Arch. Mech., 36, 3, pp. 407–414, Warszawa 1984

One-dimensional model of composite material

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TWO PARALLEL different elastic rods interact with each other. The denisty of the interaction force is assumed to be proportional to the difference of displacements. The dispersion curve consists of the acoustical and optical branches. It is shown that for small times the disturbance propagates either with speed c_1 or with speed c_2 where c_1, c_2 are the propagation speeds in the first and the second rod, respectively. For large times the disturbance propagates with new speed c, $2c^2 = c_1^2+c_2^2$.

Dwa różne równoległe pręty oddziaływują na siebie siłą rozłożoną w sposób ciągły. Gęstość tej siły jest proporcjonalna do różnicy przemieszczeń. Krzywa dyspersyjna ma gałąź akustyczną i gałąź optyczną. Pokazano, że dla małych czasów zaburzenie propaguje się z prędkością c_1 lub c_2, gdzie c_1, c_2 są odpowiednio prędkościami propagacji w pierwszym lub drugim pręcie. Dla dużego czasu zaburzenie propaguje się z prędkością c, $2c^2 = c_1^2+c_2^2$.

Два разных параллельных стержня воздействуют на себя силой распределенной непрерывным образом. Плотность этой силы пропорциональна разницы перемещений. Дисперсионная кривая имеет акустическую и оптическую ветви. Показано, что для малых времен возмущение распространяется со скоростью c_1 или c_2, где c_1, c_2 — это соответственно скорости распространения в первом или во втором стержнях. Для большого времени возмущение распространяется со скоростью c, $2c^2 = c_1^2+c_2^2$.

THERE EXISTS large literature on statics of composite materials. However, there are few results concerning the dynamics. The existing results for propagation of the discontinuity surface are misleading because they do not describe the real dynamics of the composite material. When considering the discontinuity waves, very careful examination of the transport equation is necessary.

1. Model of the composite material

Two parallel elastic rods have equal cross-sections and equal densities but different elastic moduli E_1 and $E_2 > E_1$, Fig. 1. Denote by u, v the axial displacements in the first and second rod, respectively. It is assumed that the rods interact with each other by force $\alpha(u-v)$, $\alpha = \text{const}$. As the approximation for such a system may serve two elastic wires connected by an elastic layer or a system of elastic springs.

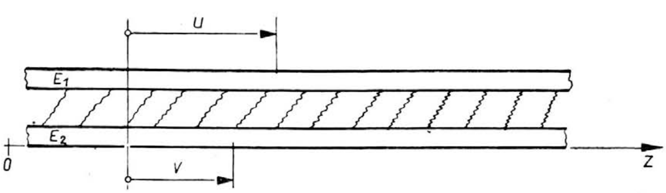

FIG. 1.

Assuming that the strain is one-dimensional

$$u = u(x, t), \qquad v = v(x, t),$$

the equations of motion are

$$\text{(1.1)} \qquad \begin{aligned} E_1 u_{,zz} + \alpha(v-u) &= \varrho u_{,tt}, \\ E_2 v_{,zz} + \alpha(u-v) &= \varrho v_{,tt}. \end{aligned}$$

Denoting

$$\text{(1.2)} \qquad \xi = \frac{1}{c_1}\sqrt{\frac{\alpha}{\varrho}}\, z, \quad \tau = \sqrt{\frac{\alpha}{\varrho}}\, t, \quad c_1^2 = \frac{E_1}{\varrho}, \quad q = \frac{E_2}{E_1},$$

and assuming $\alpha \neq 0$ we have

$$\text{(1.3)} \qquad \begin{aligned} u_{,\xi\xi} + (v-u) &= u_{,\tau\tau}, \\ qv_{,\xi\xi} + (u-v) &= v_{,\tau\tau}. \end{aligned}$$

The case $\alpha = 0$ is trivial. Already this simple model has interesting properties; we do not intend to consider obvious generalizations.

The initial problem will be considered later. Here we look for the solution of Eq. (1.3) of the form

$$\text{(1.4)} \qquad \begin{aligned} u &= Ae^{i(k\xi \pm \omega\tau)}, \\ v &= pAe^{i(k\xi \pm \omega\tau)}, \end{aligned}$$

where A, p, k, ω are constants. Substitution into Eq. (1.3) leads to the algebraic equations

$$\text{(1.5)} \qquad \begin{aligned} \omega^2 - k^2 - 1 + p &= 0, \\ 1 + (\omega^2 - k^2 q - 1)p &= 0, \end{aligned}$$

having the solutions

$$\text{(1.6)} \qquad \omega_{1,2}^2 = \frac{1}{2}\left[k^2(q+1) + 2 \mp \sqrt{k^4(q-1)^2 + 4}\right],$$

$$\text{(1.7)} \qquad p_{1,2} = k^2 + 1 - \omega_{1,2}^2.$$

The formulae given above describe the monochromatic sinusoidal wave. The dispersion relation $\omega(k)$ given by Eq. (1.6) has two branches, Fig. 2. The lower branch $\omega_1(k)$ starts at the point $(0, 0)$. In accord with Eq. (1.7) for $k \to 0$ we have $p_1 \to 1$, hence $v \to u$. The upper branch $\omega_2(k)$ starts at the point $(0, \sqrt{2})$. For $k \to 0$ we have $p_2 \to -1$, hence $v \to -u$. It is, therefore, inherently connected with the relative motion of the rods. Because of this fact the upper branch may be called the optical branch, in contrast to the acoustical branch $\omega_1(k)$.

In order to compare with the results of the next section, we calculate the phase, group and propagation speeds for the system considered above. The phase and group speeds are respectively

$$\text{(1.8)} \qquad \begin{aligned} U_{p_{1,2}} &= \frac{\omega}{k} = \frac{1}{k\sqrt{2}}\sqrt{(q+1)k^2 + 2 \mp \sqrt{(q-1)^2 k^4 + 4}}, \\ U_{g_{1,2}} &= \frac{d\omega}{dk} = \frac{k}{2\omega_{1,2}}\left[q + 1 \mp \frac{k^2(q-1)^2}{\sqrt{k^4(q-1)^2 + 4}}\right]. \end{aligned}$$

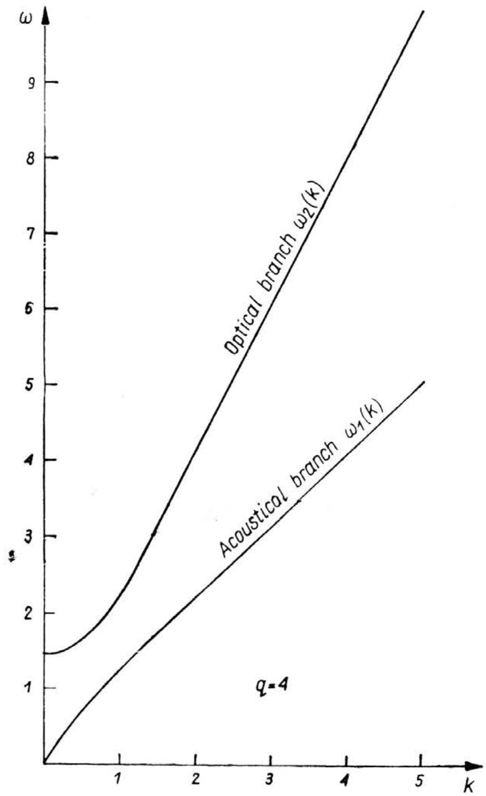

FIG. 2.

The upper sign holds for the acoustical and the lower sign holds for the optical branch. For $k \to 0$ we have

$$U_{p_1} \to \sqrt{(q+1)/2}\,, \qquad U_{p_2} \to \infty,$$
$$U_{g_1} \to \sqrt{(q+1)/2}\,, \qquad U_{g_2} \to 0.$$

Consider in turn the weak discontiniuity wave. Denote the speed of the discontinuity surface $\mathscr{S}$ (wave front) by U. If the displacements u, v and their first derivatives are continuous at $\mathscr{S}$, then the jumps of the second derivatives satisfy the compatibility relations

(1.9)
$$[\![u_{,\xi\xi}]\!] = K_1, \qquad [\![v_{,\xi\xi}]\!] = K_2,$$
$$[\![u_{,\xi\tau}]\!] = -K_1 U, \qquad [\![v_{,\xi\tau}]\!] = -K_2 U,$$
$$[\![u_{,\tau\tau}]\!] = K_1 U^2, \qquad [\![v_{,\tau\tau}]\!] = K_2 U^2.$$

Equations (1.3) being satisfied at both sides of S, we have

(1.10)
$$[\![u_{,\xi\xi}]\!] = [\![u_{,\tau\tau}]\!],$$
$$q[\![v_{,\xi\xi}]\!] = [\![v_{,\tau\tau}]\!].$$

It was taken into account that $[\![u]\!] = [\![v]\!] = 0$. Substitution into the relations (1.9) leads to the equations

$$\begin{aligned} K_1 &= U^2 K_1, \\ qK_2 &= U^2 K_2. \end{aligned} \tag{1.11}$$

It follows that either

$$U = U_1 = 1, \quad K_2 = 0, \tag{1.12}$$

or

$$U = U_2 = \sqrt{q}, \quad K_1 = 0. \tag{1.13}$$

Each of the six speeds (1.8), (1.12) and (1.13) has some physical sense. The most important is the speed that describes the behaviour of the structure as a whole. This will be discussed in the next chapter.

2. Initial problem

Consider the motion following the static deformation

$$\begin{aligned} u(\xi, 0) &= v(\xi, 0) = g(\xi), \\ u_{,\tau}(\xi, 0) &= v_{,\tau}(\xi, 0) = 0. \end{aligned} \tag{2.1}$$

By assumption the function $g(\xi)$ is given and may be decomposed into the (cosine) Fourier series

$$g(\xi) = \sum_{n=0}^{\infty} C_n \cos n k_0 \xi. \tag{2.2}$$

Try

$$\begin{aligned} \tilde{u}(\xi, \tau) &= A[\cos(k\xi - \omega_1 \tau) + \cos(k\xi + \omega_1 \tau)] \\ &\quad + B[\cos(k\xi - \omega_2 \tau) + \cos(k\xi + \omega_2 \tau)], \\ \tilde{v}(\xi, \tau) &= Ap_1[\cos(k\xi - \omega_1 \tau) + \cos(k\xi + \omega_1 \tau)] \\ &\quad + Bp_2[\cos(k\xi - \omega_2 \tau) + \cos(k\xi + \omega_2 \tau)]. \end{aligned} \tag{2.3}$$

If ω_1, ω_2, p_1, p_2 are functions of the wave number k as in Eqs. (1.6) and (1.7), then the displacements (2.3) satisfy the equations of motion (1.1) for each A, B.

In accord with Eq. (2.3) there is

$$\begin{aligned} \tilde{u}(\xi, 0) &= 2(A+B)\cos k\xi, \\ \tilde{v}(\xi, 0) &= 2(Ap_1 + Bp_2)\cos k\xi, \\ \tilde{u}_{,\tau}(\xi, 0) &= \tilde{v}_{,\tau}(\xi, 0) = 0. \end{aligned} \tag{2.4}$$

The initial condition $(2.1)_2$ is therefore automatically satisfied. In order to satisfy the condition (2.1), assume that the whole set of functions (2.3) with the parameters A_n, B_n, k_n is considered. There follow the equations for A_n, B_n

$$\begin{aligned} 2A_n + 2B_n &= C_n, \\ 2A_n p_{1n} + 2B_n p_{2n} &= C_n, \end{aligned}$$

which lead to the relations

$$A_n = \frac{p_{2n}-1}{2(p_{2n}-p_{1n})} C_n, \qquad B_n = -\frac{p_{1n}-1}{2(p_{2n}-p_{1n})} C_n. \tag{2.5}$$

Basing on Eqs. (2.3) and (2.5) the displacement field for each $g(\xi)$ may be calculated. Take in particular

$$g(\xi) = 1+\frac{4}{\pi}\left(\frac{1}{1}\cos k_0\xi-\frac{1}{3}\cos 3k_0\xi+\frac{1}{5}\cos 5k_0\xi+\ldots\right). \tag{2.6}$$

This function equals 0 or 2 and has the period equal to $2\pi/k_0$, Fig. 3. In the neighbourhood of $\xi = T/4$ it equals the double of the Heaviside function. Many other series possess the same property, but Eq. (2.6) is exceptionally convenient because of the regular structure.

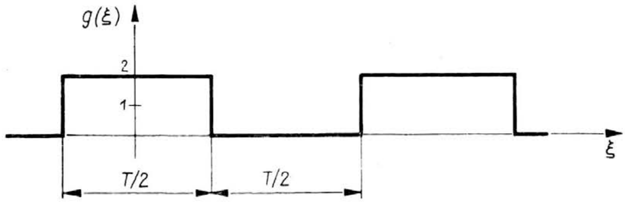

FIG. 3.

Finally, taking into account the relations (2.5) for $g(\xi)$ given by Eq. (2.6) we have

$$\begin{aligned} u(\xi, \tau) &= 1+\frac{2}{\pi}(D_1-D_3+D_5-D_7+\ldots), \\ v(\xi, \tau) &= 1+\frac{2}{\pi}(E_1-E_3+E_5-E_7+\ldots), \end{aligned} \tag{2.7}$$

$$\begin{aligned} D_n &= \frac{1}{n}\left\{\frac{p_{2n}-1}{p_{2n}-p_{1n}}[\cos(nk_0\xi-\omega_{1n}\tau)+\cos(nk_0\xi+\omega_{1n}\tau)\right. \\ &\qquad \left. -\frac{p_{1n}-1}{p_{2n}-p_{1n}}\cos(nk_0\xi-\omega_{2n}\tau)+\cos(nk_0\xi+\omega_{2n}\tau)]\right\}, \\ E_n &= \frac{1}{n}\left\{p_{1n}\frac{p_{2n}-1}{p_{2n}-p_{1n}}[\cos(nk_0\xi-\omega_{1n}\tau)+\cos(nk_0\xi+\omega_{1n}\tau)]\right. \\ &\qquad \left. -p_{2n}\frac{p_{1n}-1}{p_{2n}-p_{1n}}[\cos(nk_0\xi-\omega_{2n}\tau)+\cos(nk_0\xi+\omega_{2n}\tau)]\right\}. \end{aligned} \tag{2.8}$$

The formulae (2.7) give $u(\varrho, t)$, $v(\varrho, t)$ for initial deformation given by Eqs. (2.1). It should be added that the Fourier transformation allows to obtain the formal solution as the integrals

$$u^*(\xi, \tau) = \int_0^\infty d\alpha \frac{p_2-1}{p_2-p_1}\frac{\sin\alpha b}{\alpha}[\cos(\alpha\xi-\omega_1\tau)+\cos(\alpha\xi+\omega_1\tau)] \tag{2.9}$$

(2.9) [cont.]

$$-\int_0^\infty d\alpha \frac{p_1-1}{p_2-p_1}\frac{\sin\alpha b}{\alpha}[\cos(\alpha\xi-\omega_2\tau)+\cos(\alpha\xi+\omega_2\tau)],$$

$$v^*(\xi,\tau)=\int_0^\infty d\alpha p_1 \frac{p_2-1}{p_2-p_1}\frac{\sin\alpha b}{\alpha}[\cos(\alpha\xi-\omega_1\tau)+\cos(\alpha\xi+\omega_1\tau)]$$

$$-\int_0^\infty d\alpha p_2 \frac{p_1-1}{p_2-p_1}\frac{\sin\alpha b}{\alpha}[\cos(\alpha\xi-\omega_2\tau)+\cos(\alpha\xi+\omega_2\tau)],$$

which satisfy the equations of motion (1.3), provided ω_1, ω_2, p_1, p_2 are functions of α as in Eqs. (1.6) and (1.7). They satisfy the inital conditions

$$u^*(\xi,0)=v^*(\xi,0)=\int_0^\infty \frac{\sin\alpha b}{\alpha}\cos\alpha d\alpha=\begin{cases}\pi/2 & \text{for} \quad |\xi|>b,\\ 0 & \text{for} \quad |\xi|<b.\end{cases}$$

The integrands being very complex functions of α it is impossible to perform analytically the integration.

Cutting the summation on C_{39}, D_{39} the wave profile was found for $q = 2.25$, $k = 1$, Fig. 4. This value of q corresponds to the ratio of speeds $U_2/U_1 = 1.5$, cf. Eqs. (1.12), (1.13). Similarly, as in the case of string, the wave propagates to the left and to the right. It is seen that the propagation speeds along the first rod and the second rod are 1 and 1.5, respectively. Note that for $\xi > \pi/2$ there is $v < 1$. The interaction with the first rod leads to the smaller displacement as compared with that in the first rod.

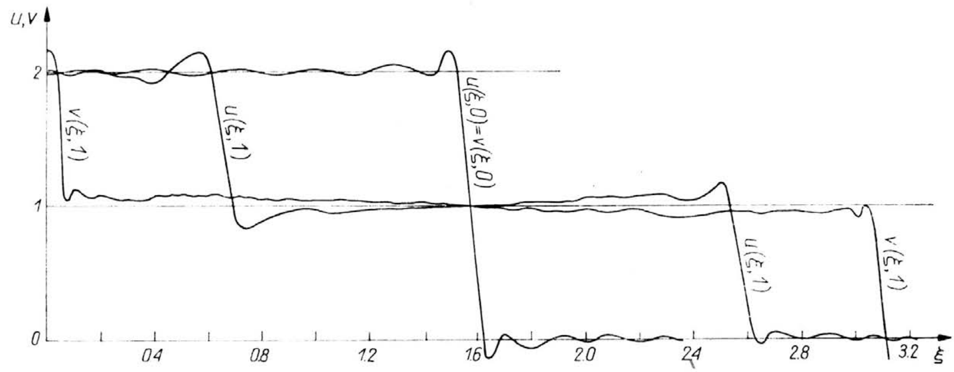

FIG. 4.

The picture changes drastically after the wave travels a large distance. Figure 5 shows the wave profile for $k_0 = 0.01$, $\xi = 2.25$, $\tau = 25, 50, 75, 100$. Approximately there is

$$u(\xi,\tau)\approx v(\xi,\tau).$$

The difference $u-v$ is very small and never exceeds 0.015 e.g. for $\tau = 50$ there is

$$u(210)=0.053, \quad v(210)=0.048,$$
$$u(220)=0.414, \quad v(220)=0.414,$$
$$u(230)=-1.064, \quad v(230)=-1.060.$$

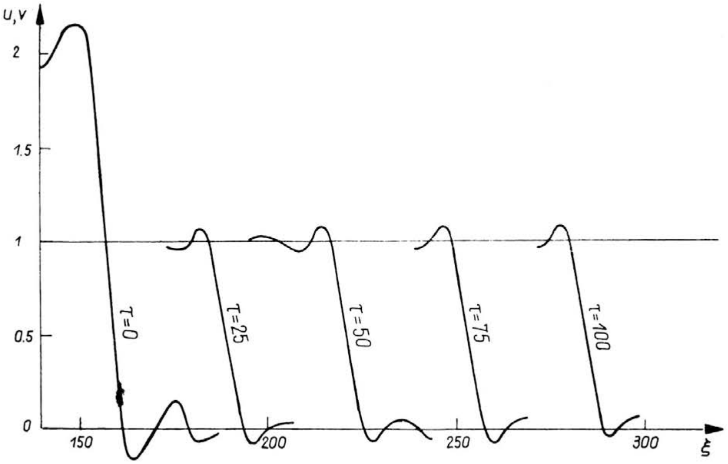

FIG. 5.

The propagation speed c equals approximately

$$c \approx 1.27. \tag{2.10}$$

It is an unexpected value; compare with the results of Sect. 1.

In order to explain the relation (2.10) eliminate u or v from Eqs. (1.3) to obtain the fourth-order equations

$$\begin{aligned} &qu_{,\xi\xi\xi\xi}-(q+1)u_{,\xi\xi\tau\tau}+u_{,\tau\tau\tau\tau}-(q+1)u_{,\xi\xi}+2u_{,\tau\tau} = 0,\\ &qv_{,\xi\xi\xi\xi}-(q+1)v_{,\xi\xi\tau\tau}+v_{,\tau\tau\tau\tau}-(q+1)v_{,\xi\xi}+2v_{,\tau\tau} = 0. \end{aligned} \tag{2.11}$$

Looking for the solutions of the form

$$u = f[\nu(\xi-c\tau)] \tag{2.12}$$

we have

$$\nu^4[q-(q+1)c^2+c^4]f^{\mathrm{IV}}+\nu^2[-(q+1)+2c^2]f^{\mathrm{II}} = 0.$$

For $v \to \infty$ there is

$$c^2 = 1 \quad \text{or} \quad c^2 = q \quad \text{or} \quad f^{\mathrm{IV}} = 0. \tag{2.13}$$

For $v \to 0$ there is

$$c^2 = \frac{q+1}{2} \quad \text{or} \quad f^{\mathrm{II}} = 0. \tag{2.14}$$

The last value corresponds exactly to the relation (2.10). It follows that the speed (2.10) corresponds to the slowly changing profile, cf. Eq. (2.9) for $v \to 0$.

There remains unansvered the question what is the mechanism of passing from the propagation speeds $v_1 = 1$, $v_2 = 1.5$ to the propagation speed $c = 1.27$. In order to illustrate this fact Fig. 6 shows the wave front for τ between the small values (as on Fig. 4) and large values (as on Fig. 5). For $k_0 = 0.1$ $u(\xi, \tau)$ is shown as the solid curve and $v(\varrho, \tau)$

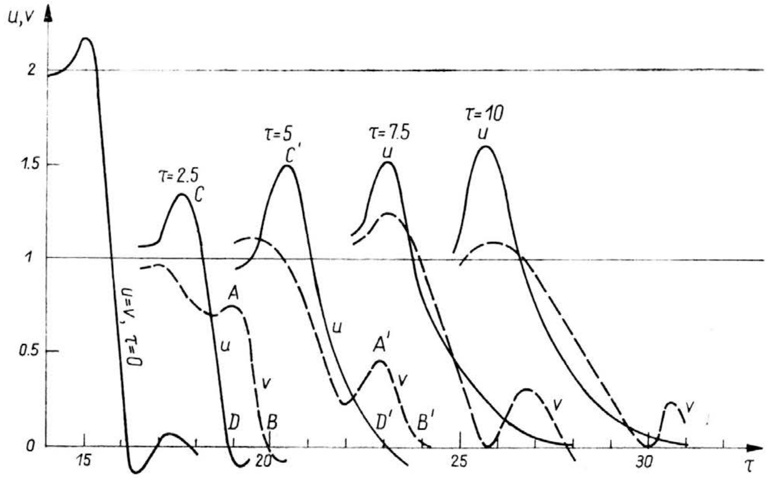

FIG. 6.

is shown as the dotted curve. At $\tau = 2.5$ the amplitude AB of the faster wave equals about 0.75 of that at $\tau = 0$ (equal 1). At the expense of the faster wave the amplitude CD of the slower wave increases. At $\tau = 5$ the amplitude $A'B'$ of the faster wave equals 0.45 and at $\tau = 7.5$ only 0.3 of the amplitude of the slower wave.

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Received November 11, 1983.