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Stability and instability of the thermodiffusive equilibrium in anisotropic magnetohydrodynamics (*)

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LINEAR instability and nonlinear Liapunov energy stability of the thermodiffusive equilibrium in anisotropic M.H.D. are considered and stabilizing–instabilizing effects of Hall's and ion-slip currents are proved.

Rozważono niestatecznośc oraz stateczność nieliniową energii w sensie Lapunova dla równowagi termodyfuzyjnej w magnetohydrodynamice anizotropowej; wykazano stabilizujące i destabilizujące efekty zjawiska Halla i prądów poślizgów jonowych.

Рассмотрены неустойчивость, а также нелинейная устойчивость энергии в смысле Ляпунова для термодиффузного равновесия в анизотропной магнитогидродинамике; показаны стабилизирующие и дестабилизирующие эффекты явления Холла и токов ионных скольжений.

1. Introduction

IN THESE LAST years we have published some papers [1]–[10] on the anisotropic M.H.D. In some of these papers, taking into account Hall's and ion-slip currents, we have proved the existence and uniqueness of steady flows in plane layers and in coaxial cylinders and we have compared these flows with the classical Couette, Poiseuille, Hartmann flows of the hydrodynamics and isotropic M.H.D. Moreover, since, as it is well known, the effective physical technical realization and, overall, the persistence of the motions, strongly depend on their stability, we have studied the stability of these flows.

Nevertheless, in almost all of these papers the fluid is considered in isothermal evolution. In the present work, in greater adherence to the physical situation, we consider the fluid not only electrical but also thermal conducting and we study the linear and nonlinear stability of the thermodiffusive equilibrium and the effects of the electroanisotropic currents on the stability.

2. Statement of the problem

In the dynamics of electrical conducting fluids in a magnetic field it is not right to neglect the anisotropic character of the electroconductivity when the mass fraction of the neutral particles is near to unity and the product of the cyclotron frequency of the charged particles

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for the average time between the collisions exceeds considerably the unity; this is the case, for instance, of a weakly ionized gas in a strong magnetic field [11, 12]. In these cases Hall's and ion-slip currents must be taken into account. The equations for these flows, in the nonrelativistic M.H.D. and in the Boussinesq approximation, for the incompressible case, are

$$
\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{v}\cdot\nabla\mathbf{v} - \frac{1}{\varrho_0}\nabla\pi + \nu\Delta_2\mathbf{v} + \frac{\mu_e}{\varrho_0}\nabla\times\mathbf{H}\times\mathbf{H} + [1-\alpha(T-T_0)]\mathbf{g},\\
(2.1)\qquad &\frac{\partial \mathbf{H}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{H}) + \eta_e\Delta_2\mathbf{H} + \beta_1\nabla\times\mathbf{H}\times\nabla\times\mathbf{H} + \beta_2\nabla\times(\mathbf{H}\times\mathbf{H}\times\nabla\times\mathbf{H}),\\
&\frac{\partial T}{\partial t} = -\mathbf{v}\cdot\nabla T + k_T\Delta_2 T,\\
&\nabla\cdot\mathbf{v} = \nabla\cdot\mathbf{H} = 0.
\end{aligned}
$$

To these equations, of course, suitable initial and boundary conditions (depending on the nature of the boundary) must be added. In these equations $\mathbf{v}$, $\mathbf{H}$, T, π, $\mathbf{g}$, are velocity, magnetic, thermal total pressure and gravitational fields, respectively; ϱ_0 and T_0 are fixed density and temperature of reference; μ_e is the magnetic permeability; ν, η_e are kinematic and magnetic viscosities and α, K_τ are volumetric expansion and thermal conductivity coefficients. Moreover Hall's and ion-slip currents are present with the coefficients β_1 and β_2.

Let the fluid be at rest in a thermodiffusive equilibrium in a fixed and rigid bounded domain S, under a constant temperature gradient $\mathbf{c}$ and in a constant magnetic field H_0; that is, let us censider the solution:

$$
(2.2)\qquad \mathbf{v} = 0, \quad \mathbf{H} = \mathbf{H}_0, \quad \nabla T = \mathbf{c}, \quad \nabla p_0 = \varrho_0\mathbf{g}[1-\alpha(T-T_0)]
$$

of the equations in S. The aim of the paper is to study the stability of this thermodiffusive equilibrium and the effects of Hall's and ion-slip currents on the stability.

3. Linear instability

Let S be a layer of boundary Σ, with the two walls $x_3 = 0$ and $x_3 = d(> 0)$, both free, not electrical but thermal conductors, in a constant orthogonal magnetic field $\mathbf{H}_0 = H_0\mathbf{e}_3$, where $\mathbf{e}_3$ is normal to Σ and vertically positive upwards. The linearized equations for small perturbations

$$
(3.1)\qquad \{\mathbf{u}(\mathbf{x}/t);\ \mathbf{h}(\mathbf{x}/t);\ \theta(\mathbf{x}/t)\}
$$

to the thermodiffusive equilibrium, of $\{\mathbf{u}, \mathbf{h}, \theta\}$ kinetic, magnetic and thermal fields respectively, after we have introduced the vorticity vector and the current denisty vector

$$
\mathbf{\Omega} = \nabla\times\mathbf{u}, \quad \mathbf{j} = \nabla\times\mathbf{h},
$$

are the following [9]:

$$
(3.2)\qquad \frac{\partial h_3}{\partial t} = H_0\frac{\partial u_3}{\partial x_3} + (\eta_e + \beta_2 H_0^2)\Delta_2 h_3 - \beta_1 H_0\frac{\partial j_3}{\partial x_3},
$$

(3.2) [cont.]

$$\frac{\partial \Omega_3}{\partial t} = \nu \Delta_2 \Omega_3 + \frac{\mu_e H_0}{\varrho} \frac{\partial j_3}{\partial x_3},$$

$$\frac{\partial}{\partial t} \Delta_2 u_3 = \nu \Delta_4 u_3 + \frac{\mu_e H_0}{\varrho} \frac{\partial}{\partial x_3} \Delta_2 h_3 + g\alpha \left(\frac{\partial^2 \theta}{\partial x_1^2} + \frac{\partial^2 \theta}{\partial x_2^2} \right),$$

$$\frac{\partial j_3}{\partial t} = H_0 \frac{\partial \Omega_3}{\partial x_3} + \eta_e \Delta_2 j_3 + \beta_1 H_0 \frac{\partial}{\partial x_3} \Delta_2 h_3 + \beta_2 H_0^2 \frac{\partial^2 j_3}{\partial x_3^2},$$

$$\frac{\partial \theta}{\partial t} = \beta u_3 + K_T \Delta_2 \theta$$

with the boundary conditions

$$u_3 = \theta = j_3 = h_3 = \frac{\partial \Omega_3}{\partial x_3} = \frac{\partial^2 u_3}{\partial x_3^2} = 0. \tag{3.3}$$

For the perturbations of normal mode

$$\{u_3, \theta, \Omega_3, j_3, h_3\} = \{W(x_3), \Theta(x_3), Z(x_3), X(x_3), K(x_3)\} \exp[i(k_1 x_1 + k_2 x_2 + pt],\tag{3.4}$$

if we introduce the wave number $k = \sqrt{(k_1^2 + k_2^2)}$ and the nondimensional parameters

$$a = Kd, \quad \sigma = pd^2/\nu, \quad p_1 = \nu/K_T, \quad p_2 = \nu/\eta_e, \quad \beta_H = \beta_1 H_0/\eta_e, \quad \beta_I = \beta_2 H_0^2/\eta_e;$$

assuming the nondimensional derivative $D = d. (d/dx_3)$, from Eq. (3.2) it follows

$$[(D^2 - a^2)(1 + \beta_1) - p_2 \sigma] K = -\frac{H_0 d}{\eta_e} DW + \beta_H d D X,$$

$$(D^2 - a^2 - \sigma) Z = -\frac{\mu_e H_0 d}{\varrho_0 \nu} DX,$$

$$(D^2 - a^2)(D^2 - a^2 - \sigma) W + \frac{\mu_e H_0 d}{\varrho_0 \nu} D(D^2 - a^2) K = \frac{g\alpha a^2 d^2}{\nu} \Theta, \tag{3.5}$$

$$[(1 + \beta_I) D^2 - a^2 - p_2 \sigma] X = -\frac{H_0 d}{\eta_e} DZ - \frac{\beta_H}{d} D(D^2 - a^2) K,$$

$$(D^2 - a^2 - p_1 \sigma) \Theta = -\frac{\beta d^2}{k_T} W$$

with the boundary conditions

$$W = \Theta = DZ = D^2 W = X = 0, \quad x_3 = 0, 1. \tag{3.6}$$

If we have the onset of instability as stationary convection, the marginal state will be characterized by $\sigma = 0$, then from Eq. (3.5) we have

$$\begin{aligned}&[(D^2 - a^2)^3 + Ra^2] \big(\{(D^2 - a^2)[(1 + \beta_I) D^2 - a^2] - M^2 D^2\} (D^2 - a^2)(1 + \beta_I) \\ &+ (D^2 - a^2)^2 \beta_H^2 D^2 \big) W = M^2 (D^2 - a^2)^2 \{[(1 + \beta_I) D^2 - a^2](D^2 - a^2) - M^2 D^2\} D^2 W,\end{aligned} \tag{3.7}$$

where

$$M = H_0 d \sqrt{\mu_e / \varrho_0 \nu \eta_e} \quad \text{and} \quad R = g\alpha\beta d^4 / \nu K_T$$

are Hartmann and Rayleigh numbers, respectively. It is easy to prove that not only W, but all its even derivatives must be zero on the boundary. Therefore, for perturbations of the type sinusoidal waves

$$W = A \sin(n\pi x_3),$$

we obtain the following Rayleigh function:

$$R(n, x, M, \beta_H, \beta_I) = \pi^4 \frac{x+n^2}{x} \times \frac{\{(n^2+x)[(1+\beta_I)n^2+x]+M_1^2 n^2\}\{(n^2+x)^2(1+\beta_I)+M_1^2 u^2\}+\beta_H^2 n^2(n^2+x)^3}{\{(n^2+x)[(1+\beta_I)n^2+x]+M_1^2 u^2\}(1+\beta_I)+\beta_H^2 n^2(n^2+x)} \tag{3.8}$$

with

$$M_1^2 = M^2/\pi^2, \qquad x = a^2/\pi^2.$$

By comparison, it easily follows

$$R(1, x, M, \beta_H, \beta_I) < R(n, x, M, \beta_H, \beta_I); \quad \forall n > 1, \quad \forall x, M, \beta_H, \beta_I$$

and therefore from Eq. (3.8), with $n = 1$, we have the critical Rayleigh functions for the linear instability:

$$R, R_M, R_H, R_{HI}, \tag{3.9}$$

in the purely hydrodynamic case ($M = \beta_H = \beta_I = 0$), in the isotropic M.H.D. case ($\beta_H = \beta_I = 0$), in the anisotropic case and in the presence of the only Hall's current ($\beta_I = 0$), in the presence of both Hall's and ion-slip currents, respectively. Finally the comparison between these functions and the proof of the existence and uniqueness of the minimum of the Rayleigh function (3.8) allow us to state the following:

THEOREM. *Let $R^{(c)}$, $R_M^{(c)}$, $R_H^{(c)}$, $R_{HI}^{(c)}$ be the critical Rayleigh numbers related to the critical Rayleigh functions* (3.9), *respectively. We have*

$$R^{(c)} < R_{HI}^{(c)} < R_H^{(c)} < R_M^{(c)}. \tag{3.10}$$

Therefore Hall's and ion-slip currents, at least for the linear stability of the thermodiffusive equilibrium, have a stabilizing effect with respect to the hydrodynamic case, but they have an instabilizing effect with respect to the isotropic M.H.D. *case.*

Now, at this step, we cannot know if this linear stability implies nonlinear stability, since, for that is at our knowledge, there does not exist a linearization principle in anisotropic M.H.D. like in the hydrodynamic case [13] and in isotropic M.H.D. [14]. Therefore we shall investigate directly the nonlinear stability of the thermodiffusive equilibrium solution.

4. Nonlinear energy stability

Lest us suppose that the fluid lies in a domain S, bounded in at least one dimension, like a layer, and the boundary is conducting, rigid and fixed Σ. With Eqs. (2.1) we must consider now the following boundary conditions:

$$\mathbf{v}|_\Sigma = 0, \quad \mathbf{H}\times\mathbf{n}|_\Sigma = \mathbf{H}_\Sigma, \quad T = T_\Sigma, \tag{4.1}$$

where $\mathbf{n}$ is the outward unit vector and $\mathbf{H}_\Sigma$, $\mathbf{T}_\Sigma$ are fixed.

If we introduce nondimensional quantities:

$$\mathbf{x} = x^* d, \quad t = t^* d^2/\nu, \quad \mathbf{u} = u^*\nu/d, \quad \mathbf{H} = \mathbf{H}^* H_0, \quad T = T^*\nu c d/k_T,$$

with d a comparison length the nonlinear equations for the perturbations belonging to a suitable class of regularity

$$\{\mathbf{u}(\mathbf{x}/t);\quad \mathbf{h}(x/t);\quad \theta(\mathbf{x}/t);\quad p(\mathbf{x}/t)\}, \tag{4.2}$$

are the following (the stars are left out):

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} &= -\mathbf{u}\cdot\nabla\mathbf{u}-\nabla p+\Delta_2\mathbf{u}+\sigma M^2(\nabla\times\mathbf{h})\times(\mathbf{e}_3+\mathbf{h})+Ra\theta\mathbf{e}_3,\\
\frac{\partial \mathbf{h}}{\partial t} &= \nabla\times[\mathbf{u}\times(\mathbf{e}_3+\mathbf{h})]+\sigma\Delta_2\mathbf{h}+\beta_H\nabla\times[(\mathbf{e}_3+\mathbf{h})\times\nabla\times\mathbf{h}]\\
&\qquad +\beta_I\nabla\times\{(\mathbf{e}_3+\mathbf{h})\times[(\mathbf{e}_3+\mathbf{h})\times\nabla\times\mathbf{h}]\},\\
\frac{\partial\theta}{\partial t} &= -\mathbf{u}\cdot(\mathbf{C}+\nabla\theta)+\frac{1}{P_r}\Delta_2\theta,\\
\nabla\cdot\mathbf{u} &= 0,\qquad \nabla\cdot\mathbf{h}=0,
\end{aligned}\tag{4.3}$$

with the boundary conditions

$$\mathbf{u}|_\Sigma = \mathbf{h}\times\mathbf{n}|_\Sigma = \theta|_\Sigma = 0. \tag{4.4}$$

In Eqs. (4.3)

$$\sigma = R_m/Re,\qquad P_r = \nu/k_T,\qquad R_a = g\alpha c d^4/\nu K_T,$$
$$\beta_H = \beta_1 H_0/\nu,\qquad \beta_I = \beta_2 H_0^2/\nu,$$

with R_e, R_m, P_r, R_a, β_H, β_I, Reynolds, magnetic Reynolds, Prandtl, Rayleigh, Hall and ion-slip numbers, respectively.

Let us assume the functional (like total energy of perturbations)

$$V(t) = \frac{1}{2}\int_S (u^2+\sigma^2M^2h^2+P_r\theta^2)dS \tag{4.5}$$

as a measure of the perturbations (4.2). From Eqs. (4.3) taking into account the boundary conditions (4.4), the transport and the divergence theorems and the well-known vectorial identities, we obtain the following Reynolds equation:

$$\frac{dV}{dt} = \mathcal{J}-\mathcal{D} \tag{4.6}$$

with the quadratic functionals

$$\mathcal{J} = \int_S (R_a-1)\theta u_3 dS, \tag{4.7}$$

$$\mathcal{D} = \int_S \{[(\nabla\mathbf{u})^2+\sigma^2M^2(\nabla\times\mathbf{h})^2+\sigma M^2\beta_I[(\mathbf{e}_3+\mathbf{h})\times\nabla\times\mathbf{h}]^2+(\nabla\theta)^2\}dS. \tag{4.8}$$

Thanks to Eqs. (4.6), (4.7) and (4.8) we can establish the following three theorems:

1. *The isothermal anisotropic* M.H.D. *equilibrium*

$$\{\mathbf{v}=0,\quad \mathbf{H}=\mathbf{H}_0,\quad T=T_0,\quad \nabla p_0=\varrho_0\mathbf{g}\} \tag{4.9}$$

is unconditionally asymptotically stable with respect to the measure $\int_S (u^2+\sigma^2M^2h^2)dS$ of the isothermal perturbations $\{\mathbf{u},\mathbf{h},\theta=0\}$.

2. *Hall's current has no effect on the nonlinear energy stability of the thermodiffusive equilibrium* (2.2)

3. *The ion-slip current can only have a stabilizing effect on the nonlinear energy stability of the thermodiffusive equilibrium* (2.2).

Proof of the first theorem: If we put $W(t) = \frac{1}{2}\int\limits_S (u^2+\sigma^2 M^2 h^2)dS$, from Eqs. (4.6), (4.7) and (4.8) it follows

$$\frac{dW}{dt} \leqslant -\int\limits_S [(\nabla \mathbf{u})^2 + \sigma^2 M^2 (\nabla \times \mathbf{h})^2] dS. \tag{4.10}$$

Therefore we have

$$W(t) \leqslant W(0)\exp(-mt),$$

with $m = \min(\delta, \gamma)$, where δ, γ, are the constants of the problems

$$\delta = \min_S \frac{\int\limits_S (\nabla \mathbf{A})^2 dS}{\int\limits_S A^2 dS}, \quad \gamma = \min_S \frac{\int\limits_S (\nabla \times \mathbf{B})^2 dS}{\int\limits_S B^2 dS}, \tag{4.11}$$

$$(\mathbf{A}, \mathbf{B}) \in C^{(1)}(S), \quad [\nabla \cdot \mathbf{A}]_S = [\nabla \cdot \mathbf{B}]_S = 0, \quad [A]_\Sigma = [\mathbf{B} \times \mathbf{n}]_\Sigma = 0$$

with the related well-known variational problems [15, 16].

The proofs of the second and the third theorem are quite clear.

Finally, we shall give now a further theorem that will be a sufficient condition to ensure nonlinear Liapunov stability of the solution (2.2).

Let us assume the following functionals:

$$X_1^2 = \int\limits_S u^2 dS, \quad X_2^2 = \int\limits_S (\nabla \times \mathbf{h})^2 dS, \quad X_3^2 = \int\limits_S (\mathbf{h} \times \nabla \times \mathbf{h})^2 dS, \quad X_4^2 = \int\limits_S \theta^2 dS. \tag{4.12}$$

From Eqs. (4.6), (4.7) and (4.8) and suitable well-known integral inequalities, with the constants δ and γ given by Eq. (4.11) it is easy to find the Liapunov functional

$$\mathcal{J} = -\delta X_1^2 - \sigma M^2(\sigma - \beta_I) X_2^2 - \sigma M^2 \beta_I X_3^2 - \gamma X_4^2 + 2\sigma M^2 \beta_I X_2 X_3 + (R_a+1) X_1 X_4, \tag{4.13}$$

such that

$$\frac{dV}{dt} \leqslant \mathcal{J}. \tag{4.14}$$

Therefore the conditions for the quadratic form (4.13) to be negative allow us to prove the following theorem [1, 17]

4. *The conditions*

$$R_a + 1 < 2\sqrt{\delta\gamma},$$

$$\beta_I < \frac{1}{2}\sigma$$

ensure the asymptotic exponential stability of the thermodiffusive equilibrium (2.2), *with respect to the total energy measure* $V(t)$.

REMARK. Of course, the optimum stability conditions and the connection between linear and nonlinear stability, that are solved in the hydrodynamic case and in the isotropic M.H.D. case, [18, 19, 20], in the field of the Anisotropic Magnetohydrodynamics are open problems; at the stage of the present work I have not answer to these questions.

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