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Study of corotational rates for kinematic hardening in finite deformation plasticity

J. E. PAULUN (HANNOVER) and R. B. PĘCHERSKI (WARSZAWA)

THE AIM of the paper is to study corotational rates for kinematic hardening in finite deformation plasticity. The simple shear processes under both prescribed kinematical and shear traction boundary conditions were considered. This enabled a comparison to be made between existing theories, and resulted in an analysis for eliminating the unwanted oscillations in the shear stress and unsatisfactory prediction of material behaviour in the case of simple shear traction. This was achieved by introducing an influence function ω of accumulated plastic strain which produces a retardation of the material spin **W**. The modified spin was then formulated for the three-dimensional case. The merits of the present proposal vis à vis the existing theories are discussed.

Celem pracy jest zbadanie współobrotowych prędkości tensora naprężenia i parametru wzmocnienia występujących w teorii skończonych deformacji plastycznych ze wzmocnieniem kinematycznym. Zbadano problem czystego ścinania dany kinematycznie w prędkości ścinania oraz problem ścinania dany w naprężeniach ścinających. Umożliwiło to porównanie istniejących teorii i pozwoliło zaproponować modyfikację współobrotowej prędkości, która prowadzi do eliminacji niepożądanych oscylacji naprężenia ścinającego. Usuwa ona również niezadowalający opis zachowania się materiału w przypadku prostego ścinania danego w naprężeniach ścinających. Uzyskano to wprowadzając pewną funkcję wpływu ω, zależną od zakumulowanego odkształcenia plastycznego, która powoduje opóźnienie spinu materialnego **W**. Następnie określono postać zmodyfikowanego spinu dla przypadku trójwymiarowego. Przedyskutowano zalety przedstawionej propozycji w porównaniu z istniejącymi teoriami.

Целью работы является исследование совращательных скоростей тензора напряжений и параметра упрочнения, выступающих в теории конечных пластических деформаций с кинематическим упрочнением. Исследованы задача чистого сдвига, заданная кинематически в скорости сдвига и задача сдвига, заданная в напряжениях сдвига. Это дает возможность сравнить существующие теории и позволяет предложить модификацию совращательной скорости, которая приводит к исключению лишних осцилляций напряжения сдвига. Исключает она тоже неудовлетворительное описание поведения материала в случае простого сдвига, заданного в напряжениях сдвига. Это получено, вводя некоторую функцию влияния ω, зависящую от накопленной пластической деформации, которая вызывает замедление материального спина **W**. Затем определен вид модифицированного спина для трехмерного случая. Обсуждены достоинства представленного предположения по сравнению с существующими теориями.

1. Introduction

THE QUESTION of unwanted oscillatory stresses generated by simple shear to large deformation in plastic materials with kinematic hardening and Zaremba–Jaumann derivative raised by NAGTEGAAL and DE JONG [16] was analysed by several authors and remains the subject of discussion. Recently, LEE *et al.* [10] and ONAT ([17, 18]) as well as DAFALIAS ([3, 4]) and LORET [13] proposed appropriate modifications of the corotational rates of kinematic hardening tensor (back-stress) $\boldsymbol{\alpha}$ and Cauchy stress $\boldsymbol{\sigma}$. Similar questions were discussed also by FRESSENGEAS and MOLINARI [6]. Different aspects of proper choice of the objective stress rate in finite deformation problems were studied recently by ATLURI [1], JOHNSON and BAMMANN [8] and MOSS [15] as well as by SIMO and PISTER [20] and SOWERBY and CHU [21]. The growing body of the literature devoted to this subject becomes visible.

In [10] the modified corotational rate is expressed by means of the spin of material lines which are instantaneously coincident with the principal direction of **α** having the largest absolute eigenvalue. ONAT ([17, 18]) defined the modified corotational rate by means of the spin equal to the antisymmetric part of **Dα** multiplied by a constant, where **D** is the rate of plastic deformation. The proper choice of the constant leads to the non-oscillatory solution of simple shear problem. Similar expression for the corotational rate was obtained independently by DAFALIAS ([3, 4]) and LORET [13], who interpreted it as the corotational rate associated with material substructure. The qualitative analysis of the solution of the simple shear problem presented in [17] and [18] as well as in [4] and [13] shows that in non-oscillatory solution the shear stress is unbounded and increases monotonically when the deformation increases. At the same time the normal stress approaches an asymptotic upper bound, and the principal directions of **α** tend towards the bisector directions of the x_1-x_2 coordinate axes. This differs from the solution given in [10] where the both components of stress are unbounded and increase monotonically with shear strain, while the maximum principal direction of **α** inclines towards the axis x_1.

The observed qualitative discrepancies reveal that the problem is not completely explained and requires further elucidation. Furthermore, in most of papers an example of simple shear process under prescribed kinematic boundary conditions was considered only. This led the authors to the overhasty conclusion that the proposed modifications of corotational rate provide satisfactory results in general. According to our opinion this is not always true.

The aim of the paper is to study critically the discussed theories and propose a suitable form of corotational rate for kinematic hardening in finite deformation plasticity. The unified analysis of the system of differential equations, describing the problem of simple shear, led us to the conclusion that a retardation of the material spin **W** provides a non-oscillatory solution. This was achiewed by means of an influence function ω decreasing with the shear strain, γ. The function ω was specified in such a way that the angular velocity $\dot{\gamma}$, corresponding to the spin **W** in simple shear, is reduced to the angular velocity $\dot{\phi}$ of a single material line element. The influence function ω, formulated in the simple shear process, was applied in the generalization to three-dimensional case, and modified objective corotational rate associated with material substructure was obtained. The analysis of the results for simple shear and also for simple shear traction shows that the theory proposed herein reveals some advantages in comparison with the other theories under consideration. It has been shown, in particular, that these theories fail in the proper prediction of the shear stress–shear strain characteristic and of the Swift effect corresponding to torsion of thin-walled tube which is modelled by the simple shear problem under prescribed shear traction boundary conditions. This hitherto overlooked fact seems to be important for proper formulation of corotational rate of kinematic hardening in finite deformation plasticity.

2. Basic equations of rigid-plastic material with kinematic hardening

For the sake of clarity consider the simplest constitutive model of finite plastic deformation in the well-known form of rigid-plastic material with the Huber–Mises yield critetion

$$F = \frac{3}{2}(\mathbf{s}-\boldsymbol{\alpha})\cdot(\mathbf{s}-\boldsymbol{\alpha})-\sigma_0^2 = 0, \tag{2.1}$$

where $\mathbf{s}$ represents the deviator of the Cauchy stress $\boldsymbol{\sigma}$, $\boldsymbol{\alpha}$ is the deviatoric kinematic hardening tensor (back-stress), and the constant σ_0 corresponds to the tensile yield strength.

The incompressible plastic deformation is described by the associated flow rule

$$\mathbf{D} = \lambda(\mathbf{s}-\boldsymbol{\alpha}), \tag{2.2}$$

and the evolution equation for the Prager type kinematic hardening

$$\mathring{\boldsymbol{\alpha}} = \frac{2}{3}h\mathbf{D}, \tag{2.3}$$

where h denotes the plastic tangent modulus, and

$$\mathring{\boldsymbol{\alpha}} := \dot{\boldsymbol{\alpha}} - \boldsymbol{\Omega}\boldsymbol{\alpha} + \boldsymbol{\alpha}\boldsymbol{\Omega} \tag{2.4}$$

represents the Zaremba–Jaumann type corotational rate associated with an skewsymmetric tensor $\boldsymbol{\Omega}$ (which will be specified as a certain spin tensor). For $\boldsymbol{\Omega} = \mathbf{W}$ (material spin) $\mathring{\boldsymbol{\alpha}}$ corresponds to the known Zaremba–Jaumann rate.

According to the consistency condition

$$\dot{F} = 3(\mathbf{s}-\boldsymbol{\alpha})\cdot(\dot{\mathbf{s}}-\dot{\boldsymbol{\alpha}}) = 3(\mathbf{s}-\boldsymbol{\alpha})\cdot(\mathring{\mathbf{s}}-\mathring{\boldsymbol{\alpha}}) = 0 \tag{2.5}$$

and due to Eqs. (2.2) and (2.3), the plastic flow rule takes the form

$$\begin{gathered} \mathbf{D} = \frac{9}{4h\sigma_0^2}[(\mathbf{s}-\boldsymbol{\alpha})\cdot\mathring{\mathbf{s}}](\mathbf{s}-\boldsymbol{\alpha}), \\ \mathring{\mathbf{s}} = \dot{\mathbf{s}} - \boldsymbol{\Omega}\mathbf{s} + \mathbf{s}\boldsymbol{\Omega}. \end{gathered} \tag{2.6}$$

Due to the application of the corotational rates (2.4) and $(2.6)_2$, the constitutive equations (2.3) and $(2.6)_1$ are objective.

3. The unified analysis of the equations of finite simple shear

A simple shear in the direction x_1 of the coordinate axes (x_1, x_2) is defined by the displacements

$$u_1 = \dot{\gamma}tx_2, \quad u_2 = u_3 = 0, \quad \dot{\gamma} = \text{const}, \tag{3.1}$$

and the resulting velocity field

$$v_1 = \dot{\gamma}x_2, \quad v_2 = v_3 = 0, \tag{3.2}$$

where γ and $\dot{\gamma}$ are shear strain and shear strain rate, respectively. From the velocity field $\mathbf{V}$ the velocity gradient $\mathbf{L}$, the rate of deformation $\mathbf{D}$ and the material spin $\mathbf{W}$ can be calculated:

$$\mathbf{L} = \dot{\gamma}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{D} = \frac{\dot{\gamma}}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{W} = \frac{\dot{\gamma}}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3.3}$$

The basic equation of the problem is the evolution equation for $\boldsymbol{\alpha}$ which, according to (2.3) and (2.4), has the form

$$\dot{\boldsymbol{\alpha}} = \frac{2}{3} h\mathbf{D} + \boldsymbol{\Omega}\boldsymbol{\alpha} - \boldsymbol{\alpha}\boldsymbol{\Omega}, \quad \boldsymbol{\alpha}(0) = 0. \tag{3.4}$$

It is to be noted that in finite simple shear each spin $\boldsymbol{\Omega}$ which occurs in (3.4) can be expressed as follows

$$\boldsymbol{\Omega} = \omega\mathbf{W} = \frac{1}{2}\omega\dot{\gamma}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{3.5}$$

where ω is a scalar positive function of the shear strain γ with the property, $\omega(0) = 1$.
Thus, Eq. (3.4) can take the form

$$\begin{bmatrix} \dot{\alpha}_{11} & \dot{\alpha}_{12} \\ \dot{\alpha}_{12} & \dot{\alpha}_{22} \end{bmatrix} = \frac{1}{3} h\dot{\gamma}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{2}\omega\dot{\gamma}\begin{bmatrix} 2\alpha_{12} & \alpha_{22}-\alpha_{11} \\ \alpha_{22}-\alpha_{11} & -2\alpha_{12} \end{bmatrix}. \tag{3.6}$$

The assumption of incompressibility and the initial condition $(3.4)_2$ yields $\alpha_{33} = 0$ and $\alpha_{11} = -\alpha_{22}$. This leads to the following system of differential equations

$$\begin{aligned} \dot{\alpha}_{11} &= \dot{\gamma}\omega\alpha_{12}, & \alpha_{11}(0) &= 0, \\ \dot{\alpha}_{12} &= \frac{1}{3} h\dot{\gamma} - \dot{\gamma}\omega\alpha_{11}, & \alpha_{12}(0) &= 0. \end{aligned} \tag{3.7}$$

Upon differentiation of Eq. $(3.7)_2$ and the substitution of $(3.7)_1$, it yields the following second-order differential equation of simple shear

$$\ddot{\alpha}_{12} - \frac{\dot{\omega}}{\omega}\dot{\alpha}_{12} + (\dot{\gamma}\omega)^2\alpha_{12} = -\frac{1}{3} h\dot{\gamma}\frac{\dot{\omega}}{\omega}. \tag{3.8}$$

Similar equation was considered recently by DAFALIAS [3] who used the function $z = \frac{1}{2}\omega$ for the unified representation of the following special cases: $z = 1/2$ — Zaremba–Jaumann rate, $z = 2/(\gamma^2+4)$ — the rate associated with the spin $\dot{\mathbf{R}}\mathbf{R}^T$ introduced by GREEN and NAGHDI [7], where $\mathbf{R}$ is the orthogonal tensor of the polar decomposition of the deformation gradient $\mathbf{F} = \mathbf{R}\mathbf{U}$, and $z = \sin^2\theta$ — the rate proposed in [10], where θ is the angle between the x_1-axis and the eigenvector of $\boldsymbol{\alpha}$ with the maximum positive eigenvalue, $z = 1/2(1-\varrho\alpha_{11})$ — the rate proposed independently by Onat, Dafalias and Loret, where ϱ is a constant which has the dimension of $(\text{stress})^{-1}$. In [3] the analytical solutions in closed form for $z = 1/2$ and $z = 2/(\gamma^2+4)$ are presented.

In this case Eq. (3.8) can be solved analytically by means of the substitution

$$\phi := \int \omega\, d\gamma, \quad \dot{\phi} = \dot{\gamma}\omega \tag{3.9}$$

which transforms (3.8) as follows

$$\alpha''_{12} + \alpha_{12} = -\frac{1}{3} h\dot{\gamma}\frac{\ddot{\phi}}{\dot{\phi}^3}, \tag{3.10}$$

where $\alpha''_{12} := d^2\alpha_{12}/d^2\phi$.

The homogeneous solution of the linear differential Eq. (3.10) has the form

$$\alpha^h_{12} = C_1 \sin\phi + C_2 \cos\phi \tag{3.11}$$

which, in general, has oscillatory character.

The main reason for considering the function ω, and throughout (3.9) the function ϕ, stems from the fact that a particular choice of ω leads to a non-oscillatory solution of Eq. (3.10). This can be fulfilled in different ways. It seams to be physically plausible to take ϕ as the transformation of the infinite domain of simple shear strain $0 < \gamma < \infty$ into the region $0 < \phi < \frac{\pi}{2}$. This provides the non-oscillatory solution of (3.10). In such a case ϕ can be interpreted as the shear angle

$$\phi = \arctan\gamma. \tag{3.12}$$

The substitution of Eqs. (3.9) into (3.12) yields the specification of the function ω

$$\omega = \frac{1}{1+\gamma^2}, \tag{3.13}$$

which can be called the influence function.

Due to (3.12) and (3.13),

$$\omega = \cos^2\phi \tag{3.14}$$

and the final differential equation of finite simple shear (3.10) takes the form

$$\alpha''_{12} + \alpha_{12} = \frac{2}{3} h \tan\phi (1+\tan^2\phi) \tag{3.15}$$

with the initial conditions

$$\alpha_{12}(0) = 0, \quad \alpha'_{12}(0) = \frac{1}{3} h. \tag{3.16}$$

The idea of using the transformation (3.12) is consistent with the requirement stated in [10] that, instead of the constant angular velocity $\dot{\gamma}/2$ generating the unlimited rotation of $\boldsymbol{\alpha}$ as $t \to \infty$, the angular velocity of the material line elements, which can only rotate by no more than π as $t \to \infty$, should occur in the proper formulation of the spin $\boldsymbol{\Omega}$. In [10] the spin is expressed in terms of the angular velocity of continuously changing material line elements coinciding instantaneously with the maximum eigenvector of $\boldsymbol{\alpha}$. On the other hand, the angular velocity of single material line elements can be used directly to formulate the modified spin. In such a case the angular velocity $\dot{\phi}$ is given by

$$\dot{\phi} = \dot{\gamma}\cos^2\phi = \frac{\dot{\gamma}}{1+(\tan\phi_0+\gamma)^2}, \tag{3.17}$$

where ϕ_0 is the initial angle of the material line element with respect to the axis x_2.

The spin of the material line element with the initial angle ϕ_0:

$$\boldsymbol{\Omega} = \chi\dot{\phi}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{3.18}$$

where χ takes into account the initial condition

$$\boldsymbol{\Omega}_0 = \frac{\dot{\gamma}}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \chi\dot{\gamma}\cos^2\phi_0\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.19}$$

leading to $\chi = 1/2(1+\tan^2\phi_0)$, takes, due to (3.17), the following form

$$\boldsymbol{\Omega} = \frac{\dot{\gamma}}{2}\,\frac{1+\tan^2\phi_0}{1+(\gamma+\tan\phi_0)^2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3.20}$$

Hence, in general, the influence function associated with the material line element with the initial angle ϕ_0 is given by

$$\omega = \frac{1+\tan^2\phi_0}{1+(\tan\phi_0+\gamma)^2}. \tag{3.21}$$

Due to (3.17) the values of ω given by (3.21) are proportional to the angular velocity of the material line element.

The approach presented in [10] leads to the different specification of the influence function ω. In this case the system of differential equations of simple shear takes the form

$$\begin{aligned} \dot{\alpha}_{11} &= 2\dot{\gamma}\sin^2\theta\alpha_{12}, \\ \dot{\alpha}_{12} &= \frac{1}{3}h\dot{\gamma}-2\dot{\gamma}\sin^2\theta\alpha_{11}. \end{aligned} \tag{3.22}$$

The comparison of Eqs. (3.22) with Eq. (3.7) yields

$$\omega = 2\sin^2\theta, \tag{3.23}$$

where

$$\theta = \frac{1}{2}\arctan\frac{\alpha_{12}}{\alpha_{11}}. \tag{3.24}$$

Figure 1 displays the shape of the influence functions given by (3.13) and (3.23). For he comparison, in Fig. 1 are also shown the influence function pertinent to the spin $\dot{\mathbf{R}}\mathbf{R}^T$

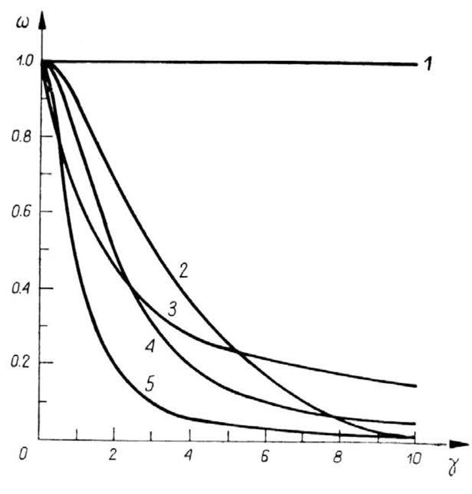

FIG. 1. The influence function ω versus shear strain Curves *1* for $\omega = 1$ — Zaremba–Jaumann derivative, *2* for $\omega = 1-\varrho\alpha_{11}$ $\varrho = 0.5$, Refs. [4, 13, 17, 18], *3* for $\omega = 1-\frac{\alpha_{11}}{(\alpha_{11}^2+\alpha_{12}^2)^{1/2}}$, Refs. [10, 11], *4* for $\omega = \frac{4}{4+\gamma^2}$ Refs. [3, 5], and *5* for $\omega = \frac{4}{1+\gamma^2}$.

of the principal directions of the stretching tensor **U** discussed in [3] and in [5] as well as in [6, 7, 8 and 21]

$$\omega = \frac{4}{4+\gamma^2}, \tag{3.25}$$

and the function ω corresponding to the rate considered in [4], [13], [17] and [18]:

$$\omega = 1-\varrho\alpha_{11}. \tag{3.26}$$

The functions (3.23) and (3.25) were determined numerically by the solution of the system of diffetential equations (3.7) for the specified influence function ω. Let us observe that, also in the cases given by (3.23), (3.25) and (3.26), the values of the influence function ω are proportional to the angular velocity of the maximum principal direction of $\boldsymbol{\alpha}$, the angular velocity of the maximum principal direction of **U** and the angular velocity associated with the spin $(\mathbf{D}\boldsymbol{\alpha}-\boldsymbol{\alpha}\mathbf{D})$, respectively. Figure 1 shows that the specified functions of ω decrease in γ what produces retardation of the material spin **W** in the course of deformation process.

4. Analytical solution of the problem of simple shear for the corotational rate related to $\omega = (1+\tan^2\phi_0)/[1+(\tan\phi_0+\gamma)^2]$

Equation (3.8), the specification of the influence function ω given by (3.21) and the substitution

$$\phi = \arctan(\tan\phi_0+\gamma), \qquad \dot{\phi} = \frac{\dot{\gamma}\omega}{1+\tan^2\phi_0} \tag{4.1}$$

lead to the following equation:

$$\alpha''_{12}+(1+\tan^2\phi_0)\alpha_{12} = \frac{2}{3}h\tan\phi(1+\tan^2\phi). \tag{4.2}$$

In the simplest case, when $\phi_0 = 0$, Eq. (4.2) transforms into Eq. (3.15).

The analytical solution of the differential equation (4.2) can be given according to KAMKE [9] (p. 413, equation 2,36b) in the form

$$\alpha_{12} = C_1\cos[(1+\tan^2\phi_0)\phi]+C_2\sin[(1+\tan^2\phi_0)\phi] + \frac{2h}{3(1+\tan^2\phi_0)}\int\limits_{\phi_0}^{\phi}\frac{\sin\tau}{\cos^3\tau}\sin[(1+\tan^2\phi_0)(\phi-\tau)]\,d\tau. \tag{4.3}$$

The constants C_1, C_2 can be determined from the initial conditions

$$\alpha_{12} = 0, \qquad \alpha'_{12} = \frac{h}{3}(1+\tan^2\phi_0), \tag{4.4}$$

whereas the component α_{11} of the back-stress tensor $\boldsymbol{\alpha}$ can be calculated from Eq. $(3.7)_2$ and relations (3.17) and (3.21)

$$\alpha_{11} = \frac{1+\tan(\phi_0+\gamma)^2}{3(1+\tan^2\phi_0)}h-\alpha'_{12}\frac{1}{1+\tan^2\phi_0}. \tag{4.5}$$

This is the general solution of the simple shear problem with the corotational rate related to the material line element with an initial angle ϕ_0. For particular values of ϕ_0, e.g. $\phi_0 = 0$ and $\phi_0 = \frac{\pi}{4}$, the integral in (4.3) has the closed analytical form and, in such a case, the closed analytical solution exists. In the simplest case, when $\phi_0 = 0$, the solution of the simple shear problem can be expressed after transformations as follows:

$$\alpha_{12} = \frac{1}{3} h \left(\tan\phi - \sin\phi - \cos\phi \ln \frac{1-\sin\phi}{\cos\phi} \right), \tag{4.6}$$

$$\alpha_{11} = \frac{1}{3} h \left(\cos\phi - 1 - \sin\phi \ln \frac{1-\sin\phi}{\cos\phi} \right). \tag{4.7}$$

From the Equations (2.1), (2.2) and (3.3)

$$s_{11} = \alpha_{11}, \qquad s_{12} = \frac{\sigma_0}{\sqrt{3}} + \alpha_{12}, \tag{4.8}$$

and the solution of the simple shear problem for $\phi_0 = 0$ in terms of s_{12}, s_{11} and γ takes the form

$$s_{12} = \frac{\sigma_0}{\sqrt{3}} + \frac{h}{3(1+\gamma^2)^{1/2}} \{\gamma(1+\gamma^2)^{1/2} - \gamma - \ln[(1+\gamma^2)^{1/2} - \gamma]\}, \tag{4.9}$$

$$s_{11} = \frac{h}{3(1+\gamma^2)^{1/2}} \{1 - (1-\gamma^2)^{1/2} - \gamma \ln[(1+\gamma^2)^{1/2} - \gamma]\}. \tag{4.10}$$

For the comparison, the closed analytical solution of the simple shear problem for $\phi_0 = \frac{\pi}{4}$ is presented

$$\alpha_{12} = \frac{2}{3} h \Big\{ \cos\phi \sin\phi [1/2 - 1/2 \tan^2\phi - \ln(2\cos^2\phi)] + (\sin^2\phi - \cos^2\phi) \left(\tan\phi + \phi - \frac{1}{2} - \frac{\pi}{4} \right) \Big\}, \tag{4.11}$$

$$\alpha_{11} = \frac{1}{6} h \{ 1 + \tan^2\phi + (\sin^2\phi - \cos^2\phi)[1 - 3\tan^2\phi - 2\ln(2\cos^2\phi)] + 2\sin\phi\cos\phi(\tan^2\phi - 5\tan\phi - 4\phi + \pi + 2) \}, \tag{4.12}$$

or in terms of s_{11}, s_{12} and γ:

$$s_{11} = \frac{1}{6} h \Big\{ 1 + (1+\gamma)^2 + \frac{(1+\gamma)^2 - 1}{1+(1+\gamma)^2} \left[1 - 3(1+\gamma)^2 - 2\ln\left(\frac{2}{1+(1+\gamma)^2} \right) \right] + \frac{2(1+\gamma)}{1+(1+\gamma)^2} [(1+\gamma)^3 - 5(1+\gamma) - 4\arctan(1+\gamma) + \pi + 2] \Big\}, \tag{4.13}$$

$$s_{12} = \frac{\sigma_0}{\sqrt{3}} + \frac{1}{6} h \Big\{ \frac{2(1+\gamma)}{1+(1+\gamma)^2} \left[1 - (1+\gamma)^2 - 2\ln\left(\frac{2}{1+(1+\gamma)^2} \right) \right] + \frac{(1+\gamma)^2 - 1}{1+(1+\gamma)^2} [4(1+\gamma) - 4\arctan(1+\gamma) - 2 - \pi] \Big\}. \tag{4.14}$$

The solution given by (4.9) and (4.10) as well as by (4.13) and (4.14) are displayed in Fig. 2. The curves were calculated for the material constants taken from [10] for an aluminium alloy: $\sigma_0 = 207$ (MPa) and $h = 310$ (MPa). It is visible that the formulation of the modified corotational rate in terms of the influence function (3.21) associated with

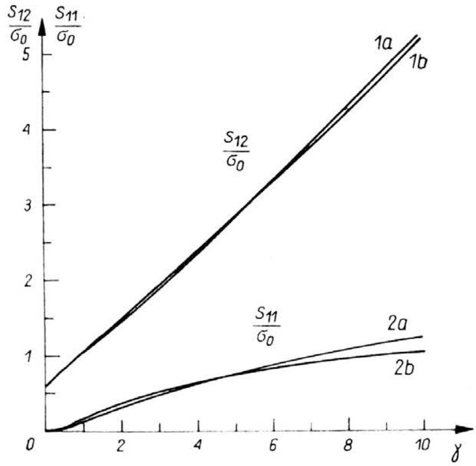

FIG. 2. Stress versus shear strain. The curves $1a$—s_{12} for $\phi_0 = 0$ and $1b$—s_{12} for $\phi_0 = \pi/4$, $2a$—s_{11} for $\phi_0 = \pi/4$ and $2b$—s_{11} for $\phi_0 = 0$.

the angular velocity of a single material line element leads to the prediction of unbounded, continuously increasing shear stress and normal stress in shear strain. It can be observed, furthermore, that the results corresponding to the both values of the initial angle are similar. This motivates us to chose the simplest form of the influence function (3.21), corresponding to $\phi_0 = 0$, as representative for any material line element in the case of simple shear.

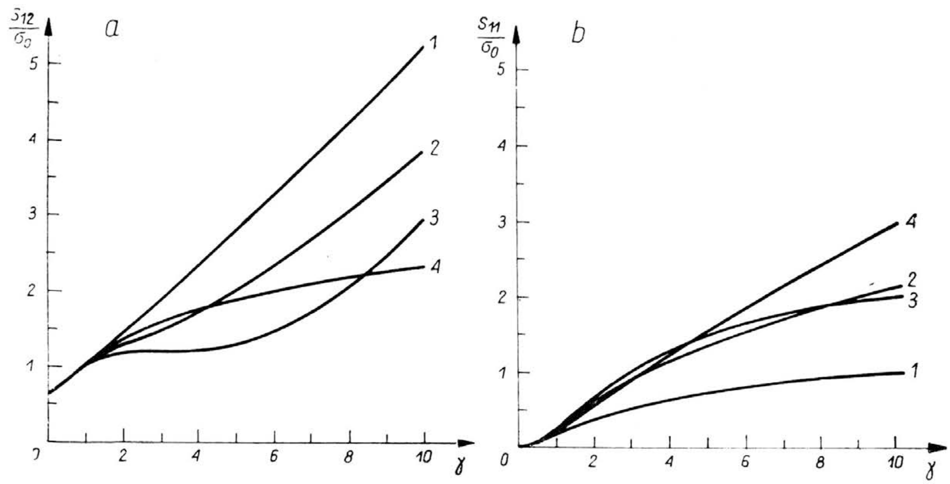

FIG. 3. Shear stress versus shear strain (a), normal stress versus shear strain (b). Curves 1 for $\omega = \dfrac{1}{1+\gamma^2}$, 2 for $\omega = \dfrac{4}{4+\gamma^2}$, Refs. [3, 5], 3 for $\omega = 1-\varrho\alpha_{11}$ $\varrho = 0.5$ Refs. [4, 13, 17, 18], and 4 for

$$\omega = 1 - \frac{\alpha_{11}}{(\alpha_{11}^2+\alpha_{12}^2)^{1/2}},$$ Refs. [10, 11].

Figures 3a and 3b show the comparison of the solution (4.9) and (4.10) — curve *1*, with the values of the stress deviator components s_{12} and s_{11} determined numerically by the solution of the problem of simple shear according to the theory associated with the spin $\dot{\mathbf{R}}\mathbf{R}^T$ — curve *2*, and the theories based on the spin $(\mathbf{D}\boldsymbol{\alpha}-\boldsymbol{\alpha}\mathbf{D})$ and the function (3.26), for $\varrho = 0.5$ — curve *3*, as well as the theory of LEE *et al.* [10] — curve *4*. According to LORET, DAFALIAS and ONAT, the nonsocillatory solution of the problem of simple shear can be obtained for an array of values of the constants ϱ. Some of these results were discussed in [4] and [13]. In each case, however, the normal component α_{11} of the back-stress has an upper bound equal to $1/\varrho$, what does not seem to be justified physically.

The inclinations to the axis x_1 of the direction of the maximum absolute principal component of $\boldsymbol{\alpha}$ for the before mentioned theories are displayed in Fig. 4. It is remarkable

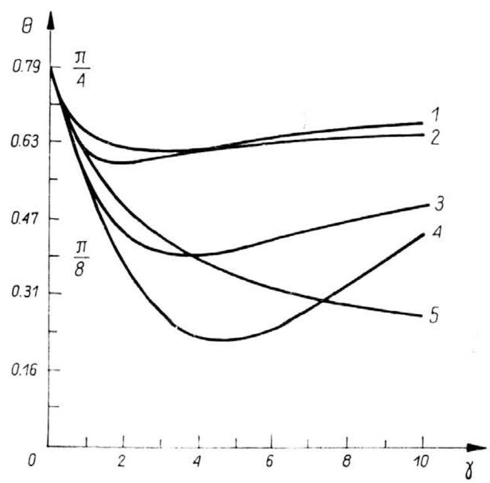

FIG. 4. The inclination angle of the direction of the maximum absolute principal component of $\boldsymbol{\alpha}$ versus shear strain, curves *1* and 2 for $\omega = \frac{1}{1+\gamma^2}$ and $\phi_0 = 0$, and $\phi_0 = \pi/4$, respectively, curves *3* and *4* for $\omega = 1-\varrho\alpha_{11}$, for $\varrho = 1$ and $\varrho = 0.5$, respectively, Refs. [4, 13, 17, 18], and curve *5* for $\omega = 1-\frac{\alpha_{11}}{(\alpha_{11}^2+\alpha_{12}^2)^{1/2}}$, Refs. [10, 11].

that, according to [10], the maximum principal direction of $\boldsymbol{\alpha}$ inclines towards the axis x_1, whereas in the other cases this direction tends towards the bisector direction of the x_1-x_2 coordinate frame in the limit when $\gamma \to \infty$.

5. Formulation of modified objective rate of stress corotational with substructure

To formulate the modified spin $\boldsymbol{\Omega}$ for the three-dimensional case, the relation (3.5) and the form of the influence function ω given in (3.13) should be generalized.

Consider

$$\boldsymbol{\Omega}^* := \mathbf{W}-\boldsymbol{\Omega} \tag{5.1}$$

and substitute it into Eq. (2.4). It follows that

$$\overset{\circ}{\boldsymbol{\alpha}} = \overset{\nabla}{\boldsymbol{\alpha}} + \boldsymbol{\Omega}^*\boldsymbol{\alpha} - \boldsymbol{\alpha}\boldsymbol{\Omega}^*, \tag{5.2}$$

where

$$\overset{\nabla}{\boldsymbol{\alpha}} = \dot{\boldsymbol{\alpha}} - \mathbf{W}\boldsymbol{\alpha} + \boldsymbol{\alpha}\mathbf{W} \tag{5.3}$$

is the Zaremba–Jaumann derivative. The rate $\overset{\circ}{\boldsymbol{\alpha}}$ is objective provided $\boldsymbol{\Omega}^*$ is objective. The question arises as to what is the simplest, physically plausible, objective representation of $\boldsymbol{\Omega}^*$.

It is commonly recognized in the theory of finite inelastic deformations of crystalline solids that the material moves with respect to the underlying crystall lattice, whereas the lattice itself undergoes elastic deformation and relative rigid-body rotations provided the phenomenon of lattice misorientation occurs (cf. PĘCHERSKI [19]). Such variables as the back-stress tensor $\boldsymbol{\alpha}$ and the stress tensor $\boldsymbol{\sigma}$ are "carried" by material substructure associated with the crystal lattice. Therefore, it seems reasonable to define the objective rate by means of rates corotational with the material substructure. Basing on microstructural considerations, MANDEL [14] devoloped the theory of plasticity in which the substructure corotational rate is defined in terms of the spin of the triad of director vectors attached to the substructure. According to Mandel's theory, constitutive relations are required not only for the rate of plastic deformation but also for the plastic spin. The general form of macroscopic constitutive relations for the plastic spin was discussed by DAFALIAS [4] and LORET [13] who used the representation theorem for isotropic second-rank antisymmetric tensor valued functions. In the case of plastic deformation with kinematic hardening, the plastic spin $\mathbf{W}^p$ can be expressed in the following simplified form (cf. [4] and [13]):

$$\mathbf{W}^p = \bar{\eta}(\boldsymbol{\alpha}\mathbf{D} - \mathbf{D}\boldsymbol{\alpha}), \tag{5.4}$$

where $\bar{\eta}$ is a scalar function of the isotropic invariants of $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}$.

Let us observe that the spin $\boldsymbol{\Omega}^*$ is determined by the plastic spin $\mathbf{W}^p$ and, due to (5.4), can be represented in the following way

$$\boldsymbol{\Omega}^* = \eta(\boldsymbol{\alpha}\mathbf{D} - \mathbf{D}\boldsymbol{\alpha})_u, \tag{5.5}$$

where

$$(\boldsymbol{\alpha}\mathbf{D} - \boldsymbol{\alpha}\mathbf{D})_u := \frac{(\boldsymbol{\alpha}\mathbf{D} - \mathbf{D}\boldsymbol{\alpha})}{\left(\mathrm{tr}[(\boldsymbol{\alpha}\mathbf{D} - \mathbf{D}\boldsymbol{\alpha})^2]\right)^{1/2}}. \tag{5.6}$$

It then remains to determine the function η. For the simple shear we have

$$\boldsymbol{\Omega} = \frac{1}{2}\dot{\gamma}\omega\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{W} = \frac{\dot{\gamma}}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{5.7}$$

and

$$(\boldsymbol{\alpha}\mathbf{D} - \mathbf{D}\boldsymbol{\alpha})_u = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{5.8}$$

According to Eqs. (5.1) and (5.5), the relations (5.7) and (5.8) yield

$$\eta = \frac{\sqrt{2}}{2}(1-\omega)\dot{\gamma}, \tag{5.9}$$

or from Eq. (3.13)

$$\eta = \frac{\sqrt{2}}{2}\frac{\gamma^2}{1+\gamma^2}\dot{\gamma}. \tag{5.10}$$

Relation (5.10) represents the difference between the constant angular speed produced by the material spin **W** and the angular velocity of the material line element lying initially along the axis x_2. Accordingly, (5.10) pertains to the angular velocity corresponding to the spin $\boldsymbol{\Omega}^*$ and appears to be the function of accumulated plastic strain γ. Also in the three-dimensional generalization of (5.10), an additional scalar parameter corresponding to the accumulated plastic strain is necessary to provide the proper description of kinematic hardening.

From Eq. (3.3),

$$\dot{\varepsilon}_{eq} = \sqrt{\frac{2}{3}\mathbf{D}\cdot\mathbf{D}} = \frac{\dot{\gamma}}{\sqrt{3}} \quad \text{and} \quad \gamma = \sqrt{3}\,\varepsilon_{eq}. \tag{5.11}$$

Consequently, the equivalent plastic strain can play the role of such a parameter. According to (5.10) and (5.11),

$$\eta = \frac{3}{2}\frac{3\varepsilon_{eq}^2}{1+3\varepsilon_{eq}^2}\dot{\varepsilon}_{eq}. \tag{5.12}$$

This is the generalized form of the angular velocity corresponding to the spin $\boldsymbol{\Omega}^*$ and, accordingly, to the plastic spin $\mathbf{W}^p$.

Thus, due to (5.1) and (5.4), the modified spin takes the form

$$\boldsymbol{\Omega} = \mathbf{W} - \eta(\boldsymbol{\alpha}\mathbf{D} - \mathbf{D}\boldsymbol{\alpha})_u, \tag{5.13}$$

and according to (2.3) and (2.4) we have

$$\overset{\nabla}{\boldsymbol{\alpha}} = \frac{2}{3}h\mathbf{D} - \eta[(\boldsymbol{\alpha}\mathbf{D}-\mathbf{D}\boldsymbol{\alpha})_u\boldsymbol{\alpha} - \boldsymbol{\alpha}(\boldsymbol{\alpha}\mathbf{D}-\mathbf{D}\boldsymbol{\alpha})_u]. \tag{5.14}$$

This is the evolution equation for kinematic hardening tensor $\boldsymbol{\alpha}$ with the modified substructure corotational rate which, due to (5.12), becomes nonlinearly dependent upon the accumulated plastic strain. Equation (5.14) provides a practical specification of a more general theory proposed by DAFALIAS [4] and LORET [13].

When

$$\eta = c\sqrt{\mathrm{tr}[(\boldsymbol{\alpha}\mathbf{D}-\mathbf{D}\boldsymbol{\alpha})^2]}, \qquad c = \text{const}, \tag{5.15}$$

Eq. (5.13) becomes similar to that considered by ONAT ([17, 18]), DAFALIAS [4] and LORET [13] as the simplest formulation of the substructure corotational rate which was applied for the solution of simple shear problem.

6. Solution of the problem of simple shear traction

Recently LEE and WERTHEIMER [11] have applied the modified corotational rate of stress and back-stress, proposed in [10], in the analysis of the problem of simple shear prescribed by shear traction boundary conditions corresponding to torsion of a cylindrical tube. This example can be considered as a test problem which may be used for the comparison of different theories discussed herein, as well as for experimental verification. The problem of simple shear traction for finite plastic deformation with kinematic hardening was considered previously by LEHMANN [12], who also revealed anomalous material reaction in the both, simple shear and simple shear traction problems.

Let us consider a thin-walled cylindrical tube loaded by torque only. Due to static determinacy and assumption of uniform stress distribution around the tube, the applied Cauchy stress tensor can be given in the following truncated form

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix}. \tag{6.1}$$

By symmetry, no rotation of planes normal to the axis of the tube occur, so that the deformation gradient $\mathbf{F}$ has the components

$$\mathbf{F} = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}, \tag{6.2}$$

and due to plastic incompressibility we have

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} \\ D_{12} & -D_{11} \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} 0 & D_{12} \\ -D_{12} & 0 \end{bmatrix}. \tag{6.3}$$

From (2.2) and (6.3) the back-stress tensor $\boldsymbol{\alpha}$ has the form

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & -\alpha_{11} \end{bmatrix}. \tag{6.4}$$

In the case of simple shear traction the modified spin $\boldsymbol{\Omega}$ can be expressed as follows

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & \zeta \\ -\zeta & 0 \end{bmatrix}, \tag{6.5}$$

where ζ is the general expression for the spin component pertinent to different specifications of the influence function ω.

Due to (6.3), (6.4) and (6.5), the evolution equation (2.3) yields

$$\begin{bmatrix} \dot{\alpha}_{11} & \dot{\alpha}_{12} \\ \dot{\alpha}_{12} & -\dot{\alpha}_{11} \end{bmatrix} = \frac{2}{3} h \begin{bmatrix} D_{11} & D_{12} \\ D_{12} & -D_{11} \end{bmatrix} + 2\zeta \begin{bmatrix} \alpha_{12} & -\alpha_{11} \\ -\alpha_{11} & -\alpha_{12} \end{bmatrix}. \tag{6.6}$$

According to Eqs. $(2.6)_2$ and (6.5), the corotational rate of the Cauchy stress deviator $\mathbf{s}$ is given by

$$\mathring{\mathbf{s}} = \begin{bmatrix} -2\zeta\tau & \dot{\tau} \\ \dot{\tau} & 2\zeta\tau \end{bmatrix}, \tag{6.7}$$

and due to Eq. $(2.6)_1$

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{12} & -D_{11} \end{bmatrix} = \frac{3}{2h\tau_0^2} [\dot{\tau}(\tau-\alpha_{12}) + 2\zeta\tau\alpha_{11}] \begin{bmatrix} -\alpha_{11} & \tau-\alpha_{12} \\ \tau-\alpha_{12} & \alpha_{11} \end{bmatrix} \tag{6.8}$$

where $\tau_0 = \dfrac{\sigma_0}{\sqrt{3}}$.

It follows from Eq. (5.13) specified for the simple shear traction problem, that

$$\zeta = D_{12} - \frac{\sqrt{2}}{2}\eta, \tag{6.9}$$

where η is given by (5.12).

In the case considered by LEE and WERTHEIMER [11], it is seen that

$$\zeta = 2D_{12}\sin^2\theta + 2D_{11}\sin\theta\cos\theta, \tag{6.10}$$

whereas for the spin proposed by ONAT ([17], [18]) and DAFALIAS [4] as well as LORET [13]:

$$\zeta = D_{12} + \varrho(D_{11}\alpha_{12} - D_{12}\alpha_{11}), \quad \varrho = \text{const}. \tag{6.11}$$

Inserting ζ, given by (6.9), (6.10) and (6.11), into Eqs. (6.6) and (6.8), the three systems of equations can be obtained which describe the simple shear traction problem for different corotational rates. These systems of equations have been solved numerically for the applied shear traction τ as the independent variable increasing linearly in time with $\dot{\tau} = 1 \left|\frac{\text{MPa}}{\text{s}}\right|$ and for the initial conditions

$$\alpha_{11}(0) = \alpha_{12}(0) = 0, \quad D_{11} = 0, \quad D_{12} = \frac{3\dot{\tau}}{2h}. \tag{6.12}$$

The material constants were chosen the same as in the problem of simple shear discussed before.

The ensuing components α_{12} and α_{11} of the back-stress tensor $\boldsymbol{\alpha}$ are shown in Fig. 5. It is visible that the components α_{12} pertinent to the different theories discussed have similar values. The larger discrepancies occur in the case of the components α_{11} representing the second-order effect.

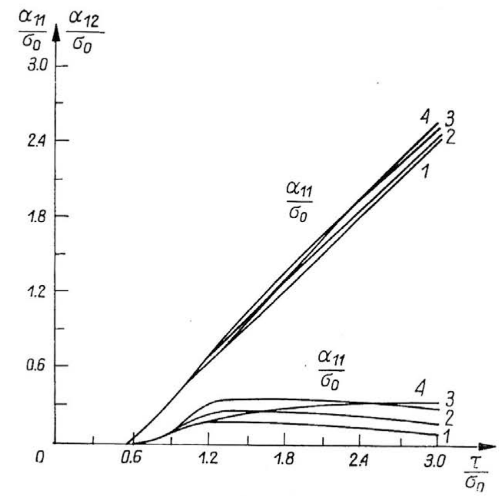

FIG. 5. Components of the back-stress tensor $\boldsymbol{\alpha}$ versus applied shear stress. Curves *1* for ζ given by Eq. (6.9), *2* for ζ given by Eq. (6.11), $\varrho = 1$, *3* for ζ given by Eq. (6.11), $\varrho = 0.5$, and *4* for ζ given by Eq. (6.10).

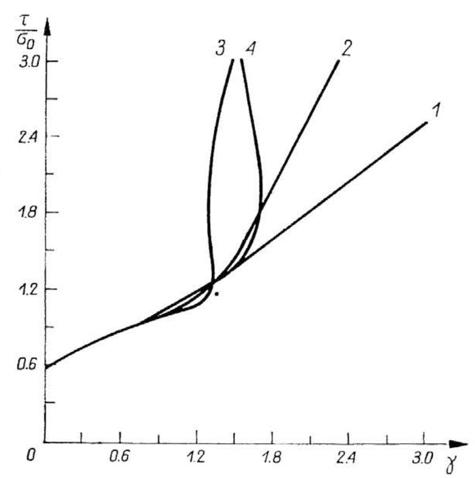

FIG. 6. Shear stress versus shear strain. Curves *1* for ζ given by Eq. (6.9), *2* for ζ given by Eq. (6.11), $\varrho = 1$, *3* for ζ given by Eq. (6.11), $\varrho = 0.5$, and *4* for ζ given by Eq. (6.10).

From the solution of the simple shear traction problem, modelling torsion of thin-walled tube, the material characteristic $\tau-\gamma$ can be obtained. Such a relation for different theories is displayed in Fig. 6. The curves *1* and *2* corresponding respectively to the modified spin (6.9) and (6.11) for $\varrho = 1$, increase monotonically with shear strain γ, whereas the curves *3* and *4* pertinent to the modified spin (6.10) and (6.11) for $\varrho = 0.5$, reveal decrease in γ when τ increases upon a certain value. Although it is difficult to find the results of the direct experimental test of the mentioned characteristics for large strains, the shape of the curves *3* and *4* does not seem to be realistic. The study of the change of the component H_{22} of the displacement gradient as a function of shear strain γ which corresponds to the axial plastic elongation of the twisted tube, the so-called Swift effect, provides a better possibility of experimental verification. The experimental investigations of the Swift effect were reported recently by BILLINGTON [2]. The results obtained for twisted tubes of copper, aluminum and iron shown in [2] reveal the continuous increase in H_{22} with an increase in γ. Although it is not possible to compare the results quantitatively because of different material constants, it is possible to observe that the values of the displacement gradient components H_{22} always remain much smaller than the corresponding

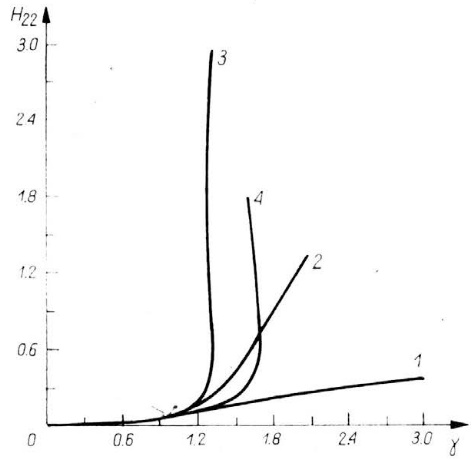

FIG. 7. Plastic elongation versus shear strain. Curves *1* for ζ given by Eq. (6.9), *2* for ζ given by Eq. (6.11), $\varrho = 1$, *3* for ζ gien by Eq. (6.11), $\varrho = 0.5$, and *4* for ζ given by Eq. (6.10).

values of γ. Figure 7 shows the dependence of H_{22} on the shear strain γ predicted by the theories discussed. It is visible that the curve 1 corresponding to the modified spin (6.9) conforms rather well with this observation.

7. Discussion and conclusions

The study of corotational rates shows that the choice or formulation of appropriate rate in finite deformation problems is not based solely on the question how to avoid the unwanted oscillatory stresses, but refers rather to proper constitutive description of anisotropic hardening in the course of finite plastic deformation process. The theory originated by Mandel which is based on constitutive relations for plastic spin and substructure corotational rate is applied herein.

The unified analysis of the system of differential equations describing the problem of simple shear led us to the conclusion that a retardation of the material spin **W** provides a non-oscillatory solution. This was made with use of the function ω decreasing in γ. As it is depicted in Fig. 1, the function ω is pertinent to each corotational rate under consideration. This motivated us to call ω an influence function.

Equation (5.5) with the derived function η, given in (5.12), provides the practical and non-trivial specification of the general representation of the constitutive relation for plastic spin derived in [4] and [13]. Although the derived form of η is neither unique nor general, it is hitherto, according to authors knowledge, the only formulation which leads to satisfactory solution of the simple shear traction problem and simulation of the Swift effect. It can be concluded that the nonlinear generalization of the evolution equation for $\boldsymbol{\alpha}$ provides a reasonable theoretical prediction of the material behaviour. Further studies on proper formulation of such an equation are necessary. It should be related with the search for nonlinear specification of the constitutive equation for plastic spin.

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Note added in proof

The concept of plastic spin and its application for the finite deformation problems of plasticity with anisotropic hardening is studied intensively by many researches and several worth noting papers have been appeared after the completion of the present work. Some of them, e.g. DAFALIAS [1, 2], REED and ATLURI [6] and LORET [3], are refered and discussed thoroughly in our forthcoming publications, PAULUN and PĘCHERSKI [4] and PĘCHERSKI [5]. Although the applications of the plastic spin concept have been considered for different kinds of initial and induced anisotropies, [1, 2], there is still a gap between the theoretical representations of the plastic spin constitutive relation and practical applications for proper prediction of material reaction at finite plastic deformations and anisotropic hardening. The proposed approximation of the general constitutive equation for plastic spin, given in (5.5)—(5.6) and (5.12), can appear to be helpful in filling this gap. Similarily as in [6], the experimental results of the torsion of thin walled tube obtained by Swift have been used, [4], to verify the theory with modified spin (5.12)—(5.14). The theoretical prediction of the normal strain E_{11} versus shear strain γ, computed with use of (6.9) and the material parameters corresponding to the test of Swift, conforms very well to the experimental points. This provides stronger justification for the derived relations for plastic spin and corotational rate. The theory presented in [1,2], as well as, the derived relations for plastic spin (5.12)—(5.14) make a basis for the modelling of deformation instability phenomena, where the new concept of perturbed plastic spin plays the crucial rôle, [5]. The results presented in [3] are also discussed, where the effect of plastic spin on the onset of localisation is studied.

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INSTITUT FÜR BAUMECHANIK AND NUMERISCHE MECHANIK
UNIVERSITÄT HANNOVER, FRG

and

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

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