413.

A MEMOIR ON ABSTRACT GEOMETRY.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLX. (for the year 1870), pp. 51—63. Received October 14,—Read December 16, 1869.]

I SUBMIT to the Society the present exposition of some of the elementary principles of an Abstract m-dimensional Geometry. The science presents itself in two ways,—as a legitimate extension of the ordinary two- and three-dimensional geometries; and as a need in these geometries and in analysis generally. In fact whenever we are concerned with quantities connected together in any manner, and which are, or are considered as variable or determinable, then the nature of the relation between the quantities is frequently rendered more intelligible by regarding them (if only two or three in number) as the coordinates of a point in a plane or in space: for more than three quantities there is, from the greater complexity of the case, the greater need of such a representation; but this can only be obtained by means of the notion of a space of the proper dimensionality; and to use such representation, we require the geometry of such space. An important instance in plane geometry has actually presented itself in the question of the determination of the number of the curves which satisfy given conditions: the conditions imply relations between the coefficients in the equation of the curve; and for the better understanding of these relations it was expedient to consider the coefficients as the coordinates of a point in a space of the proper dimensionality.

A fundamental notion in the general theory presents itself, slightly in plane geometry, but already very prominently in solid geometry; viz. we have here the difficulty as to the form of the equations of a curve in space, or (to speak more accurately) as to the expression by means of equations of the twofold relation between the coordinates of a point of such curve. The notion in question is that of a k-fold relation,—as distinguished from any system of equations (or onefold relations) serving for the expression of it, and as giving rise to the problem how to express such relation by means of a system of equations (or onefold relations). Applying to the case of solid geometry my conclusion in the general theory, it may be mentioned that I regard the twofold relation of a curve in space as being completely and precisely expressed by means of a system of equations ($P=0$, $Q=0$, ... $T=0$), when no one of the functions P, Q, ... T is a linear function, with constant or variable *integral* coefficients, of the others of them, and when *every surface whatever* which passes through the curve has its equation expressible in the form $U = AP + BQ \ldots + KT$, with constant or variable integral coefficients, A, B, ... K. It is hardly necessary to remark that all the functions and coefficients are taken to be rational functions of the coordinates, and that the word integral has reference to the coordinates.

Article Nos. 1 to 36. *General Explanations; Relation, Locus, &c.*

1. Any m quantities may be represented by means of $m+1$ quantities as the ratios of m of these to the remaining $(m+1)$th quantity, and thus in place of the absolute values of the m quantities we may consider the ratios of $m+1$ quantities.

2. It is to be noticed that we are *throughout* concerned with the ratios of the $m+1$ quantities, not with the absolute values; this being understood, any mention of the ratios is in general unnecessary; thus I shall speak of a relation between the $m+1$ quantities, meaning thereby a relation as regards the ratios of the quantities; and so in other cases. It may also be noticed that in many instances a limiting or extreme case is sometimes included, sometimes not included, under a general expression; the general expression is intended to include whatever, having regard to the subject-matter and context, can be included under it.

3. Postulate. We may conceive between the $m+1$ quantities a *relation*[1].

4. A relation is either *regular*, that is, it has a definite manifoldness, or, say, it is a k-fold relation; or else it is *irregular*, that is, composed of relations not all of the same manifoldness. As to the word "composed," see *post*, No. 14.

5. The ratios are determined (not in general uniquely) by means of a m-fold relation; and a relation cannot really be more than m-fold. But the notion of a more than m-fold relation has nevertheless to be considered. A relation may be, either in mere appearance or else according to a provisional conception thereof, more than m-fold, and be really m-fold or less than m-fold. Thus a relation expressed by $m+1$ or more

[1] The whole difficulty of the subject is (so to speak) in the analytical representation of a relation; without solving it, the theories of the text cannot be exhibited analytically with equivalent generality; and I have for this reason presented them in an abstract form without analytical expression or commentary. But it is perfectly easy to obtain analytical illustrations; a onefold relation is expressed by an equation $P=0$; and (although a k-fold relation is not in general expressible by k equations) any k independent equations $P=0$, $Q=0$, &c. constitute a k-fold relation. Thus, No. 4, an instance of an irregular relation is $MP=0$, $MQ=0$, viz. this is satisfied by the satisfaction either of the onefold relation $M=0$, or of the twofold relation $P=0$, $Q=0$. And *post*, Nos. 14 and 21, the relation composed of the two onefold relations $P=0$ and $Q=0$ is the onefold relation $PQ=0$; the relation aggregated of the same two relations is the twofold relation $P=0$, $Q=0$.

equations is in general and *primâ facie* more than m-fold; but if the equations are not independent, and equivalent to m or fewer equations, then the relation will be m-fold or less than m-fold. Or the given relation may depend on parameters, and so long as these are arbitrary be really more than m-fold; but the parameters may have to be, and be accordingly, so determined that the relation shall be m-fold or less than m-fold. A more than m-fold relation is said to be superdeterminate.

6. A system of any number of onefold relations, whether independent or dependent, and if more than m of them, whether compatible or incompatible, is termed a 'Plexus,' viz. if the number of onefold relations be $=\theta$, then the plexus is θ-fold. A θ-fold plexus constitutes a relation which is at most θ-fold, but which may be less than θ-fold.

7. Every relation whatever is expressible, and that *precisely*, by means of a plexus; but for the expression of a k-fold relation we may require a more than k-fold plexus.

8. Postulate. We may conceive a m-dimensional space, the indetermination of the ratios of $m+1$ coordinates, and *locus in quo* of the point, the unique determination of these ratios. More generally we may conceive any number of spaces, each of its own dimensionality, and existing apart by itself.

9. Conversely, any $m+1$ quantities may be taken as the coordinates of a point in a m-dimensional space.

10. The $m+1$ coordinates may have a k-fold relation; it appears (*ante*, No. 5) that the case $k>m$, or where the relation is more than m-fold, is not altogether excluded; but this is not now under consideration. The two limiting cases $k=0$ and $k=m$ will be presently mentioned; the remaining case is $k>0<m$; the system of points the coordinates of which satisfy such a relation constitutes a k-fold or $(m-k)$-dimensional locus. And k is the manifoldness, $m-k$ the dimensionality, of the locus.

11. If $k=m$, that is, if the ratios are determined, we have the point-system, which, if the determination be unique, is a single point. The expression "a locus" may extend to include the point-system, and therefore also the point. If $k=0$, that is, if the coordinates are not connected by any relation, we have the original m-dimensional space.

12. We may say that the m-dimensional space is the *locus in quo* not only of the points in such space, but of the locus determined by any relation whatever between the coordinates; and in like manner that any $(m-k)$dimensional locus in such space is a $(m-k)$dimensional space, a *locus in quo* of the points thereof, and of every locus determined by a relation between the coordinates, implying the k-fold relation which corresponds to the $(m-k)$dimensional locus.

13. There is not any locus corresponding to a relation which is really more than m-fold; hence in speaking of the locus corresponding to a given relation, we either assume that the relation is not more than m-fold, or we mean the locus, *if any*, corresponding to such relation.

14. Any two or more relations may be *composed* together, and they are then factors of a single *composite* relation; viz. the composite relation is a relation satisfied if, and not satisfied unless, some one of the component relations be satisfied.

15. The foregoing notion of composition is, it will be noticed, altogether different from that which would at first suggest itself. The definition is defective as not explaining the composition of a relation any number of times with itself, or elevation thereof to power; which however must be admitted as part of the notion of composition.

16. A k-fold relation which is not satisfied by any other k-fold relation, and which is not a power, is a *prime* relation. A relation which is not prime is composite, viz. it is a relation composed of prime factors each taken once or any other number of times; in particular, it may be the power of a single prime factor. Any prime factor is single or multiple according as it occurs once or a greater number of times.

17. A relation which is either prime, or else composed of prime factors each of the same manifoldness, is a regular relation; a k-fold relation is *ex vi termini* regular. An irregular relation is a composite relation the prime factors whereof are not all of the same manifoldness.

18. A prime k-fold relation cannot be implied in any prime k-fold relation different from itself. But a prime k-fold relation may be implied in a prime more-than-k-fold relation,—or in a composite relation, regular or irregular, each factor whereof is more than k-fold; and so also a composite relation, regular or irregular, each factor whereof is at most k-fold, may be implied in a composite relation, regular or irregular, each factor whereof is more than k-fold. In a somewhat different sense, each factor of a composite relation implies the composite relation.

19. A composite relation is satisfied if any particular one of the component relations is satisfied; but in order to exclude this case we may speak of a composite relation as being satisfied *distributively*; viz. this will be the case if, in order to the satisfaction of the composite relation, it is *necessary* to consider *all* the factors thereof, or, what is the same thing, when the reduced relation obtained by the omission of any one factor whatever is *not always satisfied*. And when the composite relation is satisfied distributively, the several factors thereof are satisfied *alternatively*; viz. there is no one which is throughout unsatisfied.

20. A composite onefold relation is never distributively implied in a prime k-fold relation—that is, a prime k-fold relation implies only a prime onefold relation, or at least only implies a composite onefold relation improperly, in the sense that it implies a certain prime factor of such composite onefold relation. Conversely, every k-fold relation which implies distributively a composite onefold relation is composite.

21. Any two or more relations may be *aggregated* together into, and they are then constituents of, a single *aggregate* relation; viz. the aggregate relation is only satisfied when all the constituent relations are satisfied. The aggregate relation implies each of the constituent relations.

22. There is no meaning in aggregating a relation with itself; such aggregation only occurs accidentally when two relations aggregated together become one and the same relation; and the aggregate of a relation with itself is nothing else than the original relation.

23. A onefold relation is not an aggregate, but is its own sole constituent; a more than onefold relation may always be considered as an aggregate of two or more constituent relations. The constituent relations determine, they in fact constitute, the aggregate relation; but the aggregate relation does not in any wise determine the constituent relations. Any relation implied in a given relation may be considered as a constituent of such given relation.

24. The aggregate of a k-fold and a l-fold relation is in general and at most a $(k+l)$fold relation; when it is a $(k+l)$fold relation, the constituent relations are independent, but otherwise, viz. if the aggregate relation is, or has for factor, a less than $(k+l)$fold equation, the constituent relations are dependent or interconnected.

25. Passing from relations to loci, we may say that the composition of relations corresponds to the *congregation* of loci, and the aggregation of relations to the *intersection* of loci.

26. For, first, the locus (if any) corresponding to a given composite relation is the congregate of the loci corresponding to the several prime factors of the given relation, the locus corresponding to a single factor being taken once, and the locus corresponding to a multiple factor being taken a number of times equal to the multiplicity of the factor.

27. And, secondly, the locus (if any) corresponding to a given aggregate relation is the locus common to and contained in each of the loci corresponding to the several constituent relations respectively; or, what is the same thing, it is the intersection of these several loci.

28. It may be remarked that a k-fold locus and a l-fold locus where $k+l>m$ (or where the aggregate relation is more than m-fold) have not in general any common locus.

29. Any onefold relation implied in a given k-fold relation is said to be in *involution* with the k-fold relation, and so in a system of onefold relations, if any relation be implied in the other relations, or, what is the same thing, in the relation aggregated of the other relations, then the system is said to be in *involution*; a system not in involution is said to be *asyzygetic*.

30. Consider a given k-fold relation, and, in conjunction therewith, a system of any number of onefold relations each implied in the given k-fold relation. We may omit from the system any relation implied in the remaining relations, and so successively until we arrive at an asyzygetic system. Consider now any other onefold relation implied in the given k-fold relation; this is either implied in the system of onefold relations, and it is then to be rejected, or if it is not implied in the system, it is to

be added on to and made part of the system. It may happen that, in the system thus obtained, some one relation of the original system is implied in the remaining relations of the new system; but if this is so the implied relation is to be rejected; the new system will in this case contain only as many relations as the original system, and in any case the new system will be asyzygetic. Treating in the same manner every other onefold relation implied in the given k-fold relation, we ultimately arrive at an asyzygetic system of onefold relations, such that every onefold relation implied in the given k-fold relation is implied in the asyzygetic system. The number of onefold relations will be at least equal to k (for if this were not so we should have the given k-fold relation as an aggregate of less than k onefold relations); but it may be greater than k, and it does not appear that there is any [assignable] superior limit to the number of onefold relations of the asyzygetic system.

31. The system of onefold relations is a precise equivalent of the given k-fold relation. Every set of values of the coordinates which satisfies the given k-fold relation satisfies the system of onefold relations; and reciprocally every set of values which satisfies the system of onefold relations satisfies the given k-fold relation. But if we omit any one or more of the onefold relations, then the reduced system so obtained is not a precise equivalent of the given k-fold relation; viz. there exist sets of values satisfying the reduced system, but not satisfying the given k-fold relation.

32. In fact consider a k-fold relation the aggregate of less than all of the onefold relations of the asyzygetic system, and in connexion therewith an omitted onefold relation; this omitted relation is not implied in the aggregate, and it constitutes with the aggregate not a $(k+1)$fold, but only a k-fold relation. This happens as follows, viz. the omitted relation is a factor of a composite onefold relation distributively implied in the aggregate; hence the aggregate is composite, and it implies distributively a composite onefold relation composed of the omitted relation and of an associated onefold relation; that is, the aggregate will be satisfied by values which satisfy the omitted relation, and also by values which (not satisfying the omitted relation) satisfy the associated relation just referred to.

33. Selecting at pleasure any k of the onefold relations of the asyzygetic system, being such that the aggregate of the k relations is a k-fold relation, we have a composite k-fold relation wherein each of the remaining onefold relations is alternatively implied; viz. each remaining onefold relation is a factor of a composite onefold relation implied distributively in the composite k-fold relation. Hence considering the $k+1$ onefold relations, viz. any $k+1$ relations of the asyzygetic system, each one of these is implied alternatively in the aggregate of the remaining k relations; and we may say that the $k+1$ onefold relations are in *convolution*.

34. More generally any $k+1$ or more, or all the relations of the asyzygetic system are in *convolution*, that is, any relation of the system is alternatively implied in the aggregate of the remaining relations, or indeed in the aggregate of any k relations (not being themselves in convolution) of the remaining relations of the asyzygetic

system. It may be added that, besides the relations of the system, there is not any onefold relation alternatively implied in the asyzygetic system.

35. The foregoing theory has been stated without any limitation as to the value of k, and it has I think a meaning even when k is $>m$; but the ordinary case is $k \not> m$. Considering the theory as applying to this case, I remark that the last proposition, viz. that no reduced system is a precise equivalent of the given k-fold relation, is generally true only on the assumption of the existence or quasi-existence of sets of values satisfying a more than m-fold relation. For let k be $\not> m$, and, on the contrary, assume, as we usually do, that it is not in general possible to satisfy a more than m-fold relation between the coordinates; the number of relations in the system may be $>m+1$; and if this is so, then selecting any $m+1$ relations of the system, it may very well happen that the given k-fold relation is not satisfied by any sets of values other than those which satisfy the $m+1$ relations,—that is, that the $m+1$ relations are a precise equivalent of the given k-fold relation. But even in this case the consideration of the entire system of the onefold relations is not the less advantageous; and I say in general that the given k-fold relation has for its precise and complete equivalent the asyzygetic system of onefold relations.

36. {In illustration of the foregoing Nos. 29 to 35, I remark that, for the functions or equations $P=0$, $Q=0$, $R=0$, &c., if we have identically $AP+BQ+CR+\ldots=0$, where the factors $A, B, C, \ldots$ are integral functions of the coordinates, and where some one of these factors, say, A, is a constant (or if we please $=1$), then the system of functions or equations is in involution; or, to speak more accurately, the function or equation $P=0$ is in involution with the remaining functions or equations $Q=0$, $R=0, \ldots$. But when the factors $A, B, C, \ldots$ are no one of them constant, then we have a convolution. If $P=0$ is in involution with the remaining equations $Q=0$, $R=0, \ldots$, then $P=0$ is implied in these equations, and the relations $(Q=0, R=0, \ldots)$ and $(P=0, Q=0, R=0, \ldots)$ are equivalent to each other. But in the case of a convolution where

$$AP+BQ+CR+\ldots=0,$$

then the relation the equations $Q=0$, $R=0, \ldots$ imply $AP=0$, that is, $A=0$ or else $P=0$; or, what is the same thing, the relation $(Q=0, R=0, \ldots)$ is a relation composed of the two relations $(A=0, Q=0, R=0, \ldots)$ and $(P=0, Q=0, R=0, \ldots)$. In the k-fold relation expressed by the more than k equations $(P=0, Q=0, R=0, \ldots)$, selecting any k of these equations which are not in convolution, and uniting thereto any one of the remaining equations, we have a convolution of $k+1$ equations; and when a k-fold relation is precisely expressed by means of a system of k or more equations $(P=0, Q=0, \ldots)$, then every equation $\Omega=0$ implied in the given relation, or, what is the same thing, the equation of any onefold locus passing through the locus given by the k-fold relation is in involution with the equations $P=0$, $Q=0, \ldots$, that is, we have identically $\Omega=AP+BQ+CR+\ldots$, $A, B, C, \ldots$ being integral functions of the coordinates.}

Article Nos. 37 to 42. *Omal Relation; Order.*

37. A k-fold relation may be linear or omal. If $k=m$, the corresponding locus is a point; if $k<m$ the locus is a k-fold, or $(m-k)$dimensional omaloid; the expression omaloid used absolutely denotes the onefold or $(m-1)$dimensional omaloid; the point may be considered as a m-fold omaloid.

38. A m-fold relation which is not linear or omal is of necessity composite, composed of a certain number M of m-fold linear or omal relations; viz. the m-fold locus corresponding to the m-fold relation is a point-system of M points, each of which may be considered as given by a separate m-fold linear or omal relation; each which relation is a factor of the original m-fold relation. The given m-fold relation, and the point-system corresponding thereto, are respectively said to be of the order M.

39. The order of a point-system of M points is thus $=M$, but it is of course to be borne in mind that the points may be single or multiple points; and that if the system consists of a point taken α times, another point taken β times, &c., then the number of points and therefore the order M of the system is considered to be $=\alpha+\beta+\ldots$.

40. If to a given k-fold relation $(k<m)$ we unite an absolutely arbitrary $(m-k)$fold linear relation, so as to obtain for the aggregate a m-fold relation, then the order M of this m-fold relation (or, what is the same thing, the number M of points in the corresponding point-system) is said to be the order of the given k-fold relation. The notion of order does not apply to a more than m-fold relation.

41. The foregoing definition of order may be more compendiously expressed as follows: viz.

Given between the $m+1$ coordinates a relation which is at most m-fold; then if it is not m-fold, join to it an arbitrary linear relation so as to render it m-fold; we have a m-fold relation giving a point-system; and the order of the given relation is equal to the number of points of the point-system.

42. The relation aggregated of two or more given relations, when the notion of order applies to the aggregate relation, that is, when it is not more than m-fold, is of an order equal to the product of the orders of the constituent relations; or, say, the orders of the given relations being $\mu, \mu', \ldots$, the order of the aggregate relation is $=\mu\mu'\ldots$.

Article Nos. 43 and 44. *Parametric Relations.*

43. We have considered so far relations which involve only the coordinates $(x, y, \ldots)$[1]; the coefficients are purely numerical, or, if literal, they are absolute constants, which either do or do not satisfy certain conditions; if they do not, the relation assumed in the first instance to be k-fold is really k-fold, or, as we may express it, the relation is

[1] The only exception is *ante*, No. 5, where, in illustration of the notion of a more than m-fold relation, mention is made of "parameters."

really as well as formally k-fold; if they do satisfy certain relations in virtue whereof the formally k-fold relation is really less than k-fold, say, it is $(k-l)$fold, then the relation is in fact to be considered *ab initio* as a $(k-l)$fold relation: there is no question of a relation being in general k-fold and becoming less than k-fold, or suffering any other modification in its form; and the notion of a more than m-fold relation is in the preceding theory meaningless.

44. But a relation between the coordinates $(x, y, \ldots)$ may involve parameters, and so long as these remain arbitrary it may be really as well as formally k-fold; but when the parameters satisfy certain conditions, it may become $(k-l)$fold, or may suffer some other modification in its form. And we have to consider the theory of a relation between the coordinates $(x, y, \ldots)$, involving besides parameters which may satisfy certain conditions, or, say simply, a relation involving variable parameters. If the number of the parameters be m', then these parameters may be regarded as the ratios of m' quantities to a remaining $m'+1$th quantity, and the relation may be considered as involving *homogeneously* the $m'+1$ parameters $(x', y', \ldots)$. And these may, if we please, be regarded as coordinates of a point in their own m'-dimensional space, or we have to consider relations between the $m+1$ coordinates $(x, y, \ldots)$ and the $m'+1$ (parameters or) coordinates $(x', y', \ldots)$. It is to be added that a relation may involve distinct sets of parameters, say, we have besides the original set of parameters, a set of $m''+1$ parameters $(x'', y'', \ldots)$ involved homogeneously. But this is a generalization the necessity for which has hardly arisen.

Article Nos. 45 to 55. *Quantics, Notation, &c.*

45. A homogeneous function of the coordinates $(x, y, \ldots)$ is represented by a notation such as

$$(*\between x, y, \ldots)^{(\cdot)}$$

(where $(*)$ indicates the coefficients and $(\cdot)$ the degree), and it is said to be a quantic; and in reference to the quantic the quantities or coordinates $(x, y, \ldots)$ are also termed *facients.* More generally a quantic involving two or more sets of coordinates, or facients, is represented by the similar notation

$$(*\between x, y, \ldots)^{(\cdot)}(x', y', \ldots)^{(\cdot)}\ldots.$$

46. The quantic is unipartite, bipartite, tripartite, &c., according as the number of sets is one, two, three, &c.; and with respect to any set of coordinates, it is binary, ternary, quaternary, ... $(m+1)$ary, according as the number of the coordinates is two, three, four, or $m+1$; and it is linear, quadric, cubic, quartic, ..., according as the degree in regard to the coordinates in question is 1, 2, 3, 4, ...

47. A quantic involving two or more sets of coordinates, and linear in regard to each of them, is said to be tantipartite; or, in particular, when there are only two sets, it is said to be lineo-linear; we may even extend the epithet lineo-linear to the case of any number of sets.

48. Instead of the general notation

$$(* \between x, y, \ldots)^{(\cdot)}(x', y', \ldots)^{(:)} \ldots$$

we may write

$$(a, \ldots \between x, y, \ldots)^{\mu}(x', y', \ldots)^{\mu'}, \ldots,$$

where the coefficients are now indicated by $(a, \ldots)$, and the degrees are $\mu, \mu', \ldots$

49. In the cases where the particular values of the coefficients have to be attended to, we write down the entire series of coefficients, or at least refer thereto by the notation $(a, \ldots)$; and it is to be understood that the coefficients expressed or referred to are each to be multiplied by the appropriate numerical coefficient, viz. for the term $x^{\alpha}y^{\beta} \ldots x'^{\alpha'}y'^{\beta'} \ldots$ this numerical coefficient is

$$= \frac{[\mu]^{\mu}[\mu']^{\mu'} \ldots}{[\alpha]^{\alpha}[\beta]^{\beta} \ldots [\alpha']^{\alpha'}[\beta']^{\beta'} \ldots}.$$

50. It is sometimes convenient not to introduce these numerical multipliers, and we then use the notation

$$(a, \ldots \between\!\!^{\urcorner} x, y, \ldots)^{\mu}(x', y', \ldots)^{\mu'} \ldots,$$

or

$$(a, \ldots \between x, y, \ldots \!^{\urcorner})^{\mu}(x', y', \ldots \!^{\urcorner})^{\mu'} \ldots.$$

In particular $(a, b, c \between x, y)^2$, $(a, b, c, d \between x, y)^3$ &c. denote respectively

$$ax^2 + 2bxy + cy^2,$$
$$ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$
&c.;

but $(a, b, c \between\!\!^{\urcorner} x, y)^2$, $(a, b, c, d \between\!\!^{\urcorner} x, y)^2$, &c. denote

$$ax^2 + bxy + cy^2,$$
$$ax^3 + bx^2y + cxy^2 + dy^3,$$
&c.,

and so $(a, b, c, f, g, h \between x, y, z)^2$ and $(a, b, c, f, g, h \between\!\!^{\urcorner} x, y, z)^2$ denote respectively

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

and

$$ax^2 + by^2 + cz^2 + fyz + gzx + hxy.$$

51. To show which are the coefficients that belong to the several terms respectively, it is obviously proper that the quantic should be once written out at full length; thus, in speaking of a ternary cubic function, we say let $U = (a, \ldots \between x, y, z)^3$

$$\begin{aligned} &= (a, b, c, f, g, h, i, j, k, l \between x, y, z)^3 \\ &= ax^3 + by^3 + cz^3 \\ &\quad + 3(fy^2z + gz^2x + hx^2y + lyz^2 + jzx^2 + kxy^2) \\ &\quad + 6lxyz, \end{aligned}$$

and the like in other cases.

52. A onefold relation between the coordinates is expressible by means of an equation of the form

$$(*\between x, y, \ldots)^{(\cdot)} = 0.$$

53. The expression "an equation" used without explanation may be taken to mean an equation of the form in question, viz. the equation obtained by putting a quantic equal to zero; the quantic is said to be the *nilfactum* of the equation. We may consequently say simply that a onefold relation between the coordinates is always expressible by an equation.

54. It is frequently convenient to denote the quantic or nilfactum by a single letter, and to use a locution such as "the equation $U = (*\between x, y\ldots)^{(\cdot)} = 0$," which really means that the single letter U stands for the quantic $(*\between x, y, \ldots)^{(\cdot)}$, so that we are afterwards at liberty to write $U=0$ as an abbreviated expression for $(*\between x, y, \ldots)^{\mu} = 0$. We may also speak of the equation or function $U=0$, meaning thereby the equation $U=0$, or the function U.

55. A k-fold relation between the coordinates is (as has been shown) equivalent to a system of k or more onefold relations; each of these is expressible by an equation $U=0$, and the k-fold relation is thus expressible by a system of k or more such equations. Representing by $((U))$ the system of functions which are the nilfacta of these equations respectively, the k-fold relations may be represented thus, $((U))=0$; or more completely, the relation being k-fold, and the number of equations being $=s$, by the notation

$$((U)\,s)\,(k\text{-fold}) = 0.$$

We may also speak of the system or relation $((U))=0$, meaning thereby the system of functions $((U))$, or the relation $((U))=0$.

Article Nos. 56 to 62. *Resultant, Discriminant, &c.*

56. In the case $k>m$, a given k-fold relation between the $m+1$ coordinates $(x, y, \ldots)$ and the parameters $(x', y', \ldots)$ leads to a $(k-m)$fold relation between the parameters. This is termed the *resultant relation* of the given k-fold relation, or when the additional specification is necessary, the *resultant relation* obtained by elimination of the coordinates $(x, y, \ldots)$.

57. Consider a k-fold relation between the $m+1$ coordinates $(x, y, \ldots)$ and the $m'+1$ coordinates $(x', y', \ldots)$. If $k \not> m$, then, considering the $(x, y, \ldots)$ as coordinates and the $(x', y', \ldots)$ as parameters, we have corresponding to the given relation a k-fold locus in the m-space; and so if $k \not> m'$, then, considering the $(x', y', \ldots)$ as coordinates, but the $(x, y, \ldots)$ as parameters, we have corresponding to the given relation a k-fold locus in the m'-space.

58. If $k > m$, but if the $(k-m)$fold resultant relation is satisfied, then the given k-fold relation becomes a m-fold linear relation between the coordinates $(x, y, \ldots)$, and is consequently satisfied by a single set of values of the coordinates. Hence, considering the given k-fold relation as implying the $(k-m)$fold resultant relation, the k-fold relation will represent a single point in the m-space, say, the *common point*.

59. A m-fold relation, or the locus, or point-system thereby represented, may have a *double* or *nodal* point, viz. two of the points of the point-system may be coincident. More generally a k-fold relation $(k \not> m)$, or the locus thereby represented, may have a *double* or *nodal* point; for let the relation if less than m-fold be made m-fold by adjoining to it a linear $(m-k)$fold relation satisfied by the coordinates of the point in question but otherwise arbitrary, then, if the point in question be a double or nodal point of the m-fold relation, or of the point-system thereby represented, the point is said to be a double or nodal point of the original k-fold relation, or of the locus thereby represented.

60. A given-fold relation $(k \not> m)$ between the $m+1$ coordinates, or the locus thereby represented, has not in general a nodal point. But if the relation involve the $m'+1$ parameters $(x', y', \ldots)$, then, if a certain onefold relation be satisfied between the parameters, there will be a nodal point. The onefold relation between the parameters is the *discriminant relation* of the given k-fold relation.

61. In the case in question, $k \not> m$, the discriminant relation is the resultant relation of a $(m+1)$fold relation which is the aggregate of the given k-fold relation with a certain relation called the *Jacobian relation*, or when the distinction is required, the *Jacobian relation* in regard to the $(x, y, \ldots)$.

62. Consider a k-fold relation $(k \not> m, \not> m')$ between the $m+1$ coordinates $(x, y, \ldots)$ and the $m'+1$ coordinates $(x', y', \ldots)$. It has been seen that to a given set of values of the $(x', y', \ldots)$ or, say, to a given point in the m'-space, there corresponds a k-fold locus in the m-space, and that to a given set of values of the $(x, y, \ldots)$, or to a given point in the m-space, there corresponds a k-fold locus in the m'-space. The k-fold locus in the m'-space may have a nodal point; this will be the case if there is satisfied between the $(x, y, \ldots)$ a certain one-fold relation, the discriminant relation of the given k-fold relation in regard to the $(x', y', \ldots)$. This onefold relation represents in the m-space a onefold locus, the *envelope* of the k-fold loci in the m-space corresponding to the several points of the m'-space. The property of the envelope is that to each point thereof there corresponds in the m'-space a k-fold locus having a nodal point.

Article Nos. 63—69. *Consecutive Points; Tangent Omals.*

63. As the notions of proximity and remoteness have been thus far altogether ignored, it seems necessary to make the following

Postulate. We may conceive a point consecutive (or indefinitely near) to a given point.

64. If the coordinates of the given point are $(x, y, \ldots)$, those of the consecutive point may be assumed to be $(x+\delta x, y+\delta y, \ldots)$, where δx, $\delta y, \ldots$ are indefinitely small in regard to $(x, y, \ldots)$.

65. It may be remarked that, taking the coordinates to be $(x+X, y+Y, \ldots)$, there is no obligation to have $(X, Y, \ldots)$ indefinitely small; in fact whatever the magnitudes of these quantities are, if only $X : Y : \ldots = x : y : \ldots$, then the point $(x+X, y+Y, \ldots)$ will be the very same with the original point, and it is therefore clear that a consecutive point may be represented in the same manner with magnitudes, however large, of $X, Y, \ldots$ But we *may* assume them indefinitely small, that is, the ratios $x+\delta x : y+\delta y, \ldots$, where δx, $\delta y, \ldots$ are indefinitely small in regard to $(x, y, \ldots)$, will represent any set of ratios indefinitely near to the ratios $(x : y, \ldots)$.

The foregoing quantities $(\delta x, \delta y, \ldots)$ are termed the increments.

66. Consider a k-fold relation between the $m+1$ coordinates $(x, y, \ldots)$, $k \not> m$; the increments $(\delta x, \delta y, \ldots)$ are connected by a linear k-fold relation.

The linear k-fold relation is satisfied if we assume the increments proportional to the coordinates—this is, in fact, assuming that the point remains unaltered. We may write $(\delta x, \delta y, \ldots) = (x, y, \ldots)$, since in such an equation only the ratios are attended to. But it may be preferable to write $(\delta x, \delta y, \ldots) = \lambda(x, y, \ldots)$. In particular if $k = m$, then the increments are connected by a linear m-fold relation; that is, the ratio of the increments is uniquely determined; and as the relation is satisfied by taking the increments proportional to the coordinates, it is clear that the values which the linear m-fold relation gives for the increments are in fact proportional to the coordinates: viz. there is not in this case any consecutive point.

67. Considering the k-fold relation as belonging to a k-fold locus in the m-space, so that $(x, y, \ldots)$ are the coordinates of a point on this locus, then if in the linear k-fold relation between the increments these increments are replaced by the coordinates $(\mathrm{x}, \mathrm{y}, \ldots)$ of a point in the m-space, then considering the original coordinates $(x, y, \ldots)$ as parameters, the locus of the point $(\mathrm{x}, \mathrm{y}, \ldots)$ is a k-fold omal locus: it is to be observed that, by what precedes, the linear k-fold relation is satisfied by writing therein the values $\mathrm{x} : \mathrm{y}, \ldots = x : y, \ldots$, that is, the k-fold omal locus passes through the original point $(x, y, \ldots)$; the k-fold omal locus is said to be the *tangent-omal* of the original k-fold locus at the point $(x, y, \ldots)$, which point is said to be the *point of contact*.

68. If in the original k-fold locus we replace $(x, y, \ldots)$ by $(\mathrm{x}, \mathrm{y}, \ldots)$, and combine therewith the k-fold linear relation, we have between the coordinates $(\mathrm{x}, \mathrm{y}, \ldots)$ a $2k$-fold relation (containing as parameters the coordinates $(x, y, \ldots)$); these parameters satisfy the original k-fold relation, and in virtue hereof the $2k$-fold relation (whether $2k$ is or is not greater than m) is satisfied by the values $\mathrm{x}, \mathrm{y}, \ldots = x : y : \ldots$; and not only so, but the point in question is a nodal or double point on the $2k$-fold locus. It also follows that the tangent-omal locus, considering in the k-fold linear relation $(x, y, \ldots)$ as parameters satisfying the original k-fold relation, has for its envelope the k-fold locus.

69. We thus arrive at the notion of the double generation of a k-fold locus, viz. such locus is the locus of the points, or, say, of the *ineunt-points* thereof; and it is also the envelope of the tangent-omals thereof. We have thus a theory of duality; I do not at present attempt to develope the theory, but it is necessary to refer to it, in order to remark that this theory is essential to the systematic development of a m-dimensional geometry; the original classification of loci as onefold, twofold,... $(m-1)$fold is incomplete, and must be supplemented with the loci reciprocally connected with these loci respectively. And moreover the theory of the singularities of a locus can only be systematically established by means of the same theory of duality; the singularities in regard to the ineunt-point must be treated of in connexion with the singularities in regard to the tangent-omal. These theories (that is, the classification of loci, and the establishment and discussion of the singularities of each kind of locus), vast as their extent is, should in the logical order precede that for which other reasons it may be expedient next to consider, the theory of Transformation, as depending on relations involving simultaneously the $m+1$ coordinates $(x, y, \ldots)$ and the $m'+1$ coordinates $(x'\ y', \ldots)$.