# Vibration of cracked structures

B. ZASTRAU (HAMBURG)

REINFORCED concrete is a material often used for construction of industrial plants and office buildings. Due to the material properties of concrete, the structural elements such as plates and beams are partially cracked. The aim of the present paper is to discuss the influence of discrete cracks with respect to the vibration behaviour of beams including the transmission of the excitation to neighbouring structures.

Beton zbrojony jest materiałem często stosowanym w konstrukcjach przemysłowych i budynkach biurowych. Własności materiałowe betonu powodują, że wykonane z niego elementy konstrukcyjne, takie jak płyty i belki, bywają częściowo spękane. Celem tej pracy jest przedyskutowanie wpływu dyskretnych szczelin na proces drgań takich elementów z uwzględnieniem również takiego problemu jak przenoszenie wymuszeń na elementy sąsiednie.

Армированный бетон является материалом часто применяемым в промышленных конструкциях и административных зданиях. Материальные свойства бетона приводят к тому, что изготовленные из него конструкционные элементы, такие как плиты и балки, бывают частично трещиноватыми. Целью этой работы является обсуждение влияния дискретных трещин на процесс колебаний таких элементов с учетом тоже такой проблемы, как перенос вынуждений на соседние элементы.

## 1. Introduction

THE ASSESSMENT of the dynamic behavior of complex structures is in general based on the knowledge of the natural frequencies. For systems with nonlinearities, such as cracks in concrete beams, a somehow different approach has to be chosen.

The aim of this paper is to show a way to evaluate the specific data that are necessary to give an assessment of the effect of mechanical vibrations for human beings according to DIN 4150. Since the interesting frequency range is rather narrow, reaching from 1 to 80 Hz, it will be possible to generate an appropriate mechanical model and to perform a time-integration for each excitation of interest. Two different integration techniques will be discussed. Corresponding to the complexity of the chosen mechanical representation, the first alternative uses spectral superposition and leads to qualitatively good results. For the calculation of quantitatively reliable results the application of the finite element method has to be preferred. Introducing so-called contact elements for the representation of the cracks, the nonlinearities can be treated on element level and an implicit time integration can be used. A final Fourier transform leads to the frequency distribution of the nonlinear vibration of the cracked structure.

## 2. Simplified description of cracked structures

Treating flexural vibrations of a column, a distinction between the pure bending situation and bending combined with constant horizontal forces has to be made. For pure

11*

bending of a symmetrically reinforced concrete column the cracks will open on both sides alternatively. Since the depth of cracks does not depend on the sign of the bending moment, the characteristic remains symmetric and nonlinearities arise only due to the nonlinear stress-strain relation of concrete. Bending with a constant horizontal load and vibration of precracked beams will lead to nonsymmetric stiffness.

According to the low tensile strength of concrete, cracks do already occur with dead load. Figure 1 shows a two-span continuous beam with an assumed crack configuration
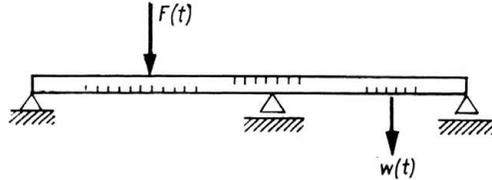


FIG. 1. Continuous beam with crack configuration according to dead load.

belonging to constant loading. It shall be assumed that the cracked regions can be mapped onto three single cracks in the place of maximum bending moments. If the nonlinear constitutive relations of concrete and steel are neglected, the problem of a simply supported beam with a single crack in the midspan, see Fig. 2, leads to the simplest possible con-
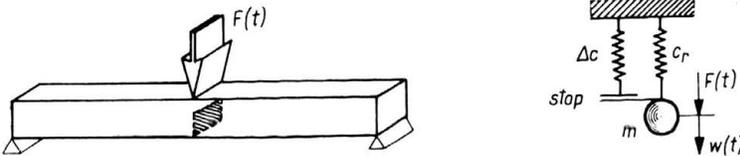


FIG. 2. Simplified crack pattern and lumped mass discrete spring representation.

centrated mass and discrete spring model. In order to describe the changing stiffness depending on the sign of the deflection $w$, a stop has to be incorporated. An additive decomposition of the global stiffness

(2.1)
$$c_u = \frac{48EI}{l^3}$$

into the reduced value

(2.2)
$$c_r = \alpha c_u$$

and the stiffness decrease due to the opening of the crack

(2.3)
$$\Delta c = (1-\alpha)c_u$$

with

(2.4)
$$c_u = c_r + \Delta c$$

will be introduced. For a beam with a rectangular cross-section the assumption of a homogeneous crack distribution with a reduced height of $k_x \cdot h$, with $0.54 > k_x > 0.09$ for reinforced concrete beams, leads to a lower bound for the stiffness reduction factor $\alpha$

(2.5)
$$1 > \alpha > \frac{I_r}{I_u} = k_x^3.$$

Since the value of $\alpha$ actually depends on the ratio of $h/l$, on the boundary conditions and even on the eigenform of the vibration, a first approximation based on a static calculation of the deflection under the load shall be used

(2.6)
$$\alpha = w_u/w_r.$$
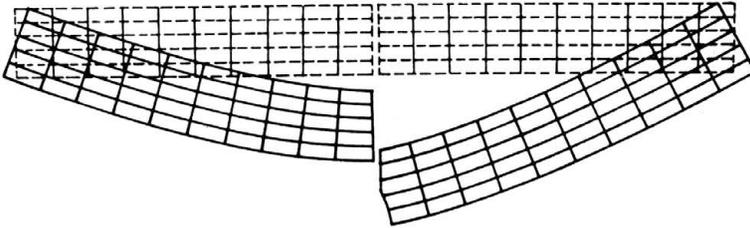


FIG. 3. Transverse displacement of an uncracked and a cracked beam.

Figure 3 gives a comparison of the deflection of one half of an uncracked beam and that with a single crack with a remaining cross-section of one half ($k_x = 0.5$). The ratio of height to span for this specific example is $1/10$. The deflectiom of the cracked structure is about two times the value of the homogeneous structure, by that leading to

$$\alpha \approx 0.5.$$

Since $\alpha$ decreases with increasing $h/l$ and vice versa, with $l$ being the distance of the curvature change of the deformation pattern, the value of $1/2$ for $\alpha$ shall be used for all qualitative calculations.

## 3. Qualitative analysis

In order to calculate the structural response to time-dependent forces, a harmonic excitation

(3.1)
$$F(t) = F \sin \Omega t + F_{\text{stat}}$$

is introduced. This load function leads, in connection with the piecewise linear force-displacement-curve, to piecewise linear equations of motion as well. The solution of the problem without a constant part is in addition linearly dependent on the amplitude of the dynamic force. The nonlinearity invoked by $F_{\text{stat}}$, being equivalent with a preopened crack, can be recognized by inspection of the characteristic as shown in Fig. 4b. Nevertheless, there exists even for this case a piecewise linear solution [3]. Using this fact the complete time history can be integrated on the basis of the continuation technique. Since
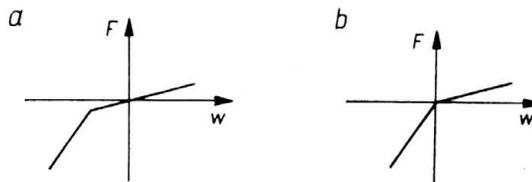


FIG. 4. Force-displacement relations for a cracked beam; a) with and b) without a gap.

only the stationary solution is of interest, the initial conditions can be chosen arbitrarily as long as damping is incorporated in the system.

In order to calculate the local response of the cracked structure and the transmission of the excitation, a mechanical model with at least two degrees of freedom has to be incorporated.
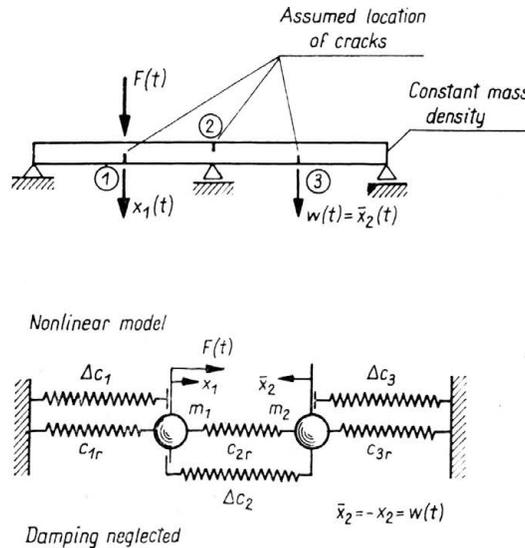


FIG. 5. Idealized continuous beam and discrete resultant mass-spring-model.

Figure 5 represents such a concentrated mass and discrete spring assembly. According to the chosen crack pattern with three single cracks, the mechanical model contains three stops as well. In order to avoid some difficult mechanisms, the deflection $w$ in the right span must correspond to the negative coordinate $x_2$. This transformation leads to the right kinematic description of crack and stop opening. For a positive value of $F(t)$ the cracks one and two will open and so will the stops connected to the springs $\Delta c_1$ and $\Delta c_2$ and so on. Before calculation of the dynamic behavior, the stiffness of the springs and the resultant masses have to be known. The comparison of the static deflection under the load and at the place of interest in the neighboring span with the displacement in $x_1$ and $x_2$ direction leads for the uncracked system to the following results

$$(3.2) \qquad\qquad c_{1u} = c_{3u} = 48 \frac{EI}{l^3}$$

and

$$(3.3) \qquad\qquad c_{2u} = 30.85 \frac{EI}{l^3}.$$

The values for $c_{iu}$ belong to a symmetric system with a total span of $2l$ and $F$ and $w$ given at the middle of each span. The lumped masses can be calculated by comparison of the eigenfrequencies of the continuous beam and the model. Using symmetry again, the mass can be given by

(3.4)
$$m_1 = m_2 = \frac{48}{\pi^4} \cdot \frac{1}{2} \cdot m_{\text{beam}} \approx \frac{1}{4} m_{\text{beam}}$$

for the first eigenfrequency being identical. The second natural frequency is simultaneously approximated with an error of less than 4%. For different locations of $F$ and $w$ the difference of the second eigenfrequency increases unfortunately and the stiffness and mass-data of the resultant model should be calculated by parameter identification. For the sake of simplicity the same stiffness reduction factor $\alpha = 0.5$ for all three cracks is assumed. Without detailed knowledge about the real damping behavior of the continuous beam, the assumption of Rayleigh damping will be possible.

The equations of motion can be written in matrix notation as

(3.5)
$$\mathbf{M\ddot{x}} + \{\alpha\mathbf{M} + \beta\mathbf{K(x)}\}\mathbf{\dot{x}} + \mathbf{K(x)x} = \mathbf{F}(t),$$

where stiffness and damping are dependent on the actual state of deformation. The system matrices are

(3.6)
$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} c_{1r}+c_{2r} & -c_{2r} \\ -c_{2r} & c_{2r}+c_{3r}, \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} F(t) \\ 0 \end{bmatrix}$$

with all cracks open. The conditions for changes in the stiffness matrix are

$$x_1 < 0 : \Delta c_1 \quad \text{to be added,}$$
$$\overline{x}_2 < 0 : \Delta c_3 \quad \text{to be added,}$$
$$x_1 + \overline{x}_2 < 0 : \Delta c_2 \quad \text{to be added.}$$

Premultiplication with the matrix of the eigenforms $\mathbf{\Phi(X)}$ decouples the system of equations

(3.7)
$$\mathbf{\Phi}^T\mathbf{M\Phi\ddot{q}} + \mathbf{\Phi}^T(\alpha\mathbf{M} + \beta\mathbf{K})\mathbf{\Phi\dot{q}} + \mathbf{\Phi}^T\mathbf{K\Phi q} = \mathbf{\Phi}^T\mathbf{F}$$

leading to $i$ equations

(3.8)
$$M_i(\mathbf{x})\ddot{q}_i + 2D_i(\mathbf{x})\omega_i(\mathbf{x})\dot{q}_i + K_i(\mathbf{x})q_i = \hat{F}_i(\mathbf{x}, t)e^{i\Omega t},$$

where $i$ is the index of the mode, $q_i$ is the generalized coordinate and $M_i$, $D_i$, $K_i$ and $F_i$ are the modal values of mass, damping, stiffness and exciting force. The solution of this quasi-single degree of freedom system can be obtained by classical means:

(3.9)
$$\mathbf{q}_i = \mathbf{q}_{ih} + \mathbf{q}_{ip},$$
$$\mathbf{q}_{ip} = \mathbf{q}_{i\,\text{stat}} + R_e\big(\mathbf{V}(\eta, D)\hat{F}e^{i\Omega t}\big),$$

where the index $h$ describes the free vibration and the particular integral noted by $p$ is split into a static part and the forced vibration with the complex frequency response function $\mathbf{V}$.

Since eigenfrequencies and modes change with each opening and closing of a stop, the amplitudes and phases of the free vibration have to be recalculated with the assumption of continuity of the displacement $\mathbf{x}$ and the velocity $\mathbf{\dot{x}}$. The last requirement can be deduced from the introduction of a massless stop into the resultant mass and spring model.

Written in terms of the real displacement, the continuity equations are

(3.10)
$$^+\mathbf{x}(t_n) = {}^-\mathbf{x}(t_n)$$

and

(3.11) $$^{+}\dot{\mathbf{x}}(t_n) = {}^{-}\dot{\mathbf{x}}(t_n),$$

with the time $t_n$ for opening and closing of any one of the stops and the minus and plus index indicating that the appropriate set of eigensolutions belonging to time interval before or after $t_n$ have to be taken.

For a system with a small number of possible crack configurations it is convenient to calculate all eigensolutions before time integration. The solution for each individual time interval, although somehow tedious, is merely a problem of complex matrix inversion and shall not be discussed here.

The applicability of the herewith stated method will be demonstrated by the following example describing a reinforced concrete beam of total length of 16 m as given in Fig. 6.
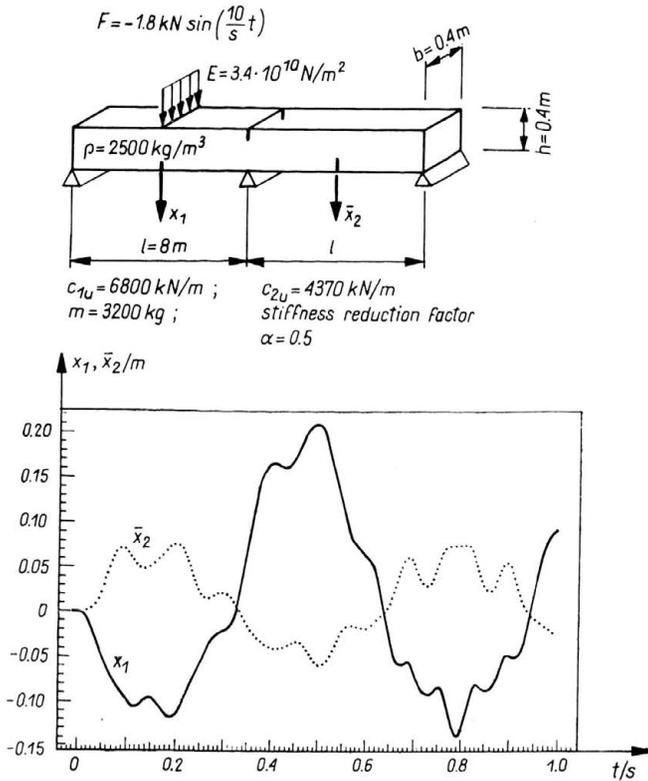


FIG. 6. Forced vibration of a continuous beam.

The eigenfrequencies are those of the continuous beam. The time history is typical for a cracked system having large amplitudes of the forced motion when the cracks are open and having smaller ones for closed cracks.

## 4. Quantitative method for calculating the time history of a two-dimensional structure

The calculation of physical systems using realistic geometrical and material properties is often done on the basis of the finite element method [1, 6], which will be used in this

context as well. Several strategies for the treatment of cracked reinforced concrete structures can be found in existing commercial program packages. A frequently used method is the introduction of an appropriate nonlinear material law representing even crack patterns with cracks in multiple directions [5].

For solitary cracks it is much simpler to generate element boundaries along the crack surfaces. Following the time-dependent deformation two different configurations occur. If the crack is open, the degrees of freedom of the neighboring nodes must be independent as long as no penetration happens. For a closed crack both sides must have a common displacement and it must be checked that the stresses between the element boundaries remain compressive stresses.

If these checks are done on the main program level stresses have to be calculated at each time increment, and after opening or closing a reorganisation of the system matrices has to be performed.

A more convenient way of describing the crack properties works with the introduction of the contact force $S_c$ as an additional variable. This leads to the following set of equilibrium equations written in internal forces $S_i$ and external forces $R$

(4.1)
$$\sum_{i=1}^{n} S_{il} + S_c + R_l = 0,$$

$$\sum_{i=1}^{n} S_{ir} - S_c + R_r = 0,$$

where the index $l$ is the degree of freedom of the left node and $r$ that of the right node, $n$ is the total number of equations. In the following text $l$ shall always indicate properties or values of the left node. The second governing equation for a closed crack is the equality of left and right displacement.

Introducing two symmetrical matrices [2]

(4.2)
$$\mathbf{H}_c = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{H}_{nc} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the element stiffness matrix for contact can be written as

(4.3)
$$\mathbf{k}_c = k^* \mathbf{H}_c,$$

with $k^*$ being an arbitrary stiffness parameter. For open cracks or no contact of the two opposite nodes the equations are decoupled by using $\mathbf{H}_{nc}$ instead of $\mathbf{H}_c$.

Element matrices for inertia or Rayleigh-damping can be derived using the same matrices $\mathbf{H}$ with arbitrary mass parameters $m^*$. The values of $\alpha$ and $\beta$ can be chosen identical with the values of the complete structure since the element itself has no damping influence.



FIG. 7. Local node numbering of contact element.

A graphic representation of the contact element is given in Fig. 7. It should be noted that, contrary to standard finite elements, the contact element may not be used in opposite direction since the sign of the contact force $S$ depends on the orientation in the coordinate system. In order to handle arbitrarily oriented cracks, local coordinates in direction of the normal to the crack surface need to be introduced.

At the first glance time integration seems to need additional contact equations since velocity and acceleration of the two connected nodes have to stay identical too. It can be shown that for implicit procedures like the Newmark method [1] where velocity and acceleration is described using displacements only, the equality of left and right displacement $U$

$$(4.4) \qquad U_l = U_r$$

simultaneously fulfills

$$(4.5) \qquad \dot{U}_l = \dot{U}_r$$

and

$$(4.6) \qquad \ddot{U}_l = \ddot{U}_r.$$

During time integration the possible change of the contact condition must be checked. As impact or release conditions hold

$$(4.7) \qquad S_c < 0$$

indicating that the contact force has become tension and according to Fig. 7b the kinematic relation becomes

$$(4.8) \qquad U_l > (x_r - x_l) + U_r = \delta + U_r.$$

The first inequality stands for opening, the second for closing of a crack. While continuation after opening the contact can be done by using unchanged displacement, velocity and acceleration simply setting the contact force zero and replacing the element matrices by $\mathbf{H}_{nc}$, some additional assumptions have to be introduced when a crash takes place. Since in general a small time interval is used, Newton's hypothesis shall be valid with the coefficient of restitution $e$ being zero, thus resulting in a common velocity $^+U_{l/r}$ of left and right node. As the contact force is an internal force, the momentum of the structure must remain unchanged during the crash. If the product of time interval and wave velocity is less than the smallest dimension of the neighboring elements, the conservation law of momentum in direction of the impact leads to

$$(4.9) \qquad {}^-\mathbf{M}^-\dot{\mathbf{U}} - {}^+\mathbf{M}^+\dot{\mathbf{U}} = 0$$

or in component form of the system mass matrix $^-\mathbf{M}$ and $^+\mathbf{M}$

$$(4.10) \qquad {}^+\dot{U}_{l/r} = \frac{{}^-\dot{U}_l \tilde{m}_l + {}^-\dot{U}_r \tilde{m}_r}{\tilde{m}_l + \tilde{m}_r}.$$

The abbreviations $\tilde{m}$ are only necessary for consistent mass matrices

$$(4.11) \qquad \tilde{m}_l = \sum_{i=1}^{n} m_{li}, \qquad \tilde{m}_r = \sum_{i=1}^{n} m_{ri}.$$

In this form the analogy of the impact of two point masses as in a lumped mass discretisation becomes apparent. The assumption of continuity of external and internal forces can be used to introduce a similar condition for the acceleration

$$(4.12) \qquad {}^+\ddot{U}_{l/r} = \frac{-\ddot{U}_l \tilde{m}_l + {}^-\ddot{U}_r \tilde{m}_r}{\tilde{m}_l + \tilde{m}_r}.$$

This equation becomes an approach if damping is incorporated into the system and the error increases with increasing damping.

For the common displacement, there is no conservation law it can be calculated from. One possibility might be an iteration within the time step to find the exact moment when the crash takes place. On avoiding this iteration, since it is very time-consuming, an estimation on an average basis must be made. Averaging the diaplacement increment like

$$(4.13) \qquad \Delta^+ U_{l/r} = \frac{\Delta^- U_l \tilde{m}_l + \Delta^- U_r \tilde{m}_r}{\tilde{m}_l + \tilde{m}_r}$$

the velocity leads to far better results than using the mean value. This could be proved by one of the test examples even for extremely small time intervals.

## 5. Numerical results

The following problems are chosen to analyse the effectiveness and reliability of the proposed contact element. The first example is introduced to show the influence of the averaging techniques and the choice of the continuation equations.
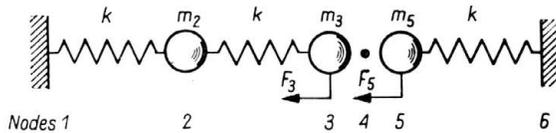
FIG. 8. Example 1. System properties: $k = 1\text{N/m}$, $m_2 = m_3 = 1$ kg, $m_5 = 100$ kg, $F_3 = F_5 = 1\text{Nh(t)}$ (unit step load).

The system represents a lumped mass, discrete spring assembly of three or when the contact element between the nodes 3 and 5 is closed, two independent degrees of freedom. The time histories of the force $F_3$ and $F_5$ are unit step functions with the instant of loading at the time $t = 0$. According to the magnitude of the masses $m_2$ and $m_3$ there should be no actual influence on the movement of the mass $m_5$. The time history of the simple average plotted in Fig. 9a leads to a not meaningful motion of node 5 in positive direction. Only the use of a weighted average according to equations (4.10), (4.12) and (4.13) as shown in Fig. 9b leads to the correct result.

The second example (Fig. 10) is constructed in order to compare the continuation technique model of Sect. 3 with a finite element model. Since the proposed weighted average rule cannot be used for simultaneous contact of more than two nodes, the resultant mass of the left span is split into two halfs, i.e. $m_2$ and $m_{11}$.

The massless stops are realized with $1\%_0$ of $m_3$.
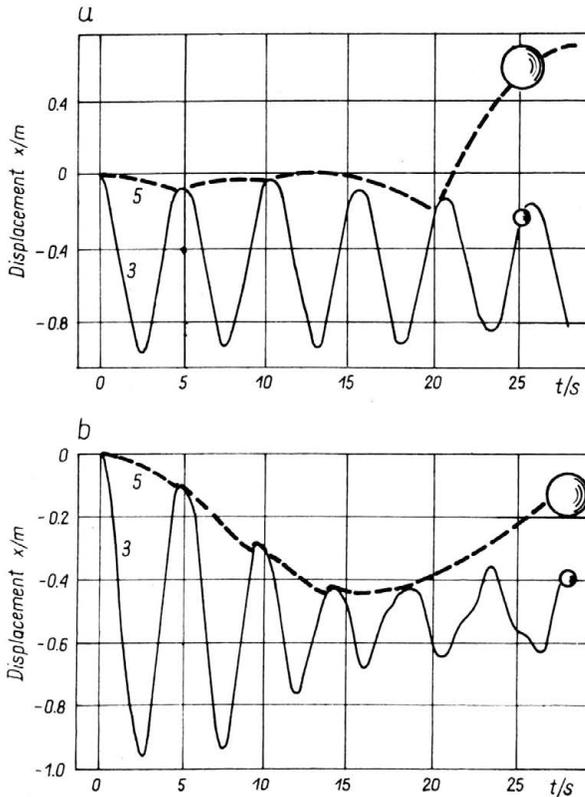
FIG. 9. Time history of the displacements of example 1; a) average, b) weighted average.
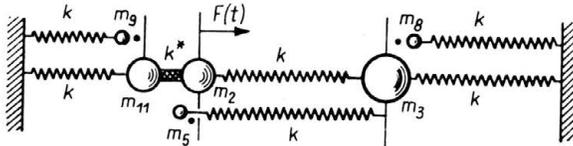


FIG. 10. Example 2. System properties, continuation technique model: $c_1 = c_2 = c_3 = 3600$ N/m, $\Delta c_i = c_i$, $m_1 = m_2 = 1800$ kg, $F(t) = 180$ kN sin (10s/t), FE-model: $k = 3600$ N/m, $k^* = 10^6$ k, $m_2 = m_{11} = 900$ kg, $m_3 = 1800$ kg, $m_9 = m_5 = m_8 = 1.8$ kg, $F = 180$ kN sin (10s/t).

The results of both calculations are plotted in Fig. 11a for the continuation technique and in Fig. 11b for the finite element solution. The displacement of $x_1$ corresponds to those of the nodes 2 and 11. The curve for $-x_2$ belongs to that of the mass $m_3$. Both systems have zero initial conditions and are subjected to harmonic excitation. The computer plots of the time histories are in good agreement thus proving the reliability of the contact element and the chosen algorithm.

Finally the results obtained for a two-dimensional discretization of a continuous beam over two spans will be discussed (Fig. 12). Basis for the time integration is a $4 \times 32$ element representation with 3 assumed cracks, each modelled with 4 contact elements.
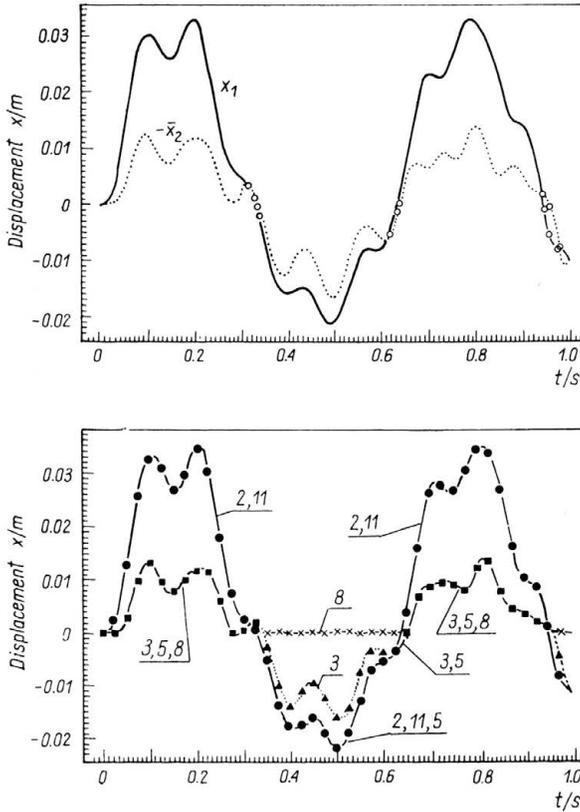
FIG. 11. Time history of the a) continuation technique model and the b) finite element model.
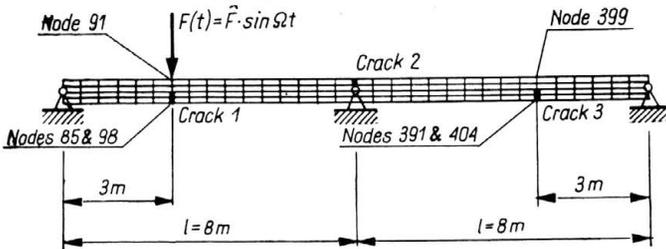


FIG. 12. Discretisation of a two-span-beam. Consistent mass matrix, $b = 0.4$ m, $h = 0.6$ m, $E = 3.4 \cdot 10^{10}$ N/m², $\varrho = 2500$ kg/m³, $\hat{F} = -1.8$ kN, $\Omega = 10$/s (50/100/314), $4 \times 32$, 8 — node-elements, $3 \times 4$, contact-elements.

The curves plotted in Fig. 13 are the time histories of the vertical displacements of the nodes 91 ($\bigcirc$) and 399 ($\triangle$). It can be recognized that the deflection under the load is much larger than in the neighboring span and that the displacement in direction of the force is about twice as large as that in the opposite direction. This fact is due to the local weakening of the structure caused by the opening of the crack underneath.
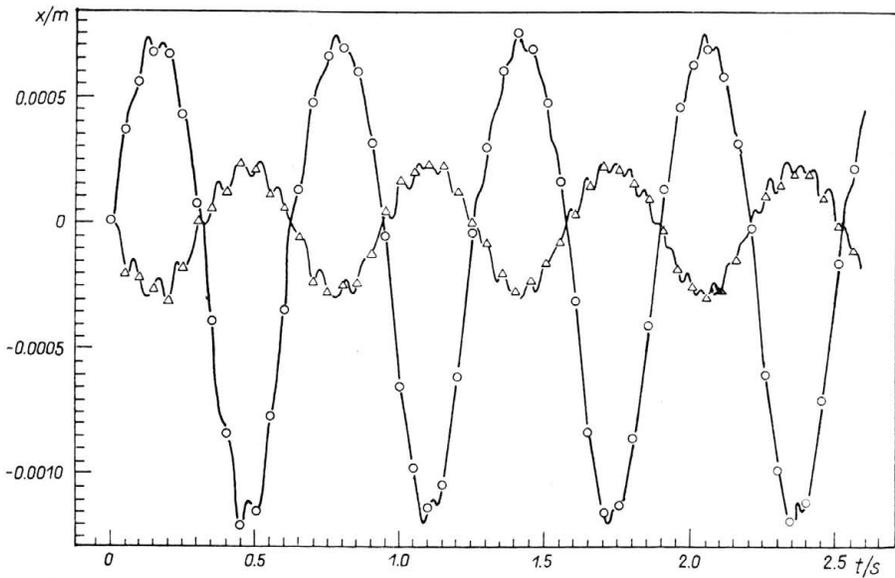
FIG. 13. Time history of the deflection of a beam calculated with a two-dimensional finite element model.
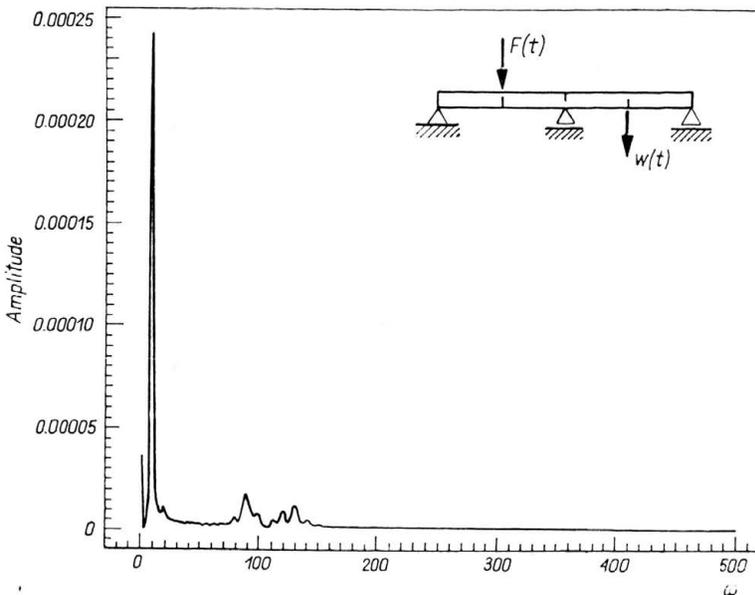


FIG. 14. Fourier transform of the deflection at node 399.

An interpretation of these data in order to give an assessment of the effect of the vibration for a human being [7] can be done after a Fourier transform [4]. The result for the vibration of the neighboring node 399 is given in Fig. 14.

Unlike to the linear problem, for which in the plotted frequency domain only 3 or 4 induced eigenvibrations can be observed, the nonlinear oscillation contains quite many frequencies. These additional peaks can be identified as eigenfrequencies of the beam

Table 1. **Main eigenfrequencies of a continuous beam with gaps.**

| Nr. | crack 1 2 3 | $\omega_1/s$ | $\omega_2/s$ | $\omega_3/s$ |
|-----|-------------|--------------|--------------|--------------|
| 1 | 0  0  0 | 84.1 | 116.2 | 326.3 |
| 2 | w  0  0 | 89.4 | 126.6 | 340.0 |
| 4 | w  w  0 | 89.8 | 142.1 | 346.4 |
| 6 | w  w  w | 97.7 | 151.1 | 348.8 |

0 — crack surfaces released, w — crack fixed

having small gaps instead of opening and closing cracks. The eigenvalues themselves, as stated in Table 1, can be obtained by a linear eigensolution, but their contribution to the nonlinear oscillation could only be calculated by time integration.

## 6. Conclusion

In this paper two different ways of calculating the response of cracked beams subjected to a time-dependent load have been discussed. The effectiveness of the first method, introducing lumped mass, discrete spring assemblies with stops for the individual crack is restricted to simple structures and a small number of cracks. The developed contact element and the proposed algorithm for the numerical integration have been successfully used even for complex configurations with a greater number of cracks. Up to now the only disadvantage of this method is the relatively long computation time. But this fact should be regarded as less important than the universal applicability of the finite element method.

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INSTITUT FÜR MECHANIK, HOCHSCHULE DER BUNDESWEHR, HAMBURG, FRG.