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Velocity and temperature distribution for a variable viscosity generalized Couette flow of a dilute suspension

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A STUDY of a plane generalized Couette flow with constant heat flux through the lower plate and isothermal upper plate is made. Concerning the suspension, the viscosity factor ν derived by Simha is used for the limiting case when the rotary Pêclet number Pe = 0. Using a nonlinear viscosity — temperature relation for the solvent, expressions for the velocity, temperature and the Nusselt number are obtained for prolate and oblate spheroids. The results are written in a general form which can be applied to any other scalar expressions for the viscosity factor ν of other suspensions.

Przeanalizowano płaski uogólniony przepływ Couette'a przy stałym strumieniu ciepła na płycie dolnej i z ustaloną temperaturą płyty górnej. Dla zawiesiny przyjęto współczynnik lepkości ν wyprowadzony przez Simhę w granicznym przypadku liczby Pècleta Pe = 0. Posługując się nieliniową zależnością lepkości od temperatury rozpuszczalnika wyprowadzono wyrażenia na prędkość, temperaturę i liczbę Nusselta dla przypadku sferoidy spłaszczonej i wydłużonej. Wyniki podano w postaci ogólnej nadającej się do zastosowania w przypadku dowolnych wyrażeń skalarnych dla współczynników ν w innych zawiesinach.

Проанализировано плоское обобщенное течение Куэтта при постоянном потоке тепла на нижней плите и с установленной температурой верхней плиты. Для взвеси принят коэффициент вязкости ν, выведенный Симгой в предельном случае числа Пекле Pe = 0. Послуживаясь нелинейной зависимостью вязкости от температуры растворителя, выведены выражения для скорости, температуры и числа Нуссельта для случая сплющенного и удлиненного сфероидов. Результаты приведены в общем виде, пригодном для применения в случае произвольных скалярных выражений для коэффициентов ν в других взвесях.

Nomenclature

A_0, B_0, C_0 constants given by Eqs. (3.10), (3.9) and (3.16),
a, b major and minor semiaxes of the spheroidal particles,
C constant of integration, Eq. (3.2),
Cp_0 specific heat at constant pressure P_0,
Ec the Eckert number,
F_1, F_2, F_3, F_4 functions given by Eqs. (3.8), (3.14) and (3.15),
h distance between the plates,
k thermal conductivity,
Q, R, J, K quantitities given by Eqs. (3.18) and (3.20),
E, L, M, G, H
N, S dimensionless parameters, Eq. (3.5),
Pe rotary Pèclet number,
$\left(\frac{dp}{dx}\right)$ constant pressure gradient along the x-axis,
Pr Prandtl number,

q, q_1 dimesnionless pressure gradient, Eqs. (2.6) and (3.3),
r_e axes ratio,
T, T^* temperature and dimensionless temperature, Eq. (2.2),
T_0 temperature of the upper plate,
V velocity of the upper plate,
u^* dimensionless velocity, Eq. (2.1),
x the x-axis along the lower plate,
y the y-axis normal to the x-axis.

Greek symbols

α dimensionless parameter, Eq. (2.7),
β constant heat flux through the lower plate, Eq. (2.4),
μ_{01} constant viscosity of the solvent at temperature T_0,
μ_0 the viscosity of the solvent,
μ^* dimensionless viscosity of the solvent,
μ^*_{eff} dimensionless viscosity of the suspension, Eq. (2.5),
ν viscosity factor for the suspension,
ξ, η dimensionless distances, Eq. (3.2),
θ^* dimensionless temperature, Eq. (2.7),
Φ volume concentration of suspended particles.

1. Introduction

A comprehensive review of research on the motion of small particles in a fluid is given by Brenner [1] and Leal [2]. For the solvent, which is a Newtonian fluid, the specific heat and thermal conductivity are relatively independent of temperature, but the viscosity decreases very markedly with temperature. Various relations between the dimensionless viscosity and temperature are used in studying the flow of a variable viscosity fluids [3–6]. The variable viscosity plane Poiseuille flow of a fluid is studied by Hausenblas [7], keeping both the walls at the same temperature. It is reconsidered by Bansal and Jain [6], taking the walls at unequal temperatures. The problem of Bansal has been extended by the author [8] to cover the case of a dilute suspension of rigid spheroidal particles.

The main purpose of the present work is to study the effects of heat transfer on the velocity and temperature profiles for the generalized Couette flow of a dilute suspension. The Nusselt number for the transfer of heat at the upper plate is calculated in Table 1. Expressions for the velocity, temperature and the Nusselt number are written in a general form which can be applied to suspensions of spherical particles [9] and near spherical particles [10]. The constant viscosity flow of the suspension and constant viscosity flow of the solvent are easily deduced.

2. Basic equations and boundary conditions

Consider a steady generalized plane Couette flow between two parallel plates, taking the x-axis along the lower one and the y-axis at a right angle to it. The dilute suspension consists of rigid spheroidal particles suspended in a Newtonian fluid of variable viscosity.

In this case the momentum and energy equations governing the motion in the dimensionless form are [8, 11]

$$\frac{d}{d\eta}\left[\mu^*_{\text{eff}}\frac{du^*}{d\eta}\right] = -2q, \tag{2.1}$$

$$\frac{d^2T^*}{d\eta^2}+\text{Ec}+\text{Pr}\cdot\mu^*_{\text{eff}}\left(\frac{du^*}{d\eta}\right)^2 = 0 \tag{2.2}$$

and the boundary conditions are

$$\eta = 0: \quad u^* = 0, \quad \eta = 1: \quad u^* = 1, \tag{2.3}$$

$$\eta = 0: \quad \frac{dT^*}{d\eta} = \beta, \quad \eta = 1: \quad T^* = 0 \tag{2.4}$$

in which $\eta = y/h$, h is the distance between the plates,

$$\mu^*_{\text{eff}} = \frac{\mu_0}{\mu_{01}}(1+\nu\Phi) = \mu^*(1+\nu\Phi). \tag{2.5}$$

μ_0 is the viscosity of the solvent which depends on the temperature, μ_{01} is the viscosity of the solvent at the constant temperature of the upper plate T_0, ν is the viscosity factor of the suspension which is given by SIMHA [12] for prolate spheroids and by KUHN and KUHN [13] for oblate spheroids and depends on the axes ratio $r_e = a/b$, a and b are the major and minor semiaxes of the spheroidal suspended particles, Φ is the volume concentration of the suspended particles, $u^* = u/V$ is the dimensionless velocity, V is the velocity of the upper plate,

$$q = -\frac{h^2}{2V\mu_{01}}\left(\frac{dP}{dx}\right), \tag{2.6}$$

(dp/dx) is the constant pressure gradient along the x-axis [14], $T^* = (T-T_0)/T_0$ is the dimensionless temperature, $\text{Ec} = V^2/(Cp_0T_0)$ is the Eckert number, Cp_0 is the specific heat at constant pressure P_0, $\text{Pr} = (\mu_{01}Cp_0)/K$ is the Prandtl number, k is the thermal conductivity, β being a constant. The empirical relation between the dimensionless viscosity of the solvent and the temperature [7]

$$\frac{1}{\mu^*} = 1+\alpha T^* = \theta^* \quad \text{(say)}, \tag{2.7}$$

where α is a parameter which depends on the nature of the solvent. It is used to solve the coupled differential equations (2.1) and (2.2).

3. Analysis

By integrating Eq. (2.1), we obtain

$$\mu^*\frac{du^*}{d\xi} = -2q_1\xi, \tag{3.1}$$

where

$$\xi = \eta - C, \tag{3.2}$$

$$q_1 = \frac{q}{(1+\nu\Phi)} \tag{3.3}$$

and C is the constant of integration. Substituting from Eq. (3.1) into Eq. (2.2) and using Eq. (2.7), we get

$$\frac{d^2\theta^*}{d\xi^2} + 4N\xi^2\theta^* = 0 \tag{3.4}$$

where

$$N = \frac{\alpha \cdot \mathrm{Ec} \cdot \mathrm{Pr} \cdot q^2}{(1+\nu\Phi)} = \frac{Sq^2}{(1+\nu\Phi)}; \qquad S = \alpha \cdot \mathrm{Ec} \cdot \mathrm{Pr} \tag{3.5}$$

is a dimensionless parameter.

The boundary conditions associated with Eq. (3.4), in view of Eqs. (2.4), (3.2) and (2.7) are

$$\xi = -C: \quad \frac{d\theta^*}{d\xi} = \alpha\beta; \qquad \xi = 1-C: \quad \theta^* = 1. \tag{3.6}$$

The solution of Eq. (3.4) satisfying the boundary conditions (3.6) is given by

$$\theta^* = A_0 F_1\{\xi^2\sqrt{N}\} + B_0 \xi F_2\{\xi^2\sqrt{N}\}, \tag{3.7}$$

where

$$F_1(t) = \sum_{n=0}^{\infty}\left\{\frac{1}{n!}\,\frac{\left(\frac{-1}{4}\right)!}{\left(\frac{-1}{4}+n\right)!}\left(\frac{it}{2}\right)^{2n}\right\}, \qquad F_2(t) = \sum_{n=0}^{\infty}\left\{\frac{1}{n!}\,\frac{\left(\frac{1}{4}\right)!\left(\frac{it}{2}\right)^{2n}}{\left(\frac{1}{4}+n\right)!}\right\} \tag{3.8}$$

$$B_0 = \frac{1 - A_0 F_1\{(1-C)^2\sqrt{N}\}}{(1-C)F_2\{(1-C)^2\sqrt{N}\}} \tag{3.9}$$

and

$$A_0 = \frac{2\sqrt{N}C^2F_2'\{C^2\sqrt{N}\} + F_2\{C^2\sqrt{N}\} - \alpha\beta(1-C)F_2\{(1-C)^2\sqrt{N}\}}{2C(1-C)\sqrt{N}F_1'\{C^2\sqrt{N}\}F_2\{(1-C)^2\sqrt{N}\} + F_1\{(1-C)^2\sqrt{N}\}\left[2C^2\sqrt{N}F_2'\{C^2\sqrt{N}\} + F_2\{C^2\sqrt{N}\}\right]}, \tag{3.10}$$

where a prime denotes differentiation with respect to t.

Equation (3.1), in view of Eqs. (2.7) and (3.7), may be written as

$$\frac{du^*}{d\xi} = -2q_1\xi\left[A_0F_1\{\xi^2\sqrt{N}\} + B_0\xi F_2\{\xi^2\sqrt{N}\}\right]. \tag{3.11}$$

The boundary conditions on u^*, in view of Eqs. (2.3) and (3.2), are

$$\xi = -C: \quad u^* = 0, \qquad \xi = 1-C: \quad u^* = 1. \tag{3.12}$$

The solution of Eq. (3.11) is given by

$$u^* = -2q_1\left[A_0\xi^2F_3\{\sqrt{N}\xi^2\} + B_0\xi^3F_4\{\xi^2\sqrt{N}\}\right] + C_0, \tag{3.13}$$

where

$$(3.14)\qquad F_3(t) = \sum_{n=0}^{\infty}\left\{\frac{1}{n!}\,\frac{\left(\frac{-1}{4}\right)!\left(\frac{it}{2}\right)^{2n}}{\left(\frac{-1}{4}+n\right)!\,(4n+2)}\right\},$$

$$(3.15)\qquad F_4(t) = \sum_{n=0}^{\infty}\left\{\frac{1}{n!}\,\frac{\left(\frac{1}{4}\right)!\left(\frac{it}{2}\right)^{2n}}{\left(\frac{1}{4}+n\right)!\,(4n+3)}\right\}.$$

The boundary conditions (3.12) when applied to Eq. (3.13), give the value of C_0 as

$$(3.16)\qquad \begin{aligned} C_0 &= 2q_1\left[A_0C^2F_3\{C^2\sqrt{N}\} - B_0C^3F_4\{C^2\sqrt{N}\}\right] \\ &= 1+2q_1\left[A_0(1-C)^2F_3\{(1-C)^2\sqrt{N}\} + B_0(1-C)^3F_4\{(1-C)^2\sqrt{N}\}\right]. \end{aligned}$$

By substituting the value B_0 from Eq. (3.9) in Eq. (3.16), the value of the constant A_0 is given by

$$(3.17)\qquad A_0 = \frac{(1-C)F_2\{(1-C)^2\sqrt{N}\}+2q_1\left[(1-C)^3F_4\{(1-C)^2\sqrt{N}\}+C^3F_4\{C^2\sqrt{N}\}\right]}{Q+R},$$

where

$$(3.18)\qquad \begin{aligned} Q &= 2q_1(1-C)F_2\{(1-C)^2\sqrt{N}\}\left[C^2F_3\{C^2\sqrt{N}\}-(1-C)^2F_3\{(1-C)^2\sqrt{N}\}\right], \\ R &= 2q_1F_1\{(1-C)^2\sqrt{N}\}\left[(1-C)^3F_4\{(1-C)^2\sqrt{N}\}+C^3F_4\{C^2\sqrt{N}\}\right]. \end{aligned}$$

Equating the two values of the constant A_0 given by Eqs. (3.10) and (3.17), the equation which will determine the constant C, for prescribed values of q_1, N and $\alpha\beta$, is given by

$$(3.19)\qquad \alpha\beta = \frac{J}{K} - \frac{(K+E)(L+M)}{K(G+H)},$$

where

$$(3.20)\qquad \begin{aligned} J &= 2C^2\sqrt{N}F_2'\{C^2\sqrt{N}\}+F_2\{C^2\sqrt{N}\}, \\ K &= (1-C)F_2\{(1-C)^2\sqrt{N}\}, \\ E &= 2q_1\left[(1-C)^3F_4\{(1-C)^2\sqrt{N}\}+C^3F_4\{C^2\sqrt{N}\}\right], \\ L &= 2C(1-C)\sqrt{N}F_1'\{C^2\sqrt{N}\}F_2\{(1-C)^2\sqrt{N}\}, \\ M &= F_1\{(1-C)^2\sqrt{N}\}\left[2\sqrt{N}C^2F_2'\{C^2\sqrt{N}\}+F_2\{C^2\sqrt{N}\}\right], \\ G &= 2q_1(1-C)F_2\{(1-C)^2\sqrt{N}\}\left[C^2F_3\{C^2\sqrt{N}\}-(1-C)^2F_3\{(1-C)^2\sqrt{N}\}\right., \\ H &= 2q_1F_1\{(1-C)^2\sqrt{N}\}\left[(1-C)^3F_4\{(1-C)^2\sqrt{N}\}+C^3F_4\{C^2\sqrt{N}\}\right]. \end{aligned}$$

From Eq. (3.19) it is not possible to calculate the value of C for given values of q_1, N and $\alpha\beta$ since C appears in the form of an infinite series. However, it is convenient to determine the values of $\alpha\beta$ for prescribed values of q_1, N and C.

The velocity and temperature profiles for $\Phi = 0.01$, $q = 2.0$ and $S = 0.25$ are drawn in Figs. 1, 2 and 3. The velocity profiles u^* are displayed in Fig. 1 for different values of r_e when the parameter $\alpha\beta = 3$ and are compared with the profiles representing the flow of a Newtonian fluid having constant or variable viscosity. It is found that in the present

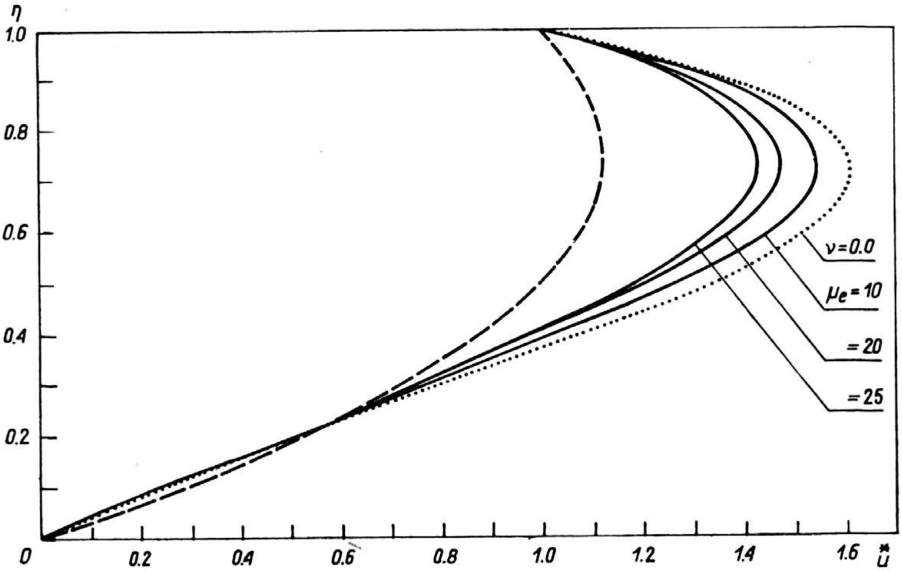

FIG. 1. The variation of u^* with η for different values of μ_e when $\alpha\beta = 3$.
— — — — Newtonian fluid with constant viscosity,
.... Newtonian fluid with variable viscosity,
———— suspension with variable viscosity.

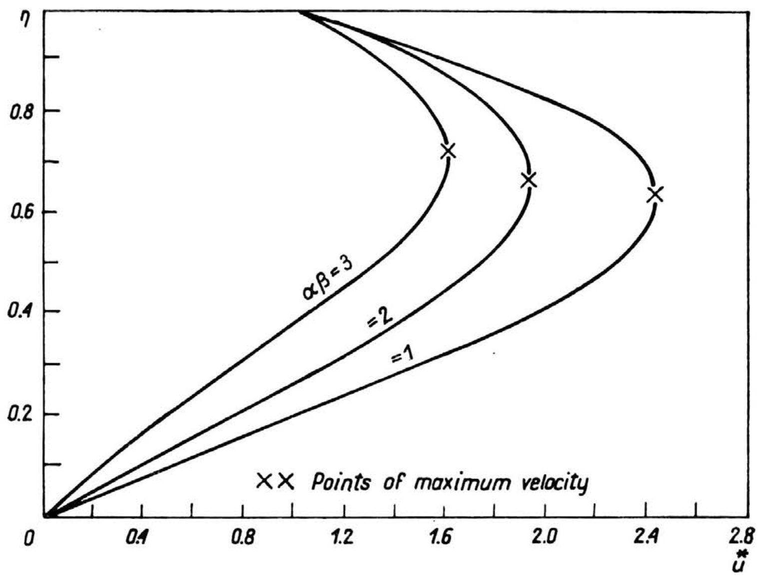

FIG. 2. The variation of u^* with η for different values of $\alpha\beta$ when $\nu = 0$.

case the magnitude of the velocity increases with the decrease of r_e and is always less than the velocity of a Newtonian fluid with variable viscosity but greater than the velocity of a Newtonian fluid with constant viscosity. The velocity profiles representing the flow of a Newtonian fluid having variable viscosity are drawn in Fig. 2 for different values of $\alpha\beta$. Figure 3 represents the variation of θ^* with η for different values of $\alpha\beta$ when $\nu = 0.0$ and $\nu = 55.19$. As shown in Figs. 2 and 3, the velocity and temperature increase with the decrease in $\alpha\beta$ and the maximum velocity moves towards the upper plate as the value of $\alpha\beta$ increases.

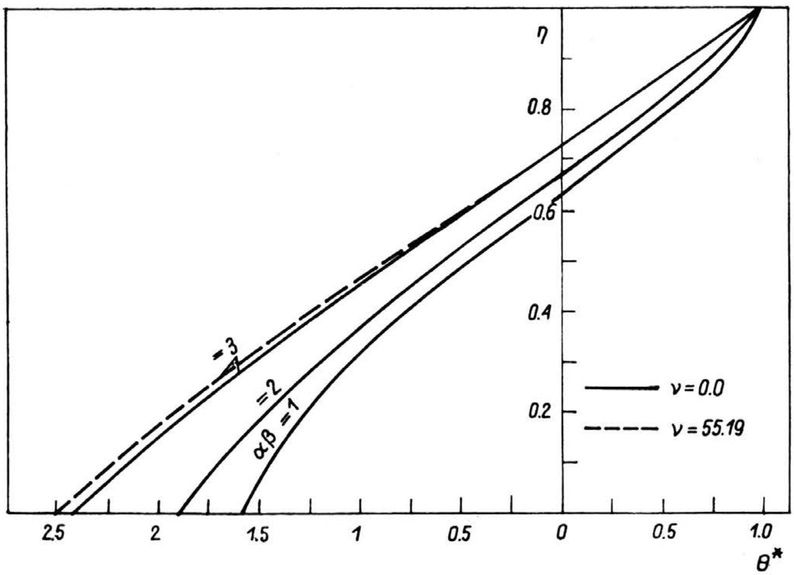

FIG. 3. The variation of θ^* with η for different values of $\alpha\beta$.

The dimensionless coefficient of heat transfer at the upper plate, viz the Nusselt number is defined as

$$\text{(3.21)} \qquad \text{Nu} = \frac{-1}{T^*_{\eta=0}} \left(\frac{dT^*}{d\eta} \right)_{\eta=1}.$$

In view of Eqs. (3.2), (2.7) and (3.7), Eq. (3.21) may be written as

$$\text{(3.22)} \qquad \text{Nu} = \frac{\begin{array}{l} 2A_0(1-C)\sqrt{N}F_1'\{(1-C)^2\sqrt{N}\} \\ \qquad + B_0\{2(1-C)^2\sqrt{N}F_2'\{(1-C)^2\sqrt{N}\} + F_2\{(1-C)^2\sqrt{N}\}\} \end{array}}{1 - A_0F_1\{C^2\sqrt{N}\} + B_0CF_2\{C^2\sqrt{N}\}}.$$

The calculated values of the parameter $\alpha\beta$ and the Nusselt number Nu for some given values of the viscosity factor ν and the constant C are given in Table 1 for $\Phi = 0.01$, $q = 2$ and $S = 0.25$.

From Table 1 it can be seen that Nu increases with the increase of $\alpha\beta$ and ν.

Table 1.

ν	C	$\alpha\beta$	Nu
0.0	-0.81	1	0.3141
0.0	-0.47	2	0.7479
0.0	-0.19	3	0.9070
13.63	-0.22	3	0.9223
38.53	-0.28	3	0.9412
55.19	-0.32	3	0.9510

4. Constant viscosity solution for the suspension

In this case the constant α in Eq. (2.7) equals zero; consequently, from Eq. (3.5), $S = N = 0$. Substituting these values in Eq. (3.20) it follows that

$$\begin{gathered} J = 1, \quad K = 1-C, \quad E = \frac{2}{3} q_1 \big((1-C)^3 + C^3\big); \\ L = 0, \quad M = 1, \quad G = q_1(1-C)(2C-1), \quad H = \frac{2}{3} q_1 \big((1-C)^3 + C^3\big). \end{gathered} \tag{4.1}$$

From Eqs. (4.1) and (3.19) we get

$$C = \frac{1+q_1}{2q_1}. \tag{4.2}$$

Substituting the value $N = 0$ in Eqs. (3.10), (3.9) and (3.16) we find

$$A_0 = 1, \quad B_0 = 0, \quad C_0 = q_1 C^2. \tag{4.3}$$

Hence Eqs. (3.7), (3.13) and (3.22) for T^*, u^* and Nu reduce to

$$T^* = \beta(\eta-1) - \frac{1}{6} \mathrm{Ec}\cdot\mathrm{Pr}(1+\nu\Phi)\,[2q_1^2\eta^4 - 4q_1(1+q_1)\eta^3 + 3(1+q_1)^2\eta^2 - (q_1^2+2q_1+3)], \tag{4.4}$$

$$u^* = \eta[1+q_1(1-\eta)], \tag{4.5}$$

$$\mathrm{Nu} = \frac{\frac{1}{3}\mathrm{Ec}\cdot\mathrm{Pr}(q_1^2+3)(1+\nu\Phi) - \beta}{\frac{1}{6}\mathrm{Ec}\cdot\mathrm{Pr}(q_1^2+2q_1+3)(1+\nu\Phi) - \beta}, \tag{4.6}$$

respectively:.

Expressions for T^*, u^* and Nu representing the flow of a Newtonian fluid having constant viscosity can be deduced from Eqs. (4.4)–(4.6) by taking the viscosity factor $\nu = 0$.

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