$$= b_1-c_1 : c_1+a_1 : -a_1-b_1.$$

(To verify this, observe that for the first set we have

$$\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}} = \frac{(b_1-c_1)^2}{-d_1(b_1-c_1)} + \frac{(c_1-a_1)^2}{c_1(a_1-d_1)} + \frac{(a_1-b_1)^2}{-b_1(a_1-d_1)},$$

$$= \frac{b_1-c_1}{-d_1} + \frac{1}{a_1-d_1}\left(c_1+\frac{a_1^2}{c_1}-b_1-\frac{a_1^2}{b_1}\right),$$

$$= \frac{b_1-c_1}{-d_1} + \frac{b_1-c_1}{a_1-d_1}\left(\frac{a_1^2}{b_1c_1}-1\right),$$

$$= -\frac{b_1-c_1}{d_1} + \frac{b_1-c_1}{a_1-d_1}\left(\frac{a_1}{d_1}-1\right), = 0;$$

and the like as regards the second set.)

189. These values of $\sqrt{l} : \sqrt{m} : \sqrt{n}$ give the equations of the two circular cubics with the foci (A, B, C, D), the equation of each of them under a fourfold form, viz., we have

$$\left(\begin{array}{cccc} \cdot\;, & d_1-c_1, & b_1-d_1, & c_1-b_1 \\ c_1-d_1, & \cdot\;, & d_1-a_1, & a_1-c_1 \\ d_1-b_1, & a_1-d_1, & \cdot\;, & b_1-a_1 \\ b_1-c_1, & c_1-a_1, & a_1-b_1, & \cdot \end{array}\right|(\sqrt{\mathfrak{A}}, \sqrt{\mathfrak{B}}, \sqrt{\mathfrak{C}}, \sqrt{\mathfrak{D}})=0 \quad \text{(first curve)},$$

and

$$\left(\begin{array}{cccc} \cdot\;, & -c_1-d_1, & d_1+b_1, & -b_1+c_1 \\ d_1+c_1, & \cdot\;, & a_1-d_1, & c_1-a_1 \\ -b_1-d_1, & d_1-a_1, & \cdot\;, & a_1+b_1 \\ b_1-c_1, & -c_1+a_1, & -a_1-b_1, & \cdot \end{array}\right|(\sqrt{\mathfrak{A}}, \sqrt{\mathfrak{B}}, \sqrt{\mathfrak{C}}, \sqrt{\mathfrak{D}})=0 \quad \text{(second curve)}.$$

190. Similarly CA and BD meet in S, and if we denote by a_2, b_2, c_2, d_2 the distances from S of the four points respectively, so that $c_2a_2 = b_2d_2 = \text{rad.}^2 S$ (observe that if as usual A, B, C, D are taken in order on the circle O, then A, C are on opposite sides of S, and similarly B, D are on opposite sides of S, so that taking a_2, b_2 positive c_2, d_2 will be negative), we have

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = c_2(b_2-d_2) : d_2(c_2-a_2) : -a_2(b_2-d_2) : -b_2(c_2-a_2),$$

and then the equations $\dfrac{l}{\mathrm{a}}+\dfrac{m}{\mathrm{b}}+\dfrac{n}{\mathrm{c}}=0$, $\sqrt{l}+\sqrt{m}+\sqrt{n}=0$, are satisfied by the two sets of values

$$\sqrt{l} : \sqrt{m} : \sqrt{n} = \quad b_2-c_2 : c_2-a_2 : a_2-b_2,$$

and

$$= -b_2-c_2 : c_2-a_2 : a_2+b_2,$$

and we have the equations of the same two cubic curves, each equation under a fourfold form, viz., these are

$$\left(\begin{array}{cccc} \cdot\;, & -c_2+d_2, & -d_2+b_2, & -b_2+c_2 \\ c_2-d_2, & \cdot\;, & d_2-a_2, & -c_2+a_2 \\ -b_2+d_2, & a_2-d_2, & \cdot\;, & -a_2+b_2 \\ b_2-c_2, & c_2-a_2, & a_2-b_2, & \cdot \end{array}\right|(\sqrt{\mathfrak{A}}, \sqrt{\mathfrak{B}}, \sqrt{\mathfrak{C}}, \sqrt{\mathfrak{D}})=0 \quad \text{(first curve)},$$

and

$$\left(\begin{array}{cccc} \cdot\;, & c_2+d_2, & -d_2+b_2, & -b_2-c_2 \\ -d_2-c_2, & \cdot\;, & a_2+d_2, & c_2-a_2 \\ -b_2+d_2, & -d_2-a_2, & \cdot\;, & a_2+b_2 \\ b_2+c_2, & -c_2+a_2, & -a_2-b_2, & \cdot \end{array}\right|(\sqrt{\mathfrak{A}}, \sqrt{\mathfrak{B}}, \sqrt{\mathfrak{C}}, \sqrt{\mathfrak{D}})=0 \quad \text{(second curve)}.$$

191. And again AB and CD meet in T, and denoting by a_3, b_3, c_3, d_3 the distances from T of the four points respectively, so that $a_3b_3 = c_3d_3 = \text{rad.}^2 T$, we have

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = b_3(c_3-d_3) : -a_3(c_3-d_3) : -d_3(a_3-b_3) : c_3(a_3-b_3);$$

and the equations $\frac{l}{\text{a}}+\frac{m}{\text{b}}+\frac{n}{\text{c}}=0$, $\sqrt{l}+\sqrt{m}+\sqrt{n}=0$, then give for $\sqrt{l}$, $\sqrt{m}$, $\sqrt{n}$ two sets of values, viz., these are

$$\sqrt{l} : \sqrt{m} : \sqrt{n} = b_3 - c_3 : c_3 - a_3 : a_3 - b_3,$$

and

$$= b_3 + c_3 : -c_3 - a_3 : a_3 - b_3;$$

and we again obtain the equations of the two cubics, each equation under a fourfold form, viz., these are

$$\left|\begin{matrix} . , & -c_3+d_3, & -d_3+b_3, & c_3-b_3 \\ -d_3+c_3, & . , & -a_3+d_3, & a_3-c_3 \\ -b_3+d_3, & -d_3+a_3, & . , & b_3-a_3 \\ b_3-c_3, & c_3-a_3, & a_3-b_3, & . \end{matrix}\right| (\sqrt{\mathfrak{A}}, \sqrt{\mathfrak{B}}, \sqrt{\mathfrak{C}}, \sqrt{\mathfrak{D}})=0,$$

and

$$\left|\begin{matrix} . , & c_3-d_3, & d_3+b_3, & -c_3-b_3 \\ d_3-c_3, & . , & -a_3-d_3, & a_3+c_3 \\ -b_3-d_3, & d_3+a_3, & . , & b_3-a_3 \\ b_3+c_3, & -c_3-a_3, & a_3-b_3, & . \end{matrix}\right| (\sqrt{\mathfrak{A}}, \sqrt{\mathfrak{B}}, \sqrt{\mathfrak{C}}, \sqrt{\mathfrak{D}})=0.$$

192. The three systems have been obtained independently, but they may of course be derived each from any other of them: to show how this is, recollecting that we have

$$RA,\ RB,\ RC,\ RD = a_1,\ b_1,\ \ c_1,\ \ d_1,$$
$$SA,\ SB,\ SC,\ SD = a_2,\ b_2,\ -c_2,\ -d_2,$$
$$TA,\ TB,\ TC,\ TD = a_3,\ b_3,\ \ c_3,\ \ d_3;$$

then to compare

$$(a_1, b_1, c_1, d_1) \text{ and } (a_2, b_2, c_2, d_2);$$

the similar triangles

$$SBC \text{ give } b_1 - c_1 : -c_2 : b_2,$$
$$SAD \quad = a_1 - d_1 : -d_2 : a_2,$$

and the similar triangles

$$RAC \text{ give } a_2 - c_2 : c_1 : a_1,$$
$$RBD \quad = b_2 - d_2 : d_1 : b_1;$$

using these equations to determine the ratios of a_2, b_2, c_2, d_2 we have

$$\frac{a_2-c_2}{b_2-d_2}=\frac{c_1}{d_1}, \text{ or } d_1a_2 - d_1c_2 - c_1b_2 + c_1d_2 = 0;$$

that is

$$b_2\left\{-c_1+d_1\frac{a_1-d_1}{b_1-c_1}\right\}+c_2\left\{-d_1+c_1\frac{a_1-d_1}{b_1-c_1}\right\}=0;$$

and hence

$$b_2(-b_1c_1+c_1^2+a_1d_1-d_1^2)+c_2(-b_1d_1+c_1d_1+a_1c_1-c_1d_1)=0,$$

that is

$$b_2(c_1^2-d_1^2)+c_2(a_1c_1-b_1d_1)=0,$$

but

$$a_1c_1-b_1d_1=\frac{b_1}{d_1}(c_1^2-d_1^2),$$

or the equation gives $b_2+\dfrac{b_1}{d_1}c_2=0$, or say $b_2:c_2=b_1:-d_1$, and this with $\dfrac{b_1-c_1}{a_1-d_1}=\dfrac{c_2}{d_2}=\dfrac{b_2}{a_2}$, gives all the ratios, or we have

$$a_2 : b_2 : c_2 : d_2 = b_1(a_1-d_1) : b_1(b_1-c_1) : -d_1(a_1-d_1) : -d_1(b_1-c_1).$$

We have then for example

$$b_2-c_2 : c_2-a_2 : a_2-b_2 = b_1-c_1 : c_1-a_1 : a_1-b_1 \text{ ; \&c.},$$

showing the identity of the forms in (a_1, b_1, c_1, d_1) and (a_2, b_2, c_2, d_2).

Article No. 193. *Transformation to a New Set of Concyclic Foci.*

193. Consider the equation

$$\sqrt{l\mathfrak{A}}+\sqrt{m\mathfrak{B}}+\sqrt{n\mathfrak{C}}=0,$$

which refers to the foci A, B, C, and taking D the fourth concyclic focus, let (A_1, D_1) be the antipoints of (A, D) and (B_1, C_1) the antipoints of (B, C); so that (A_1, B_1, C_1, D_1) are another set of concyclic foci. We have $\mathfrak{B}_1 . \mathfrak{C}_1=\mathfrak{B} . \mathfrak{C}$, and it appears, *ante* No. 104, that we can find l_1, m_1, n_1, such that identically

$$-l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}=-l_1\mathfrak{A}_1+m_1\mathfrak{B}_1+n_1\mathfrak{C}_1$$

and that $m_1n_1=mn$. The equation of the curve gives

$$-l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}+2\sqrt{mn\mathfrak{B}\mathfrak{C}}=0,$$

we have therefore

$$-l_1\mathfrak{A}_1+m_1\mathfrak{B}_1+n_1\mathfrak{C}+2\sqrt{m_1n_1\mathfrak{B}_1\mathfrak{C}_1}=0,$$

that is,

$$\sqrt{l_1\mathfrak{A}_1}+\sqrt{m_1\mathfrak{B}_1}+\sqrt{n_1\mathfrak{C}_1}=0,$$

viz., this is the equation of the curve expressed in terms of the concyclic foci A_1, B_1, C_1.

Article No. 194. *The Tetrazomal Curve, Decomposable or Indecomposable.*

194. I consider the tetrazomal curve

$$\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}+\sqrt{p\mathfrak{D}^\circ}=0,$$

where the zomals are circles described about any given points A, B, C, D as centres.

There is not, in general, any identical equation $a\mathfrak{A}^\circ + b\mathfrak{B}^\circ + c\mathfrak{C}^\circ + d\mathfrak{D}^\circ = 0$, but when such relation exists, and when we have also $\frac{l}{a} + \frac{m}{b} + \frac{n}{c} + \frac{p}{d} = 0$, then the curve breaks up into two trizomals. When the conditions in question do not subsist, the curve is indecomposable. But there may exist between l, m, n, p relations in virtue of which a branch or branches ideally contain $(z^a = 0)$ the line infinity a certain number of times, and which thus cause a depression in the order of the curve. The several cases are as follows:

Article No. 195. *Cases of the Indecomposable Curve.*

195. I. The general case; l, m, n, p not subjected to any condition. The curve is here of the order $= 8$; it has a quadruple point at each of the points I, J (and there is consequently no other point at infinity); it is touched four times by each of the circles A, B, C, D; and it has six nodes, viz., these are the intersections of the pairs of circles

$$\sqrt{m\mathfrak{B}^\circ} + \sqrt{n\mathfrak{C}^\circ} = 0, \quad \sqrt{l\mathfrak{A}^\circ} + \sqrt{p\mathfrak{D}^\circ} = 0,$$
$$\sqrt{n\mathfrak{C}^\circ} + \sqrt{l\mathfrak{A}^\circ} = 0, \quad \sqrt{m\mathfrak{B}^\circ} + \sqrt{p\mathfrak{D}^\circ} = 0,$$
$$\sqrt{l\mathfrak{A}^\circ} + \sqrt{m\mathfrak{B}^\circ} = 0, \quad \sqrt{n\mathfrak{C}^\circ} + \sqrt{p\mathfrak{D}^\circ} = 0;$$

the number of dps. is $6 + 2.6, = 18$, and there are no cusps, hence the class is $= 20$, and the deficiency is $= 3$.

II. We may have

$$\sqrt{l} + \sqrt{m} + \sqrt{n} + \sqrt{p} = 0;$$

there is in this case a single branch ideally containing $(z = 0)$ the line infinity; the order is $= 7$. Each of the points I, J is a triple point, there is consequently one other point at infinity; viz., this is a real point, or the curve has a real asymptote. There are 6 nodes as before; dps. are $6 + 2.3, = 12$; class $= 18$, deficiency $= 3$.

III. We may have

$$\sqrt{l} + \sqrt{m} = 0, \quad \sqrt{n} + \sqrt{p} = 0;$$

there are then two branches each ideally containing $(z = 0)$ the line infinity; the order is $= 6$. Each of the points I, J is a double point, and there are therefore two more points at infinity. These may be real or imaginary; viz., the curve may have (besides the asymptotes at I, J) two real or imaginary asymptotes. The circles $\sqrt{l\mathfrak{A}} + \sqrt{m\mathfrak{B}} = 0$, $\sqrt{n\mathfrak{C}} + \sqrt{p\mathfrak{D}} = 0$, each contain $(z = 0)$ the line infinity, or they reduce themselves to two lines, so that in place of two nodes we have a single node at the intersection of these lines; number of nodes is $= 5$. Hence dps. are $5 + 2.1, = 7$. Class $= 16$, deficiency $= 3$.

IV. We may have

$$\sqrt{l} : \sqrt{m} : \sqrt{n} : \sqrt{p} = a : b : c : d;$$

there is here a single branch containing ($z^2=0$) the line infinity twice; the order is $=6$. Each of the points I, J is a double point, and there are therefore two more points at infinity, that is (besides the asymptotes at I, J), there are two (real or imaginary) asymptotes. The number of nodes, as in the general case, is $=6$. Hence dps. are $6+2.1, =8$; class is $=14$; deficiency $=2$.

I notice the included particular case where the circles reduce themselves to their centres; viz., we have here the curve

$$\mathrm{a}\sqrt{\mathfrak{A}}+\mathrm{b}\sqrt{\mathfrak{B}}+\mathrm{c}\sqrt{\mathfrak{C}}+\mathrm{d}\sqrt{\mathfrak{D}}=0,$$

which (see *ante* No. 93) is in fact the curve which is the locus of the foci of the conics which pass through the four points A, B, C, D. It is at present assumed that the four points are not a circle; this case will be considered *post* No. 199. If we have BC, AD meeting in R; CA, BD in S, and AB, CD in T, then these points R, S, T are three of the six nodes. In fact, writing down the equations of the two circles

$$\mathrm{b}\sqrt{\mathfrak{B}}+\mathrm{c}\sqrt{\mathfrak{C}}=0,\quad \mathrm{a}\sqrt{\mathfrak{A}}+\mathrm{d}\sqrt{\mathfrak{D}}=0,$$

and observing that when the current point is taken at R, we have $\mathfrak{B}:\mathfrak{C}=\overline{RB}^2:\overline{RC}^2=(BAD)^2:(CAD)^2=\mathrm{c}^2:\mathrm{b}^2$, and similarly $\mathfrak{A}:\mathfrak{D}=\overline{RA}^2:\overline{RD}^2=(ABC)^2:(DBC)^2=\mathrm{d}^2:\mathrm{a}^2$, we see that each of the two circles passes through the point R, or this point is a node. Similarly, the points S and T are each of them a node.

V. If

$$\sqrt{l}=\sqrt{m}=\sqrt{n}=\sqrt{p},$$

there are here three branches, each ideally containing ($z=0$) the line infinity; the order is thus $=5$. Each of the points I, J is an ordinary point on the curve; there are besides at infinity three points, all real, or one real and two imaginary; that is (besides the asymptotes at I, J) there are three asymptotes, all real, or one real and two imaginary. Each of the circles $\sqrt{\mathfrak{A}}+\sqrt{\mathfrak{B}}=0$, &c., contains the line infinity, and is thus reduced to a line; the number of nodes is therefore $=3$. Hence also, dps. $=3$; class $=14$; deficiency $=3$.

Article No. 196. *Cases of the Indecomposable Curve, the Centres being in a Line.*

196. There are some peculiarities in the case where the centres A, B, C, D are on a line; taking as usual (a, b, c, d) for the x-coordinates or distances of the four centres from a fixed point on the line, I enumerate the cases as follows:

I. No relation between l, m, n, p; corresponds to I. *supra.*

II. $\sqrt{l}+\sqrt{m}+\sqrt{n}+\sqrt{p}=0$; corresponds to II. *supra.*

III. $\sqrt{l}+\sqrt{m}=0$, $\sqrt{n}+\sqrt{p}=0$; corresponds to III. *supra.*

IV. $\sqrt{l}+\sqrt{m}+\sqrt{n}+\sqrt{p}=0,\ a\sqrt{l}+b\sqrt{m}+c\sqrt{n}+d\sqrt{p}=0$; corresponds to IV. *supra*, viz., there is a branch ideally containing $(z^2=0)$ the line infinity twice. But, observe that whereas in IV. *supra*, in order that this might be so, it was necessary to impose on l, m, n, p three conditions giving the definite systems of values $\sqrt{l} : \sqrt{m} : \sqrt{n} : \sqrt{p} = \mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d}$, in the present case only two conditions are imposed, so that a single arbitrary parameter is left.

V. $\sqrt{l}=\sqrt{m}=\sqrt{n}=\sqrt{p}$; corresponds to V. *supra*.

VI. $\sqrt{l}+\sqrt{m}=0,\ \sqrt{n}+\sqrt{p}=0,\ a\sqrt{l}+b\sqrt{m}+c\sqrt{n}+d\sqrt{p}=0$, or what is the same thing, $\sqrt{l} : \sqrt{m} : \sqrt{n} : \sqrt{p} = c-d : d-c : b-a : a-b$; the equation is thus $(c-d)(\sqrt{\mathfrak{A}^\circ}-\sqrt{\mathfrak{B}^\circ})-(a-b)(\sqrt{\mathfrak{A}^\circ}-\sqrt{\mathfrak{B}^\circ})=0$. There is here one branch ideally containing $(z^2=0)$ the line infinity twice, and another branch ideally containing $(z=0)$ the line infinity once; order is $=5$. Each of the points I, J is an ordinary point on the curve, the remaining points at infinity are a node $(\mathfrak{A}^\circ=\mathfrak{B}^\circ,\ \mathfrak{C}^\circ=\mathfrak{D}^\circ)$, as presently mentioned, counting as three points, viz., one branch has for its tangent the line infinity, and the other branch has for its tangent a line perpendicular to the axis; or what is the same thing, there is a hyperbolic branch having an asymptote perpendicular to the axis, and a parabolic branch ultimately perpendicular to the axis. The number of nodes is $=5$, viz., there is the node $\mathfrak{A}^\circ=\mathfrak{B}^\circ,\ \mathfrak{C}^\circ=\mathfrak{D}^\circ$ just referred to; and the two pairs of nodes $\big((c-d)\sqrt{\mathfrak{A}^\circ}-(a-b)\sqrt{\mathfrak{C}^\circ}=0,\ -(c-d)\sqrt{\mathfrak{B}^\circ}+(a-b)\sqrt{\mathfrak{D}^\circ}=0\big)$ and $\big((c-d)\sqrt{\mathfrak{A}^\circ}+(a-b)\sqrt{\mathfrak{D}^\circ}=0,\ (c-d)\sqrt{\mathfrak{B}^\circ}+(a-b)\sqrt{\mathfrak{C}^\circ}=0\big)$, each pair symmetrically situate in regard to the axis. Hence also dps. $=5$; class $=10$; deficiency $=1$.

And there is apparently a seventh case, which, however, I exclude from the present investigation, viz., this would be if we had

$$\left(\begin{array}{cccc} 1, & 1, & 1, & 1, \\ a, & b, & c, & d, \\ a^2, & b^2, & c^2, & d^2, \\ a''^2, & b''^2, & c''^2, & d''^2, \end{array}\right)(\sqrt{l},\ \sqrt{m},\ \sqrt{n},\ \sqrt{p})=0,$$

that is, a, b, c, d denoting as before, if we had

$$\sqrt{l} : \sqrt{m} : \sqrt{n} : \sqrt{p} = \mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d},\quad \text{and}\quad \mathrm{a}a''^2+\mathrm{b}b''^2+\mathrm{c}c''^2+\mathrm{d}d''^2=0.$$

For observe that in this case we have

$$\mathrm{a}\mathfrak{A}^\circ+\mathrm{b}\mathfrak{B}^\circ+\mathrm{c}\mathfrak{C}^\circ+\mathrm{d}\mathfrak{D}^\circ=0,\quad\text{and}\quad \frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0;$$

that is, the supposition in question belongs to the decomposable case.

Article No. 197. *The Decomposable Curve.*

197. We have next to consider the decomposable case, viz., when we have

$$\mathrm{a}\mathfrak{A}^\circ+\mathrm{b}\mathfrak{B}^\circ+\mathrm{c}\mathfrak{C}^\circ+\mathrm{d}\mathfrak{D}^\circ=0;$$

see *ante*, Nos. 87 *et seq.*—it there appears that (unless the centres A, B, C, D are in a line) the condition signifies that the four circles have a common orthotomic circle; and when we have also

$$\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0.$$

The formulæ for the decomposition are given *ante*, Nos. 42 *et seq.* Writing therein $\mathfrak{A}^\circ$, $\mathfrak{B}^\circ$, $\mathfrak{C}^\circ$, $\mathfrak{D}^\circ$ in place of U, V, W, T respectively, it thereby appears that the tetrazomal curve $\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}+\sqrt{p\mathfrak{D}^\circ}=0$, breaks up into the two trizomal curves

$$\sqrt{l_1\mathfrak{A}^\circ}+\sqrt{m_1\mathfrak{B}^\circ}+\sqrt{n_1\mathfrak{C}^\circ}=0,\quad \sqrt{l_2\mathfrak{A}^\circ}+\sqrt{m_2\mathfrak{B}^\circ}+\sqrt{n_2\mathfrak{C}^\circ}=0,$$

where

$$\sqrt{l_1}=\sqrt{l}+\frac{\mathrm{a}}{\mathrm{d}}\frac{p}{\sqrt{l}},\qquad \sqrt{l_2}=\sqrt{l}+\frac{\mathrm{a}}{\mathrm{d}}\frac{p}{\sqrt{l}},$$

$$\sqrt{m_1}=\sqrt{m}-\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\,\frac{p}{l}\,\mathrm{b}\sqrt{n},\quad \sqrt{m_2}=\sqrt{m}+\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\,\frac{p}{l}\,\mathrm{b}\sqrt{n},$$

$$\sqrt{n_1}=\sqrt{n}+\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\,\frac{p}{l}\,\mathrm{c}\sqrt{m},\quad \sqrt{n_2}=\sqrt{n}-\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\,\frac{p}{l}\,\mathrm{c}\sqrt{m},$$

and where we have

$$\frac{l_1}{\mathrm{a}}+\frac{m_1}{\mathrm{b}}+\frac{n_1}{\mathrm{c}}=0,\quad \frac{l_2}{\mathrm{a}}+\frac{m_2}{\mathrm{b}}+\frac{n_2}{\mathrm{c}}=0.$$

Article Nos. 198 to 203. *Cases of the Decomposable Curve, Centres not in a line.*

198. I assume, in the first instance, that the centres of the circles are not in a line; we have the following cases:

I. No further relation between l, m, n, p; the order of the tetrazomal is $=8$; the order of each of the trizomals is $=4$, that is each of them is a bicircular quartic.

II. $\sqrt{l}+\sqrt{m}+\sqrt{n}+\sqrt{p}=0$; the order of the tetrazomal is $=7$, that of one of the trizomals must be $=3$.

To verify this, observe that we have

$$\sqrt{l_1}+\sqrt{m_1}+\sqrt{n_1}=\sqrt{l}+\sqrt{m}+\sqrt{n}+\frac{\mathrm{a}p}{\mathrm{d}\sqrt{l}}+\frac{\sqrt{p}}{\sqrt{l}}\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\,(\mathrm{c}\sqrt{m}-\mathrm{b}\sqrt{n}),$$

or substituting for $\sqrt{l}+\sqrt{m}+\sqrt{n}$ the value $-\sqrt{p}$, this is

$$=\frac{\sqrt{p}}{\mathrm{d}\sqrt{l}}\left\{\mathrm{a}\sqrt{p}-\mathrm{d}\sqrt{l}+\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\,(\mathrm{c}\sqrt{m}-\mathrm{b}\sqrt{n})\right\},$$

and similarly for $\sqrt{l_2}+\sqrt{m_2}+\sqrt{n_2}$, the only change being in the sign of the radical $\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}$. But from the two conditions satisfied by l, m, n, p it is easy to deduce

$$(\mathrm{a}\sqrt{p}-\mathrm{d}\sqrt{l})^2-\frac{\mathrm{ad}}{\mathrm{bc}}(\mathrm{c}\sqrt{m}-\mathrm{b}\sqrt{n})^2=0,$$

and hence one or other of the two functions

$$\sqrt{l_1}+\sqrt{m_1}+\sqrt{n_1},\ \sqrt{l_2}+\sqrt{m_2}+\sqrt{n_2} \text{ is } =0\,;$$

that is, one of the trizomal curves is a cubic.

III. $\sqrt{l}+\sqrt{p}=0$, $\sqrt{m}+\sqrt{n}=0$; order of the tetrazomal is $=6$; and hence order of each of the trizomals is $=3$. To verify this, observe that here

$$l\left(\frac{1}{\mathrm{a}}+\frac{1}{\mathrm{d}}\right)+m\left(\frac{1}{b}+\frac{1}{c}\right)=0,$$

which since $\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}=0$, gives $\dfrac{l}{m}=\dfrac{\mathrm{ad}}{\mathrm{bc}}$; so that, properly fixing the sign of the radical, we may write $\sqrt{l}+\sqrt{\dfrac{\mathrm{ad}}{\mathrm{bc}}}\sqrt{m}=0$. We have then

$$\sqrt{l_1}=\frac{\mathrm{a}+\mathrm{d}}{\mathrm{d}}\sqrt{l},\quad \sqrt{m_1}+\sqrt{n_1}=\sqrt{\frac{\mathrm{a}}{\mathrm{abc}}}\,(\mathrm{b}+\mathrm{c})\sqrt{m}\,;$$

which last equation, using $\sqrt{\dfrac{\mathrm{ad}}{\mathrm{bc}}}$ to denote as above, but properly selecting the signification of $\pm$, may be written

$$\sqrt{m_1}+\sqrt{n_1}=\pm\frac{\mathrm{b}+\mathrm{c}}{\mathrm{d}}\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\sqrt{m}.$$

Hence

$$\sqrt{l_1}\mp(\sqrt{m_1}+\sqrt{n_1})=\frac{1}{\mathrm{a}}\left\{(\mathrm{a}+\mathrm{d})\sqrt{l}+(\mathrm{b}+\mathrm{c})\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\sqrt{m}\right\}$$

$$=\frac{\mathrm{a}+\mathrm{d}}{\mathrm{d}}\left\{\sqrt{l}+\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\sqrt{m}\right\},\ =0,$$

viz., $\sqrt{l_1}\mp(\sqrt{m_1}+\sqrt{n_1})$ with a properly selected signification of the sign $\mp$ is $=0$; and similarly $\sqrt{l_2}\mp(\sqrt{m_2}+\sqrt{n_2})$ with a properly selected signification of the sign $\mp$ is $=0$; that is, each of the trizomals is a cubic.

199. IV. $\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}=\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}$ $\Big($values which, be it observed, satisfy of themselves the above assumed equation $\dfrac{l}{\mathrm{a}}+\dfrac{m}{\mathrm{b}}+\dfrac{n}{\mathrm{c}}+\dfrac{p}{\mathrm{d}}=0\Big)$; the order of the tetrazomal is $=6$; and the order of each of the trizomals is here again $=3$. We in fact have $\sqrt{l_1}=\mathrm{a}+\mathrm{d}$, $\sqrt{m_1}+\sqrt{n_1}=\mathrm{b}+\mathrm{c}$, and therefore $\sqrt{l_1}+\sqrt{m_1}+\sqrt{n_1}=0$; and similarly $\sqrt{l_2}+\sqrt{m_2}+\sqrt{n_2}=0$; that is, each of the trizomals is a cubic.

I attend, in particular, to the case where the four circles reduce themselves to the points A, B, C, D; these four points are then in a circle; and the curve under consideration is

$$\mathrm{a}\sqrt{\mathfrak{A}}+\mathrm{b}\sqrt{\mathfrak{B}}+\mathrm{c}\sqrt{\mathfrak{C}}+\mathrm{d}\sqrt{\mathfrak{D}}=0\,;$$

in the general case where the points A, B, C, D are not on a circle, this is, as has been seen, a sextic curve, the locus of the foci of the conics which pass through the four given points; in the case where the points are in a circle then the sextic breaks up into two cubics (viz., observing that the curve under consideration is $\sqrt{l\mathfrak{A}}+\sqrt{m\mathfrak{B}}+\sqrt{n\mathfrak{C}}+\sqrt{p\mathfrak{D}}=0$, where $\sqrt{l} : \sqrt{m} : \sqrt{n} : \sqrt{p} = \mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d}$, these values do of themselves satisfy the condition of decomposability $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0$), that is, the locus of the foci of the conics which pass through four points on a circle is composed of two circular cubics, each of them having the four points for a set of concyclic foci. It is easy to see why the sextic, thus defined as a locus of foci, must break up into two cubics; in fact, as we have seen, the conics which pass through the four concyclic points A, B, C, D have their axes in two fixed directions; there is consequently a locus of the foci situate on the axes which are in one of the fixed directions, and a separate locus of the foci situate on the axes which lie in the other of the fixed directions; viz., each of these loci is a circular cubic.

200. Adopting the notation of No. 188, or writing

$$RA=a_1,\ RB=b_1,\ RC=c_1,\ RD=d_1,$$

(and therefore $b_1c_1=a_1d_1$) we have

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = -d_1(b_1-c_1) : c_1(a_1-d_1) : -b_1(a_1-d_1) : a_1(b_1-c_1).$$

Moreover

$$\sqrt{l_1}=\mathrm{a}+\mathrm{d}, \quad \sqrt{l_2}=\mathrm{a}+\mathrm{d},$$

$$\sqrt{m_1}=\mathrm{b}+\sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}}, \quad \sqrt{m_2}=\mathrm{b}-\sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}},$$

$$\sqrt{n_1}=\mathrm{c}-\sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}}, \quad \sqrt{n_2}=\mathrm{c}+\sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}},$$

and we have

$$\frac{\mathrm{bcd}}{\mathrm{a}}=(a_1-d_1)^2\frac{a_1b_1c_1}{d_1}=a_1^2(a_1-d_1)^2, \quad \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}}=-a_1(a_1-d_1) \text{ suppose};$$

and thence

$$\sqrt{l_1}=(a_1-d_1)(b_1-c_1), \quad \sqrt{l_2}=(a_1-d_1)(\ \ b_1-c_1)$$

$$\sqrt{m_1}=(a_1-d_1)(c_1-a_1), \quad \sqrt{m_2}=(a_1-d_1)(\ \ c_1+a_1)$$

$$\sqrt{n_1}=(a_1-d_1)(a_1-b_1), \quad \sqrt{n_2}=(a_1-d_1)(-a_1-b_1),$$

that is

$$\sqrt{l_1} : \sqrt{m_1} : \sqrt{n_1} = b_1-c_1 : c_1-a_1 : \ \ a_1-b_1,$$

$$\sqrt{l_2} : \sqrt{m_2} : \sqrt{n_2} = b_1-c_1 : c_1+a_1 : -a_1-b_1,$$

agreeing with the formulæ No. 188.

The tetrazomal curve

$$-d_1(b_1-c_1)\sqrt{\mathfrak{A}}+c_1(a_1-d_1)\sqrt{\mathfrak{B}}-b_1(a_1-d_1)\sqrt{\mathfrak{C}}+a_1(b_1-c_1)\sqrt{\mathfrak{D}}=0$$

is thus decomposed into the two trizomals

$$(b_1 - c_1)\sqrt{\mathfrak{A}} + (c_1 - a_1)\sqrt{\mathfrak{B}} + (a_1 - b_1)\sqrt{\mathfrak{C}} = 0,$$

$$(b_1 - c_1)\sqrt{\mathfrak{A}} + (c_1 + a_1)\sqrt{\mathfrak{B}} - (a_1 + b_1)\sqrt{\mathfrak{C}} = 0.$$

201. Observe that the tetrazomal equation is a consequence of either of the trizomal equations; taking for instance the first trizomal equation, this gives the tetrazomal equation, and consequently any combination of the trizomal equation and the tetrazomal equation is satisfied if only the trizomal equation is satisfied. Multiply the trizomal equation by $-a_1 + d_1$ and add it to the tetrazomal equation; the resulting equation contains the factor a_1, and omitting this, it is

$$(b_1 - c_1)(-\sqrt{\mathfrak{A}} + \sqrt{\mathfrak{D}}) + (a_1 - d_1)(\sqrt{\mathfrak{B}} - \sqrt{\mathfrak{C}}) = 0,$$

where observe that $b_1 - c_1$ is the distance BC, and $a_1 - d_1$ the distance AD. But in like manner multiplying the second trizomal equation by $-a_1 + d_1$, and adding it to the original tetrazomal equation, the resulting equation, omitting the factor a_1, is

$$(b_1 - c_1)(-\sqrt{\mathfrak{A}} + \sqrt{\mathfrak{D}}) - (a_1 - d_1)(\sqrt{\mathfrak{B}} - \sqrt{\mathfrak{C}}) = 0;$$

viz., it is in fact the same tetrazomal equation as was obtained by means of the first trizomal equation.

The new tetrazomal equation, say

$$(b_1 - c_1)(-\sqrt{\mathfrak{A}} + \sqrt{\mathfrak{D}}) + (a_1 - d_1)(\sqrt{\mathfrak{B}} - \sqrt{\mathfrak{C}}) = 0,$$

is thus equivalent to the original tetrazomal equation; observe that it is an equation of the form $\sqrt{l\mathfrak{A}} + \sqrt{m\mathfrak{B}} + \sqrt{n\mathfrak{C}} + \sqrt{p\mathfrak{D}} = 0$, where

$$\sqrt{l} = -(b_1 - c_1),\quad \sqrt{m} = a_1 - d_1,\quad \sqrt{n} = (a_1 - d_1),\quad \sqrt{p} = b_1 - c_1,$$

and where consequently $\sqrt{l} + \sqrt{p} = 0$, $\sqrt{m} + \sqrt{n} = 0$, that is an equation of the form (198) III., decomposable, as it should be, into the equations of two circular cubics. Writing

$$\frac{-\sqrt{\mathfrak{A}} + \sqrt{\mathfrak{D}}}{a_1 - d_1} = \theta,\quad \frac{\sqrt{\mathfrak{B}} - \sqrt{\mathfrak{C}}}{b_1 - c_1} = \theta,$$

where θ is an arbitrary parameter, the curve is obtained as the locus of the intersections of two similar conics having respectively the foci (A, D) and the foci (B, C) (see Salmon, *Higher Plane Curves*, p. 174): whence we have the theorem, that if A, B, C, D are any four points on a circle, the two circular cubics which are the locus of the foci of the conics which pass through the four points A, B, C, D, are also the locus of the intersections of the similar conics, which have for their foci (A, D) and (B, C) respectively; and of the similar conics with the foci (B, D) and (C, A) respectively; and of the similar conics with the foci (C, D) and (A, B) respectively.

202. V. $\sqrt{l} = \sqrt{m} = \sqrt{n} = \sqrt{p}$. The order of the tetrazomal is $= 5$, whence those of the trizomals should be $= 3$ and $= 2$ respectively. To verify this observe that the

equation $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0$ gives $\frac{1}{\mathrm{a}}+\frac{1}{\mathrm{b}}+\frac{1}{\mathrm{c}}+\frac{1}{\mathrm{d}}=0$, and combining with $\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}=0$, these are only satisfied by one of the systems $(\mathrm{a}+\mathrm{b}=0,\ \mathrm{c}+\mathrm{d}=0)$, $(\mathrm{a}+\mathrm{c}=0,\ \mathrm{b}+\mathrm{d}=0)$, $(\mathrm{a}+\mathrm{d}=0,\ \mathrm{b}+\mathrm{c}=0)$. Selecting to fix the ideas the first of these, or writing

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d})=(\mathrm{a},\ -\mathrm{a},\ \mathrm{c},\ -\mathrm{c}),$$

so that we have identically

$$\mathrm{a}\,(A^\circ-B^\circ)+\mathrm{c}\,(C^\circ-D^\circ)=0,$$

an equation which signifies that the radical axis of the circles A, B is also the radical axis of the circles C, D; then, writing as we may do, $\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}}\left(=\sqrt{\frac{1}{c^2}}\right)=\frac{1}{c}$, we have

$$\sqrt{l_1}\ =1-\frac{\mathrm{a}}{\mathrm{c}},\qquad \sqrt{l_2}\ =1-\frac{\mathrm{a}}{\mathrm{c}},$$

$$\sqrt{m_1}=1+\frac{\mathrm{a}}{\mathrm{c}},\qquad \sqrt{m_2}=1-\frac{\mathrm{a}}{\mathrm{c}},$$

$$\sqrt{n_1}\ =1+1,\ =2,\qquad \sqrt{n_2}=1-1,\ =0.$$

Here $\sqrt{l_1}+\sqrt{m_1}-\sqrt{n_1}=0$, which gives one of the trizomals a cubic, viz., this is the trizomal

$$\left(1-\frac{\mathrm{a}}{\mathrm{c}}\right)\sqrt{\mathfrak{A}^\circ}+\left(1+\frac{\mathrm{a}}{\mathrm{c}}\right)\sqrt{\mathfrak{B}^\circ}+2\sqrt{\mathfrak{C}^\circ}=0.$$

The other trizomal reduces itself to the bizomal $\sqrt{\mathfrak{A}^\circ}+\sqrt{\mathfrak{B}^\circ}=0$, which regarded as a trizomal, or written under the form $(\sqrt{\mathfrak{A}^\circ}+\sqrt{\mathfrak{B}^\circ})^2=0$, is the line $\mathfrak{A}^\circ-\mathfrak{B}^\circ=0$ twice, viz., this is the radical axis of the circles A_1, B_1 twice; and the order is thus $=2$. By what precedes, the line in question is in fact the common radical axis of the circles A, B and of the circles C, D.

Article Nos. 203 to 205. *Cases of the Decomposable Curve, the Centres in a Line.*

203. We have yet to consider the decomposable case when the centres A, B, C, D are on a line; the equation $\mathrm{a}\mathfrak{A}^\circ+\mathrm{b}\mathfrak{B}^\circ+\mathrm{c}\mathfrak{C}^\circ+\mathrm{d}\mathfrak{D}^\circ=0$ here subsists universally, whatever be the radii a'', b'', c'', d''. We establish as before the relation $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0$. The cases are as follows:

I. No further relation between l, m, n, p; order of tetrazomal $=8$, of trizomals 4 and 4.

II. $\sqrt{l}+\sqrt{m}+\sqrt{n}+\sqrt{p}=0$; order of tetrazomal $=7$; of trizomals $=4$ and 3; same as II. *supra*.

III. $\sqrt{l}+\sqrt{p}=0$, $\sqrt{m}+\sqrt{n}=0$; order of tetrazomal $=6$; of trizomals 3 and 3; same as III. *supra*.

204. IV. $\sqrt{l}+\sqrt{m}+\sqrt{n}+\sqrt{p}=0$, $\mathrm{a}\sqrt{l}+\mathrm{b}\sqrt{m}+\mathrm{c}\sqrt{n}+\mathrm{d}\sqrt{p}=0$; order of tetrazomal $=6$; this is a remarkable case, the orders of the trizomals are either 3, 3 or else 4, 2.

To explain how this is, it is to be noticed that in the absence of any special relation between the radii, the above conditions combined with $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0$ give $\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}=\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}$[1]; when l, m, n, p have these values, the case is the same as IV. *supra*, and the orders of the trizomals are 3, 3. But if the radii of the circles satisfy the condition

$$\begin{vmatrix} 1\ , & 1\ , & 1\ , & 1 \\ a\ , & b\ , & c\ , & d \\ a^2, & b^2, & c^2, & d^2 \\ a''^2, & b''^2, & c''^2, & d''^2 \end{vmatrix}=0,$$

then the two conditions satisfy of themselves the remaining condition $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0$, and the ratios $\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}$ instead of being determinate as above, depend on an arbitrary parameter.

We have

$$\sqrt{l_1}=\sqrt{l}+\frac{\mathrm{a}}{\mathrm{d}}\frac{p}{\sqrt{l}},\quad \sqrt{m_1}=\sqrt{m}-\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}}\,\mathrm{b}\sqrt{n},\quad \sqrt{n_1}=\sqrt{n}+\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}}\,\mathrm{c}\sqrt{m},$$

and between l, m, n, p only the relations

$$\sqrt{l}+\sqrt{m}+\sqrt{n}+\sqrt{p}=0,\quad a\sqrt{l}+b\sqrt{m}+c\sqrt{n}+d\sqrt{p}=0.$$

We find first

$$\sqrt{l_1}+\sqrt{m_1}+\sqrt{n_1}=\sqrt{l}+\sqrt{m}+\sqrt{n}$$
$$+\frac{\sqrt{p}}{\sqrt{l}}\left\{\frac{\mathrm{a}}{\mathrm{d}}\sqrt{p}-\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\,(\mathrm{b}\sqrt{n}-\mathrm{c}\sqrt{m})\right\}$$
$$=-\frac{\sqrt{p}}{\sqrt{l}}\left\{\frac{1}{\mathrm{d}}(\mathrm{d}\sqrt{l}-\mathrm{a}\sqrt{p})-\sqrt{\frac{\mathrm{a}}{\mathrm{bc}}}\,(\mathrm{b}\sqrt{n}-\mathrm{c}\sqrt{m})\right\},$$

[1] Writing x^2, y^2, z^2, w^2 in place of $\sqrt{l}$, $\sqrt{m}$, $\sqrt{n}$, $\sqrt{p}$, we have to find x, y, z, w from the conditions

$$x+y+z+w=0,$$
$$ax+by+cz+dw=0,$$
$$\frac{x^2}{\mathrm{a}}+\frac{y^2}{\mathrm{b}}+\frac{z^2}{\mathrm{c}}+\frac{w^2}{\mathrm{d}}=0,$$

where the constants are connected by the relation

$$a\mathrm{a}+b\mathrm{b}+c\mathrm{c}+d\mathrm{d}=0.$$

It readily appears that the line represented by the first two equations *touches* the quadric surface in the point $x:y:z:w=\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}$, so that these are in general the only values of $\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}$. In the case next referred to in the text the line lies in the surface, and the values are not determined.

and then

$$(d-a)\sqrt{l} = (b-d)\sqrt{m} + (c-d)\sqrt{n},$$

$$(d-a)\sqrt{p} = (a-b)\sqrt{m} + (a-c)\sqrt{n},$$

whence

$$d\sqrt{l} - a\sqrt{p} = \frac{b-c}{d-a}(\mathrm{b}\sqrt{n} - \mathrm{c}\sqrt{m}),$$

and we have thus

$$\sqrt{l_1} + \sqrt{m_1} + \sqrt{n_1} = \frac{\sqrt{p}}{\mathrm{d}\sqrt{l}}\left(\frac{b-c}{d-a} - \sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\right)(\mathrm{b}\sqrt{n} - \mathrm{c}\sqrt{m});$$

and similarly

$$\sqrt{l_2} + \sqrt{m_2} + \sqrt{n_2} = \frac{\sqrt{p}}{\mathrm{d}\sqrt{l}}\left(\frac{b-c}{d-a} + \sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\right)(\mathrm{b}\sqrt{n} - \mathrm{c}\sqrt{m}):$$

(observe that in the case not under consideration $\mathrm{b}\sqrt{n} - \mathrm{c}\sqrt{m} = 0$, and therefore $\sqrt{l_1} + \sqrt{m_1} + \sqrt{n_1} = 0$, $\sqrt{l_2} + \sqrt{m_2} + \sqrt{n_2} = 0$).

In the present case we have

$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = (b-c)(c-d)(d-b) : -(c-d)(d-a)(a-c) : (d-a)(a-b)(b-d) : -(a-b)(b-c)(c-a)$,

and thence

$$\frac{\mathrm{ad}}{\mathrm{bc}} = \frac{(b-c)^2}{(d-a)^2},$$

so that only one of the two sums $\sqrt{l_1} + \sqrt{m_1} + \sqrt{n_1}$, $\sqrt{l_2} + \sqrt{m_2} + \sqrt{n_2}$ is $= 0$, viz., assuming

$$\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}} = \frac{b-c}{d-a},$$

we have $\sqrt{l_1} + \sqrt{m_1} + \sqrt{n_1} = 0$.

We have then also

$$a\sqrt{l_1} + b\sqrt{m_1} + c\sqrt{n_1} = a\sqrt{l} + b\sqrt{m} + c\sqrt{n}$$

$$+ \frac{\sqrt{p}}{\sqrt{l}}\left\{\frac{a\mathrm{a}\sqrt{p}}{\mathrm{d}} - \sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}(b\mathrm{b}\sqrt{n} - c\mathrm{c}\sqrt{m})\right\}$$

$$= -\frac{\sqrt{p}}{\sqrt{l}}\left\{\frac{1}{\mathrm{d}}(d\mathrm{d}\sqrt{l} - a\mathrm{a}\sqrt{p}) - \sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}(b\mathrm{b}\sqrt{n} - c\mathrm{c}\sqrt{m})\right\};$$

but we find

$$d\mathrm{d}\sqrt{l} - a\mathrm{a}\sqrt{p} = \frac{b-c}{d-a}(b\mathrm{b}\sqrt{n} - c\mathrm{c}\sqrt{m}),$$

and thence

$$a\sqrt{l_1} + b\sqrt{m_1} + c\sqrt{n_1} = \frac{\sqrt{p}}{\mathrm{d}\sqrt{l}}\left(\frac{b-c}{d-a} - \sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\right)(b\mathrm{b}\sqrt{n} - c\mathrm{c}\sqrt{m}), = 0,$$

in virtue of $\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}} = \frac{b-c}{d-a}$. Hence $\sqrt{l_1} : \sqrt{m_1} : \sqrt{n_1}, = b-c : c-a : a-b$, or the corresponding trizomal is a conic, but the other trizomal is a quartic.

205. V. $\sqrt{l}=\sqrt{m}=\sqrt{n}=\sqrt{p}$; order of tetrazomal is $=5$; orders of trizomals $=3$, 2; same as V. *supra.*

VI. $\sqrt{l}+\sqrt{p}=0$, $\sqrt{m}+\sqrt{n}=0$, $a\sqrt{l}+b\sqrt{m}+c\sqrt{n}+d\sqrt{p}=0$; order of tetrazomal $=5$; orders of trizomals are 3, 2.

We have here

$$\sqrt{l_1}=\frac{\mathrm{a}+\mathrm{d}}{\mathrm{d}}\sqrt{l},$$

$$\sqrt{m_1}=\sqrt{m}+\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\,\mathrm{b}\sqrt{m},$$

$$\sqrt{n_1}=\sqrt{m}+\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\,\mathrm{c}\sqrt{m},$$

or writing the values of $\sqrt{m_1}$, $\sqrt{n_1}$ in the form

$$\sqrt{m_1}=\sqrt{m}+\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\,\frac{\mathrm{b}}{\mathrm{d}}\sqrt{m},$$

$$\sqrt{n_1}=-\sqrt{m}+\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\,\frac{\mathrm{c}}{\mathrm{d}}\sqrt{m},$$

then observing that as before $l=\frac{\mathrm{ad}}{\mathrm{bc}}m$, if to fix the ideas we assume $\sqrt{l}=\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\sqrt{m}$, the equations are

$$\sqrt{l_1}=\frac{\mathrm{a}+\mathrm{d}}{\mathrm{d}}\sqrt{l}\text{ and similarly }\sqrt{l_2}=\frac{\mathrm{a}+\mathrm{d}}{\mathrm{d}}\sqrt{l}$$

$$\sqrt{m_1}=\sqrt{m}+\frac{\mathrm{b}}{\mathrm{d}}\sqrt{l},\qquad \sqrt{m_2}=\sqrt{m}-\frac{\mathrm{b}}{\mathrm{d}}\sqrt{l},$$

$$\sqrt{n_1}=-\sqrt{m}+\frac{\mathrm{c}}{\mathrm{d}}\sqrt{l},\qquad \sqrt{n_2}=\sqrt{m}-\frac{\mathrm{c}}{\mathrm{d}}\sqrt{l},$$

whence

$$\sqrt{l_1}+\sqrt{m_1}+\sqrt{n_1}=0,\quad \sqrt{l_2}-\sqrt{m_2}-\sqrt{n_2}=0.$$

We have moreover

$$a\sqrt{l_1}=\frac{a\mathrm{a}+d\mathrm{d}}{\mathrm{d}}\sqrt{l},$$

$$b\sqrt{m_1}+c\sqrt{n_1}=(b-c)\sqrt{m}+\frac{b\mathrm{b}+c\mathrm{c}}{\mathrm{d}}\sqrt{l},$$

and thence

$$a\sqrt{l_1}+b\sqrt{m_1}+c\sqrt{n_1}=(a-d)\sqrt{l}+(b-c)\sqrt{m}=0,$$

so that

$$\sqrt{l_1}:\sqrt{m_1}:\sqrt{n_1}=b-c:c-a:a-b;$$

the corresponding trizomal is thus a conic, and it has been seen that the other trizomal is a cubic.

VII. If we have $\begin{vmatrix} 1 , & 1 , & 1 , & 1 \\ a , & b , & c , & d \\ a^2 , & b^2 , & c^2 , & d^2 \\ a''^2 , & b''^2 , & c''^2 , & d''^2 \end{vmatrix} = 0$, and $\left(\begin{matrix} 1 , & 1 , & 1 , & 1 \\ a , & b , & c , & d \\ a^2 , & b^2 , & c^2 , & d^2 \\ a''^2 , & b''^2 , & c''^2 , & d''^2 \end{matrix} \right) (\sqrt{l}, \sqrt{m}, \sqrt{n}, \sqrt{p}) = 0$,

the tetrazomal has a branch ideally containing $(z^3 = 0)$ the line infinity 3 times; order is $= 5$; orders of the trizomals are 3, 2. We have here

$$\sqrt{l} : \sqrt{m} : \sqrt{n} : \sqrt{p} = \mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d},$$

and thence

$$\sqrt{l_1} = \mathrm{a} + \mathrm{d} \quad , \quad \sqrt{l_2} = \mathrm{a} + \mathrm{d} \quad ,$$

$$\sqrt{m_1} = \mathrm{b} - \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}}, \quad \sqrt{m_2} = \mathrm{b} + \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}},$$

$$\sqrt{n_1} = \mathrm{c} + \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}}, \quad \sqrt{n_2} = \mathrm{c} - \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}},$$

which give

$$\sqrt{l_1} + \sqrt{m_1} + \sqrt{n_1} = 0, \quad \sqrt{l_2} + \sqrt{m_2} + \sqrt{n_2} = 0.$$

Moreover

$$\begin{aligned} a\sqrt{l_1} + b\sqrt{m_1} + c\sqrt{n_1} &= a(\mathrm{a} + \mathrm{d}) + b\mathrm{b} + c\mathrm{c} \\ &\quad - (b - c)\sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}} \\ &= (a - d)\,\mathrm{d} - (b - c)\sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}} \\ &= \mathrm{d}\left\{(a - d) - (b - c)\sqrt{\frac{\mathrm{bc}}{\mathrm{ad}}}\right\}, \end{aligned}$$

and similarly

$$a\sqrt{l_2} + b\sqrt{m_2} + c\sqrt{n_2} = \mathrm{d}\left\{(a - d) + (b - c)\sqrt{\frac{\mathrm{bc}}{\mathrm{ad}}}\right\};$$

whence in virtue of

$$\frac{\mathrm{ad}}{\mathrm{bc}} = \frac{(b - c)^2}{(d - a)^2},$$

one of the two expressions is $= 0$; and the trizomals are thus a conic and a cubic.

Article No. 206. *The Decomposable Curve; Transformation to a different set of Concyclic Foci.*

206. Consider the decomposable case of

$$\sqrt{l\mathfrak{A}} + \sqrt{m\mathfrak{B}} + \sqrt{n\mathfrak{C}} + \sqrt{p\mathfrak{D}} = 0;$$

viz., the points A, B, C, D lie here in a circle, and we have $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0$. Taking (A_1, D_1) the antipoints of (A, D); (B_1, C_1) the antipoints of (B, C); then

$\mathfrak{A}_1\mathfrak{D}_1 = \mathfrak{A}\mathfrak{D}$, $\mathfrak{B}_1\mathfrak{C}_1 = \mathfrak{B}\mathfrak{C}$ (No. 65) and referring to the formulæ, *ante*, Nos. 100 *et seq.*, it appears that we can find l_1, m_1, n_1, p_1 such that identically

$$-l\mathfrak{A} + m\mathfrak{B} + n\mathfrak{C} - p\mathfrak{D} = -l_1\mathfrak{A}_1 + m_1\mathfrak{B}_1 + n_1\mathfrak{C}_1 - p_1\mathfrak{D}_1,$$

and moreover that $lp = l_1p_1$, $mn = m_1n_1$.

The equation of the curve gives

$$-l\mathfrak{A} + m\mathfrak{B} + n\mathfrak{C} - p\mathfrak{D} - 2\sqrt{lp\mathfrak{A}\mathfrak{D}} + 2\sqrt{mn\mathfrak{B}\mathfrak{C}} = 0,$$

which may consequently be written

$$-l_1\mathfrak{A}_1 + m_1\mathfrak{B}_1 + n_1\mathfrak{C}_1 - p_1\mathfrak{D}_1 - 2\sqrt{l_1p_1\mathfrak{A}_1\mathfrak{D}_1} + 2\sqrt{m_1n_1\mathfrak{B}_1\mathfrak{C}_1} = 0;$$

viz., this is

$$\sqrt{l_1\mathfrak{A}_1} + \sqrt{m_1\mathfrak{B}_1} + \sqrt{n_1\mathfrak{C}_1} + \sqrt{p_1\mathfrak{D}_1} = 0;$$

that is, the two trizomals expressed by the original tetrazomal equation involving the set of concyclic foci (A, B, C, D) are thus expressed by a new tetrazomal equation involving the different set of concyclic foci (A_1, B_1, C_1, D_1); and we might of course in like manner express the equation in terms of the other two sets of concyclic foci (A_2, B_2, C_2, D_2) and (A_3, B_3, C_3, D_3) respectively. It might have been anticipated that such a transformation existed, for we could as regards each of the component trizomals separately pass from the original set to a different set of concyclic foci, and the two trizomal equations thus obtained would, it might be presumed, be capable of composition into a single tetrazomal equation; but the direct transformation of the tetrazomal equation is not on this account less interesting.

ANNEX I. *On the Theory of the Jacobian.*

Consider any three curves $U=0$, $V=0$, $W=0$, of the same order r, then writing

$$J(U, V, W) = \frac{d(U, V, W)}{d(x, y, z)} = \begin{vmatrix} d_xU, & d_xV, & d_xW \\ d_yU, & d_yV, & d_yW \\ d_zU, & d_zV, & d_zW \end{vmatrix},$$

we have the Jacobian curve $J(U, V, W) = 0$, of the order $3r-3$.

A fundamental property is that if the curves $U=0$, $V=0$, $W=0$ have any common point, this is a point on the Jacobian, and not only so, but it is a node, or double point, that is, for the point in question we have $J=0$, and we have also $d_xJ=0$, $d_yJ=0$, $d_zJ=0$.

It follows that for the three curves $l\Theta + L\Phi = 0$, $m\Theta + M\Phi = 0$, $n\Theta + N\Phi = 0$ ($\Theta = 0$ of the order $r-s'$, $\Phi = 0$ of the order $r-s$, $l=0$, $m=0$, $n=0$ each of the order s', $L=0$, $M=0$, $N=0$ each of the order s) which have in common the

$(r-s')(r-s)$ points of intersection of the curves $\Theta=0$, $\Phi=0$, each of these points is a node on the Jacobian, and hence that the Jacobian must be of the form

$$J(l\Theta+L\Phi,\ m\Theta+M\Phi,\ n\Theta+N\Phi)=A\Theta^2+2B\Theta\Phi+C\Phi^2=0,$$

where obviously the degrees of A, B, C must be $r+2s'-3$, $r+s+s'-3$, $r+2s-3$ respectively. In the particular case where $s'=0$, that is where l, m, n are constants, we have $A=0$; the Jacobian curve then contains as a factor $(\Phi=0)$, and throwing this out, the curve is $B\Theta+C\Phi=0$, viz., this is a curve of the order $2r+s-3$ passing through each of $r(r-s)$ points of intersection of the curves $\Theta=0$, $\Phi=0$.

In particular, if $r=2$, $s=1$, that is, if the curves are the conics $\Theta+L\Phi=0$, $\Theta+M\Phi=0$, $\Theta+N\Phi=0$, passing through the two points of intersection of the conic $\Theta=0$ by the line $\Phi=0$, then the Jacobian is a conic passing through these same two points, viz., its equation is of the form $\Theta+\Omega\Phi=0$. This intersects any one of the given conics, say $\Theta+L\Phi=0$ in the points $\Theta=0$, $\Phi=0$, and in two other points $\Theta+\Omega\Phi=0$, $\Omega-L=0$; at *each* of the last-mentioned points, the tangents to the two curves, and the lines drawn to the two points $\Theta=0$, $\Phi=0$, form a harmonic pencil.

Although this is, in fact, the known theorem that the Jacobian of three circles is their orthotomic circle, yet it is, I think, worth while to give a demonstration of the theorem as above stated in reference to the conics through two given points.

Taking $(z=0,\ x=0)$, $(z=0,\ y=0)$ for the two given points $\Theta=0$, $\Phi=0$, the general equation of a conic through the two points is a quadric equation containing terms in z^2, zx, zy, xy; taking any two such conics

$$cz^2+2fyz+2gzx+2hxy=0,$$
$$Cz^2+2Fyz+2Gzx+2Hxy=0,$$

these intersect in the two points $(x=0,\ z=0)$, $(y=0,\ z=0)$ and in two other points; let $(x,\ y,\ z)$ be the coordinates of either of the last-mentioned points, and take $(X,\ Y,\ Z)$ as current coordinates, the equations of the lines to the fixed points and of the two tangents are

$$Xz-Zx=0,\qquad Yz-Zy=0,$$
$$(hy+gz)(Xz-Zx)+(hx+fz)(Yz-Zy)=0,$$
$$(Hy+Gz)(Xz-Zx)+(Hx+Fz)(Yz-Zy)=0,$$

whence the condition for the harmonic relation is

$$(hy+gz)(Hx+Fz)+(hx+fz)(Hy+Gz)=0,$$

that is

$$(fG+gF)z^2+(hF+fH)yz+(gH+hG)zx+2hHxy=0,$$

but from the equations of the two conics multiplying by $\tfrac{1}{2}H$, $\tfrac{1}{2}h$ and adding, we have

$$\tfrac{1}{2}(cH+hC)z^2+(hF+fH)yz+(gH+hG)zx+2hHxy=0;$$

viz., the condition is thus reduced to

$$cH+hC-2(fG+gF)=0,$$

so that this condition being satisfied for one of the points in question, it will be satisfied for the other of them. Now for the three conics

$$cz^2 + 2f\ yz + 2g\ zx + 2h\ xy = 0,$$
$$c'z^2 + 2f'\ yz + 2g'\ zx + 2h'xy = 0,$$
$$c''z^2 + 2f''yz + 2g''zx + 2h''xy = 0,$$

forming the Jacobian, and throwing out the factor z, we may write the equation in the form

$$Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0,$$

where the values are

$$C = g\,(f'c'' - f''c') + g'\,(f''c - fc'') + g''\,(fc' - f'c),$$
$$H = g\,(h'f'' - h''f') + g'\,(h''f - hf'') + g''\,(hf' - h'f),$$
$$2F = h\,(f'c'' - f''c') + h'\,(f''c - fc'') + h''\,(fc' - f'c),$$
$$2G = h\,(c'g'' - c''g') + h'\,(c''g - cg'') + h''\,(cg' - c'g);$$

and we thence obtain

$$\begin{aligned} cH + hC &= -(fg' - f'g)(c''h - ch'') + (f''g - fg'')(ch' - c'h) \\ &= 2\,(fG + gF), \end{aligned}$$

viz., the condition is satisfied in regard to the Jacobian and the first of the three conics; and it is therefore also satisfied in regard to the Jacobian and the other two conics respectively.

I do not know any general theorem in regard to the Jacobian which gives the foregoing theorem of the orthotomic circle. It may be remarked that the use in the Memoir of the theorem of the orthotomic circle is not so great as would at first sight appear: it fixes the ideas to speak of the orthotomic circle of three given circles rather than of their Jacobian, but we are concerned with the orthotomic circle less as the circle which cuts at right angles the given circles than as a circle standing in a known relation to the given circles.

ANNEX II. *On* CASEY'S *Theorem for the Circle which touches three given Circles.*

The following two problems are identical:

1. To find a circle touching three given circles.

2. To find a cone-sphere (sphere the radius of which is $=0$) passing through three given points in space.

In fact, in the first problem if we use z to denote a given constant (which may be $=0$), then taking a, a' and $i\,(z - a'')$ for the coordinates of the centre and for the radius of one of the given circles; and similarly b, b', $i\,(z - b'')$; c, c', $i\,(z - c'')$ for the other two given circles; and S, S', $i(z-S'')$ for the required circle; the equations of the given circles will be

$$(x-a)^2+(y-a')^2+(z-a'')^2=0,$$
$$(x-b)^2+(y-b')^2+(z-b'')^2=0,$$
$$(x-c)^2+(y-c')^2+(z-c'')^2=0,$$

and that of the required circle will be

$$(x-S)^2+(y-S')^2+(z-S'')^2=0.$$

In order that this may touch the given circles, the distances of its centre from the centres of the given circles must be $i(S''-a'')$, $i(S''-b'')$, $i(S''-c'')$ respectively; the conditions of contact then are

$$(S-a)^2+(S'-a')^2+(S''-a'')^2=0,$$
$$(S-b)^2+(S'-b')^2+(S''-b'')^2=0,$$
$$(S-c)^2+(S'-c')^2+(S''-c'')^2=0,$$

or we have from these equations to determine S, S', S''. But taking (a, a', a''), (b, b', b''), (c, c', c'') for the coordinates of three given points in space, and (S, S', S'') for the coordinates of the centre of the cone-sphere through these points, we have the very same equations for the determination of (S, S', S''), and the identity of the two problems thus appears.

I will presently give the direct analytical solution of this system of equations. But to obtain a solution in the form required, I remark that the equation of the cone-sphere in question is nothing else than the relation that exists between the coordinates of any four points on a cone-sphere; to find this, consider any five points in space, 1, 2, 3, 4, 5; and let $\overline{12}$, &c. denote the distances between the points 1 and 2, &c.; then we have between the distances of the five points the relation

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1, & 1 \\ 1, & 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & \overline{15}^2 \\ 1, & \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & \overline{25}^2 \\ 1, & \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & \overline{35}^2 \\ 1, & \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & \overline{45}^2 \\ 1, & \overline{51}^2, & \overline{52}^2, & \overline{53}^2, & \overline{54}^2, & 0 \end{vmatrix} = 0;$$

whence taking 5 to be the centre of the cone-sphere through the points 1, 2, 3, 4, we have $\overline{15}=\overline{25}=\overline{35}=\overline{45}=0$; and the equation becomes

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0 \end{vmatrix} = 0,$$

which is the relation between the distances of any four points on a cone-sphere; this equation may be written under the irrational form

$$\overline{23}\,.\,\overline{14}+\overline{31}\,.\,\overline{24}+\overline{12}\,.\,\overline{34}=0.$$

Taking (a, a', a''), (b, b', b''), (c, c', c''), (x, y, z) for the coordinates of the four points respectively, we have

$$\overline{23}=\sqrt{(b-c)^2+(b'-c')^2+(b''-c'')^2},\quad \overline{14}=\sqrt{(x-a)^2+(y-a')^2+(z-a'')^2},$$
$$\overline{31}=\sqrt{(c-a)^2+(c'-a')^2+(c''-a'')^2},\quad \overline{24}=\sqrt{(x-b)^2+(y-b')^2+(z-b'')^2},$$
$$\overline{12}=\sqrt{(a-b)^2+(a'-b')^2+(a''-b'')^2},\quad \overline{34}=\sqrt{(x-c)^2+(y-c')^2+(z-c'')^2},$$

or the symbols having these significations, we have

$$\overline{23}\,.\,\overline{14}+\overline{31}\,.\,\overline{24}+\overline{12}\,.\,\overline{34}=0$$

for the equation of the cone-sphere through the three points; or rather (since the rational equation is of the order 4 in the coordinates (x, y, z)) this is the equation of the pair of cone-spheres through the three given points; and similarly it is in the first problem the equation of a pair of circles each touching the three given circles respectively.

In the first problem the radii of the given circles were $i(z-a'')$, $i(z-b'')$, $i(z-c'')$ respectively; denoting these radii by α, β, γ, or taking the equations of the given circles to be

$$(x-a)^2+(y-a')^2-\alpha^2=0,$$
$$(x-b)^2+(y-b')^2-\beta^2=0,$$
$$(x-c)^2+(y-c')^2-\gamma^2=0,$$

the symbols then are

$$\overline{23}=\sqrt{(b-c)^2+(b'-c')^2-(\beta-\gamma)^2},\quad \overline{14}=\sqrt{(x-a)^2+(y-a')^2-\alpha^2},$$
$$\overline{31}=\sqrt{(c-a)^2+(c'-a')^2-(\gamma-\alpha)^2},\quad \overline{24}=\sqrt{(x-b)^2+(y-b')^2-\beta^2},$$
$$\overline{12}=\sqrt{(a-b)^2+(a'-b')^2-(\alpha-\beta)^2},\quad \overline{34}=\sqrt{(x-c)^2+(y-c')^2-\gamma^2},$$

and the equation of the pair of circles is as before

$$\overline{23}\,.\,\overline{14}+\overline{31}\,.\,\overline{24}+\overline{12}\,.\,\overline{34}=0\,;$$

where it is to be noticed that $\overline{23}$, $\overline{31}$, $\overline{12}$ are the tangential distances of the circles 2 and 3, 3 and 1, 1 and 2 respectively; viz., if α, β, γ are the radii taken positively, then these are the direct tangential distances. By taking the radii positively or negatively at pleasure, we obtain in all four equations—the tangential distances being all direct as above, or else any one is direct, and the other two are inverse; we have thus the four pairs of tangent circles.

The cone-spheres which pass through a given circle are the two spheres which have their centres in the two antipoints of the given circle; and it is easy to see that the foregoing investigation gives the following (imaginary) construction of the tangent circles; viz., given any three circles A, B, C in the same plane, to draw the tangent circles. Taking the antipoints of the three circles, then selecting any three antipoints (one for each circle) so as to form a triad, we have in all four complementary pairs of triads. Through a triad, and through the complementary triad draw two circles, these are situate symmetrically on opposite sides of the plane; and combining each antipoint of the first circle with the symmetrically situated antipoint of the second circle, we have two pairs of points, the points of each pair being symmetrically situate in regard to the plane, and having therefore an anticircle in this plane; these two anticircles are a pair of tangent circles; and the four pairs of complementary triads give in this manner the four pairs of tangent circles.

I return to the equations

$$\begin{aligned}
(x-S)^2 &+ (y-S')^2 + (z-S'')^2 = 0,\\
(a-S)^2 &+ (a'-S')^2 + (a''-S'')^2 = 0,\\
(b-S)^2 &+ (b'-S')^2 + (b''-S'')^2 = 0,\\
(c-S)^2 &+ (c'-S')^2 + (c''-S'')^2 = 0;
\end{aligned}$$

by eliminating $(S,\ S',\ S'')$ from these equations we shall obtain the equation of the pair of cone-spheres through the points $(a,\ a',\ a'')$, $(b,\ b',\ b'')$, $(c,\ c',\ c'')$. Write $x-S,\ y-S',\ z-S''=X,\ Y,\ Z$, then we have $X^2+Y^2+Z^2=0$, and, putting for shortness

$$\begin{aligned}
\mathfrak{A} &= (a-x)^2 + (a'-y)^2 + (a''-z)^2,\\
\mathfrak{B} &= (b-x)^2 + (b'-y)^2 + (b''-z)^2,\\
\mathfrak{C} &= (c-x)^2 + (c'-y)^2 + (c''-z)^2,
\end{aligned}$$

then, by means of the equation just obtained, the other three equations become

$$\begin{aligned}
\mathfrak{A} + 2\,[(a-x)\,X + (a'-y)\,Y + (a''-z)\,Z] &= 0,\\
\mathfrak{B} + 2\,[(b-x)\,X + (b'-y)\,Y + (b''-z)\,Z] &= 0,\\
\mathfrak{C} + 2\,[(c-x)\,X + (c'-y)\,Y + (c''-z)\,Z] &= 0.
\end{aligned}$$

These last equations give

$$\begin{aligned}
X : Y : Z = {} & \lambda\mathfrak{A} + \mu\mathfrak{B} + \nu\,\mathfrak{C}\\
: {} & \lambda'\mathfrak{A} + \mu'\mathfrak{B} + \nu'\,\mathfrak{C}\\
: {} & \lambda''\mathfrak{A} + \mu''\mathfrak{B} + \nu''\mathfrak{C},
\end{aligned}$$

where

$$\begin{aligned}
\lambda &= b'c'' - b''c' + (c'-b')\,z - (c''-b'')\,y,\\
\mu &= c'a'' - c''a' + (a'-c')\,z - (a''-c'')\,y,\\
\nu &= a'b'' - a''b' + (b'-a')\,z - (b''-a'')\,y,\\
\lambda' &= b''c - bc'' + (c''-b'')\,x - (c-b)\,z,\\
\mu' &= c''a - ca'' + (a''-c'')\,x - (a-c)\,z,\\
\nu' &= a''b - ab'' + (b''-a'')\,x - (b-a)\,z,\\
\lambda'' &= bc' - b'c + (c-b)\,y - (c'-b')\,x,\\
\mu'' &= ca' - c'a + (a-c)\,y - (a'-c')\,x,\\
\nu'' &= ab' - a'b + (b-a)\,y - (b'-a')\,x;
\end{aligned}$$

and the result of the elimination then is

$$(\lambda\mathfrak{A}+\mu\mathfrak{B}+\nu\mathfrak{C})^2+(\lambda'\mathfrak{A}+\mu'\mathfrak{B}+\nu'\mathfrak{C})^2+(\lambda''\mathfrak{A}+\mu''\mathfrak{B}+\nu''\mathfrak{C})^2=0.$$

But substituting for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ their values, and writing, for shortness,

$$\begin{aligned}
-i &= b'c''-b''c'+c'a''-c''a'+a'b''-a''b,\\
-j &= b''c-bc''+c''a-ca''+a''b-ab'',\\
-k &= bc'-b'c+ca'-c'a+ab'-a'b,\\
\Delta &= a(b'c''-b''c)+a'(b''c-bc'')+a''(bc'-b'c),\\
-p &= (b'c''-b''c')(a^2+a'^2+a''^2)+(c'a''-c''a')(b^2+b'^2+b''^2)+(a'b''-a''b')(c^2+c'^2+c''^2),\\
-q &= (b''c-bc'')(a^2+a'^2+a''^2)+(c''a-ca'')(b^2+b'^2+b''^2)+(a''b-ab'')(c^2+c'^2+c''^2),\\
-r &= (bc'-b'c)(a^2+a'^2+a''^2)+(ca'-c'a)(b^2+b'^2+b''^2)+(ab'-ab')(c^2+c'^2+c''^2),\\
-l &= (c-b)(a^2+a'^2+a''^2)+(a-c)(b^2+b'^2+b''^2)+(b-a)(c^2+c'^2+c''^2),\\
-m &= (c'-b')(a^2+a'^2+a''^2)+(a'-c')(b^2+b'^2+b''^2)+(b'-a')(c^2+c'^2+c''^2),\\
-n &= (c''-b'')(a^2+a'^2+a''^2)+(a''-c'')(b^2+b'^2+b''^2)+(b''-a'')(c^2+c'^2+c''^2),
\end{aligned}$$

we find

$$\begin{aligned}
&\lambda\mathfrak{A}+\mu\mathfrak{B}+\nu\mathfrak{C}\\
&\quad=-\ i(x^2+y^2+z^2)\\
&\qquad+2i(x^2+y^2+z^2)-2x(ix+jy+kz)-2\Delta x+ny-mz-p,
\end{aligned}$$

with similar expressions for $\lambda'\mathfrak{A}+\mu'\mathfrak{B}+\nu'\mathfrak{C}$, $\lambda''\mathfrak{A}+\mu''\mathfrak{B}+\nu''\mathfrak{C}$, and the result is

$$\begin{aligned}
&\{i(x^2+y^2+z^2)-2x(ix+jy+kz)-2\Delta x+ny-mz-p\}^2\\
&+\{j(x^2+y^2+z^2)-2y(ix+jy+kz)-nx-2\Delta y+lz-q\}^2\\
&+\{k(x^2+y^2+z^2)-2z(ix+jy+kz)+mx-ly-2\Delta z-r\}^2=0,
\end{aligned}$$

viz., this is

$$\begin{aligned}
&(x^2+y^2+z^2)^2(i^2+j^2+k^2)\\
&+(x^2+y^2+z^2)\{4\Delta(ix+jy+kz)+2\left(i(ny-mz)+j(lz-nx)+k(mx-ly)\right)\\
&\qquad\qquad+4\Delta^2-2(ip+jq+kr)+(l^2+m^2+n^2)\}\\
&-(lx+my+nz)^2+4(ix+jy+kz)(px+qy+rz)\\
&+4\Delta(px+qy+rz)-2\left(p(ny-mz)+q(lz-nx)+r(mx-ly)\right)\\
&+p^2+q^2+r^2=0,
\end{aligned}$$

viz., this is in the rational form the equation of the pair of cone-spheres. The function on the left-hand side must, it is clear, be save to a numerical factor the norm of

$$\begin{aligned}
&\sqrt{(b-c)^2+(b'-c')^2+(b''-c'')^2}\,.\,\sqrt{(x-a)^2+(y-a')^2+(z-a'')^2}\\
&+\sqrt{(c-a)^2+(c'-a')^2+(c''-a'')^2}\,.\,\sqrt{(x-b)^2+(y-b')^2+(z-b'')^2}\\
&+\sqrt{(a-b)^2+(a'-b')^2+(a''-b'')^2}\,.\,\sqrt{(x-c)^2+(y-c')^2+(z-c'')^2},
\end{aligned}$$

the numerical factor of the expression in question is in fact $=-4$, that is, the norm is

$$=-4(x^2+y^2+z^2)^2(i^2+j^2+k^2)+\&c.;$$

so that attending only to the highest powers in (x, y, z) we ought to have

$$\text{Norm}\,\{\sqrt{(b-c)^2+(b'-c')^2+(b''-c'')^2}+\sqrt{(c-a)^2+(c'-a')^2+(c''-a'')^2}+\sqrt{(a-b)^2+(a'-b')^2+(a''-b'')^2}\}$$
$$=-4(i^2+j^2+k^2).$$

It is easy to see that the norm is in fact composed of the terms

$$2(b'-c')^2\{\ \ (b-c)^2-(c-a)^2-(a-b)^2\},$$
$$+\,2(c'-a')^2\{-(b-c)^2+(c-a)^2-(a-b)^2\},$$
$$+\,2(a'-b')^2\{-(b-c)^2-(c-a)^2+(a-b)^2\},$$

and of the similar terms (a, b, c), (a'', b'', c''), and in (a', b', c'), (a'', b'', c''); the above written terms are $=-4$ into

$$(b'-c')^2(a-b)(a-c)+(c'-a')^2(b-c)(b-a)+(a'-b')^2(c-a)(c-b),$$

which is

$$=a'^2(b-c)^2+b'^2(c-a)^2+c'^2(a-b)^2$$
$$+\,2b'c'(a-b)(c-a)+2c'a'(b-c)(a-b)+2a'b'(c-a)(b-c),$$
$$=\{a'(b-c)+b'(c-a)+c'(a-b)\}^2$$
$$=k^2;$$

and the value of the norm is thus $=-4(i^2+j^2+k^2)$, as it should be.

ANNEX III. *On the Norm of* $(b-c)\sqrt{\mathfrak{A}^\circ}+(c-a)\sqrt{\mathfrak{B}^\circ}+(a-b)\sqrt{\mathfrak{C}^\circ}$, *when the Centres are in a Line.*

The norm of $\sqrt{U}+\sqrt{V}+\sqrt{W}$ is

$$=(1,\ 1,\ 1,\ -1,\ -1,\ -1\between U,\ V,\ W)^2,$$

whence that of $\sqrt{U+U'}+\sqrt{V+V'}+\sqrt{W+W'}$ is

$$=\ \ (1,\ 1,\ 1,\ -1,\ -1,\ -1\between U,\ V,\ W)^2$$
$$+\ \ (1,\ 1,\ 1,\ -1,\ -1,\ -1\between U',\ V',\ W')^2$$
$$+\,2(1,\ 1,\ 1,\ -1,\ -1,\ -1\between U,\ V,\ W\between U',\ V',\ W'),$$

where the last term is $=2$ into

$$U'(U-V-W)+V'(-U+V-W)+W'(-U-V+W);$$

and the norm of $\sqrt{U+U'+U''}+\sqrt{V+V'+V''}+\sqrt{W+W'+W''}$ is obviously composed in a similar manner.

Now, applying the formula to obtain the norm of

$$(b-c)\sqrt{a^2+\theta+\alpha}+(c-a)\sqrt{b^2+\theta+\beta}+(a-b)\sqrt{c^2+\theta+\gamma},$$

the expression contains six terms, two of which are at once seen to vanish; and writing for shortness („) in place of $(1, 1, 1, -1, -1, -1)$ the remaining terms are

$$\begin{aligned}
&(„)\big((b-c)^2\alpha,\ (c-a)^2\beta,\ (a-b)^2\gamma\big)^2\\
&+2\ \ („)\big((b-c)^2\alpha,\ (c-a)^2\beta,\ (a-b)^2\gamma\big)\!\big((b-c)^2a^2,\ (c-a)^2b^2,\ (a-b)^2c^2\big)\\
&+2\theta\,(„)\big((b-c)^2\alpha,\ (c-a)^2\beta,\ (a-b)^2\gamma\big)\!\big((b-c)^2\ \ ,\ (c-a)^2\ \ ,\ (a-b)^2\ \ \big)\\
&+2\theta\,(„)\big((b-c)^2a^2,\ (c-a)^2b^2,\ (a-b)^2c^2\big)\!\big((b-c)^2\ \ ,\ (c-a)^2\ \ ,\ (a-b)^2\ \ \big);
\end{aligned}$$

the first of these terms requires no reduction; the second, omitting the factor 2, is

$$\begin{aligned}
&(b-c)^2\alpha\ [\ \ (b-c)^2a^2-(c-a)^2b^2-(a-b)^2c^2]\\
&+(c-a)^2\beta\,[-(b-c)^2a^2+(c-a)^2b^2-(a-b)^2c^2]\\
&+(a-b)^2\gamma\ [-(b-c)^2a^2-(c-a)^2b^2+(a-b)^2c^2];
\end{aligned}$$

which is

$$=2(a-b)(b-c)(c-a)\,[bc\,(b-c)\,\alpha+ca\,(c-a)\,\beta+ab\,(a-b)\,\gamma].$$

Similarly the third term, omitting the factor 2θ, is

$$\begin{aligned}
&(b-c)^2\alpha\ [\ \ (b-c)^2-(c-a)^2-(a-b)^2]\\
&+(c-a)^2\beta\,[-(b-c)^2+(c-a)^2-(a-b)^2]\\
&+(a-b)^2\gamma\ [-(b-c)^2-(c-a)^2+(a-b)^2],
\end{aligned}$$

which is

$$=2(a-b)(b-c)(c-a)\,[(b-c)\,\alpha+(c-a)\,\beta+(a-b)\,\gamma],$$

and for the last term, omitting the factor 2θ, this may be deduced therefrom by writing (a^2, b^2, c^2) in place of (α, β, γ), viz., it is

$$=-2(a-b)^2(b-c)^2(c-a)^2.$$

Hence, restoring the omitted factors, and collecting, we find

$$\begin{aligned}
&\text{Norm }\{(b-c)\sqrt{a^2+\theta+\alpha}+(c-a)\sqrt{b^2+\theta+\beta}+(a-b)\sqrt{c^2+\theta+\gamma}\}\\
&=(b-c)^4\alpha^2+(c-a)^4\beta^2+(a-b)^4\gamma^2-2(c-a)^2(a-b)^2\beta\gamma-2(a-b)^2(b-c)^2\gamma\alpha-2(b-c)^2(c-a)^2\alpha\beta\\
&\quad+4\theta\,(a-b)\,(b-c)\,(c-a)\,[\ \ (b-c)\,\alpha+\ \ (c-a)\,\beta+\ \ (a-b)\,\gamma]\\
&\quad+4\ \ (a-b)\,(b-c)\,(c-a)\,[bc\,(b-c)\,\alpha+ca\,(c-a)\,\beta+ab\,(a-b)\,\gamma]\\
&\quad-4\theta\,(a-b)^2(b-c)^2(c-a)^2.
\end{aligned}$$

Hence, first writing $a-x$, $b-x$, $c-x$ in place of a, b, c; then y^2 for θ, and $(-a''^2, -b''^2, -c''^2)$ for (α, β, γ); and finally introducing z for homogeneity, we find

$$\begin{aligned}
\text{Norm } &\{(b-c)\sqrt{(x-az)^2+y^2-a''^2z^2}+(c-a)\sqrt{\;}_{,,}+(a-b)\sqrt{\;}_{,,}\} = z^2 \text{ into} \\
&z^2\big((b-c)^4a''^4+(c-a)^4b''^4+(a-b)^4c''^4 \\
&\qquad -2(c-a)^2(a-b)^2b''^2c''^2-2(a-b)^2(b-c)^2c''^2a''^2-2(b-c)^2(c-a)^2a''^2b''^2\big) \\
&-4y^2\ (b-c)(c-a)(a-b)\left[\ (b-c)a''^2+(c-a)b''^2+(a-b)c''^2\right] \\
&-4\ \ (b-c)(c-a)(a-b)\{\ (b-c)a''^2(z^2bc-zx(b+c)+x^2) \\
&\qquad\qquad +(c-a)b''^2(z^2ca-zx(c+a)+x^2) \\
&\qquad\qquad +(a-b)c''^2(z^2ab-zx(a+b)+x^2)\} \\
&-4y\ \ (b-c)^2(c-a)^2(a-b)^2,
\end{aligned}$$

so that the equation $(b-c)\sqrt{\mathfrak{A}^\circ}+(c-a)\sqrt{\mathfrak{B}^\circ}+(a-b)\sqrt{\mathfrak{C}^\circ}=0$, in its rationalised form, contains $(z^2=0)$ the line infinity twice, and the curve is thus a conic. If $a''^2=b''^2=c''^2=k''^2$, then the expression of the norm is

$$= z^2 \text{ into } -4(a-b)^2(b-c)^2(c-a)^2(y^2-k''^2z^2),$$

viz., when the three circles have each of them the same radius k'', the curve is the pair of parallel lines $y^2-k''^2z^2=0$; and in particular when $k''=0$, or the circles reduce themselves each to a point, then the curve is $y^2=0$, the axis twice.

Annex IV. *On the Trizomal Curves $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$, which have a Cusp, or two Nodes.*

The trizomal curve $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$, has not in general any nodes or cusps: in the particular case where the zomal curves are circles, we have however seen how the ratios $l : m : n$ may be determined so that the curve shall acquire a node, two nodes, or a cusp; viz., regarding a, b, c as current areal coordinates, we have here a conic $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$, the locus of the centres of the variable circle, and the solution depends on establishing a relation between this conic and the orthotomic circle or Jacobian of the three given circles. I have in my paper "Investigations in connection with Casey's Equation," *Quart. Math. Jour.* vol. VIII. (1867), pp. 334—342, [395] given, after Professor Cremona, a solution of the general question to find the number of the curves $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$, which have a cusp, or which have two nodes, and I will here reproduce the leading points of the investigation. I remark, that although one of the loci involved in it is the same as that occurring in the case of the three circles (viz., we have in each case the Jacobian of the given curves), the other two loci Σ and Δ, which present themselves, seem to have no relation to the conic of centres which is made use of in the particular case.

We have the curves $U=0$, $V=0$, $W=0$, each of the same order r; and considering a point the coordinates whereof are (l, m, n), we regard as corresponding to this point the curve $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$, say for shortness, the curve Ω, being as above a curve of the order $2r$, having r^2 contacts with each of the given curves $U=0$, $V=0$, $W=0$. As long as the point (l, m, n) is arbitrary, the curve Ω has not any node, and in order that this curve may have a node, it is necessary that the point (l, m, n) shall lie on a certain curve Δ; this being so, the node will, it is easy to see, lie on the curve J, the Jacobian of the three given curves; and the curves J and Δ will correspond to each other point to point, viz., taking for (l, m, n) any point whatever on the curve Δ, the curve Ω will have a node at some one point of J; and conversely, in order that the curve Ω may be a curve having a node at a given point of J, the point (l, m, n) must be at some one point of the curve Δ. The curve Δ has, however, nodes and cusps; each node of Δ corresponds to two points of J, viz., for (l, m, n) at a node of Δ, the curve Ω is a binodal curve having a node at each of the corresponding points of J; each cusp of Δ corresponds to two coincident points of J, viz. for (l, m, n) at a cusp of Δ, the curve Ω has a node at the corresponding point of J. The number of the binodal curves Ω is thus equal to the number of the nodes of Δ, and the number of the cuspidal curves Ω is equal to the number of the cusps of Δ; and the question is to find the Plückerian numbers of the curve Δ. This Professor Cremona accomplished in a very ingenious manner, by bringing the curve Δ into connexion with another curve Σ (viz., Σ is the locus of the nodes of those curves $lU+mV+nW=0$ which have a node), and the result arrived at is that for the curve Δ

$$\begin{aligned}
&\text{Order} &&= 3(r-1)(3r-2),\\
&\text{Class} &&= 6(r-1)^2,\\
&\text{Nodes} &&= \tfrac{3}{2}(r-1)(27r^3-63r^2+22r+16),\\
&\text{Cusps} &&= 3(r-1)(7r-8),\\
&\text{Double tangents} &&= \tfrac{3}{2}(r-1)(12r^3-36r^2+19r+16),\\
&\text{Inflexions} &&= 12(r-1)(r-2);
\end{aligned}$$

so that, finally, the number of the cuspidal curves $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$, is found to be $=3(r-1)(7r-8)$, and the number of the binodal curves of the same form is found to be $=\frac{3}{2}(r-1)(27r^3-63r^2+22r+16)$. When the given curves are conics, or for $r=2$, these numbers are $=18$ and 36 respectively; but the formulæ are not applicable to the case where the conics have a point or points of intersection in common; nor, consequently, to the case of the three circles.